06. Acceleration and Normalized Emittance*

Prof. Steven M. Lund
Physics and Astronomy Department
Facility for Rare Isotope Beams (FRIB)
Michigan State University (MSU)

MSU PHY 905 and US Particle Accelerator School
"Accelerator Physics"
Steven M. Lund and Yue Hao

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S0: Acceleration and Normalized Emittance
S0: Introduction & Equation of Motion Derivation

The *Lorentz force equation* of a charged particle is given by (MKS Units):

$$\frac{d}{dt}\mathbf{p}_i(t) = q_i\left[\mathbf{E}(\mathbf{x}_i,t) + \mathbf{v}_i(t)\times\mathbf{B}(\mathbf{x}_i,t)\right]$$

$m_i, \; q_i$ .... particle mass, charge

$\mathbf{x}_i(t)$ .... particle coordinate

$\mathbf{p}_i(t) = m_i\gamma_i(t)\mathbf{v}_i(t)$ .... particle momentum

$\mathbf{v}_i(t) = \frac{d}{dt}\mathbf{x}_i(t) = c\vec{\beta}_i(t)$ .... particle velocity

$\gamma_i(t) = \frac{1}{\sqrt{1-\beta_i^2(t)}}$ .... particle gamma factor

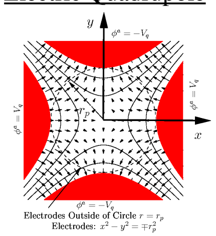
$i =$ particle index
$t =$ time

	Total		Applied		Self
Electric Field:	$\mathbf{E}(\mathbf{x},t)$	$=$	$\mathbf{E}^a(\mathbf{x},t)$	$+$	$\mathbf{E}^s(\mathbf{x},t)$
Magnetic Field:	$\mathbf{B}(\mathbf{x},t)$	$=$	$\mathbf{B}^a(\mathbf{x},t)$	$+$	$\mathbf{B}^s(\mathbf{x},t)$

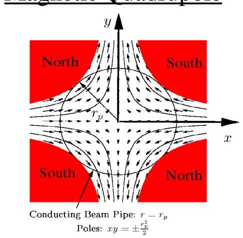
S1B: Applied Fields used to Focus, Bend, and Accelerate Beam

Transverse optics for focusing:

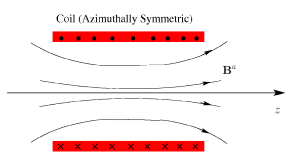
Electric Quadrupole



Magnetic Quadrupole

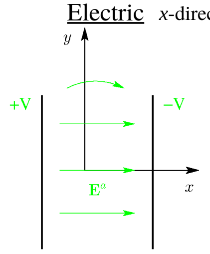


Solenoid

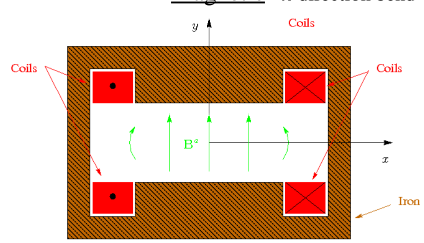


Dipole Bends:

Electric *x*-direction bend

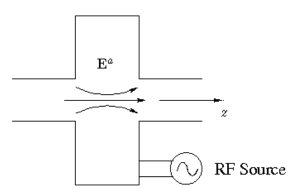


Magnetic *x*-direction bend

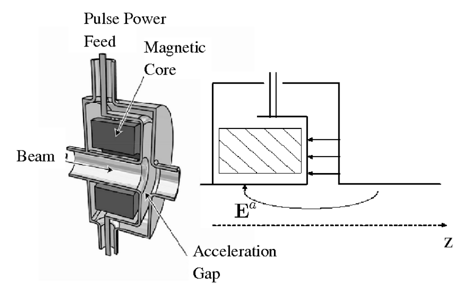


Longitudinal Acceleration:

RF Cavity



Induction Cell

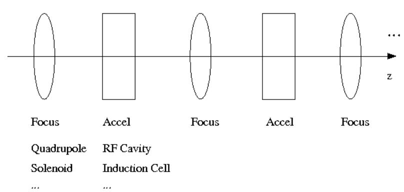


We will cover primarily transverse dynamics in initial lectures. Later lectures will cover acceleration and longitudinal physics:

- Acceleration in uences transverse dynamics – not possible to fully decouple

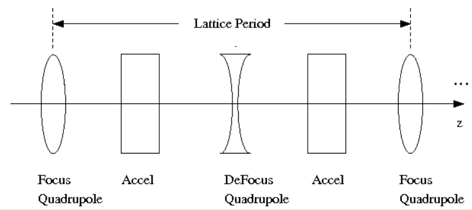
S1C: Machine Lattice

Applied   eld structures are often arraigned in a regular (periodic) lattice for beam transport/acceleration:



- Sometimes functions like bending/focusing are combined into a single element

Example – Linear FODO lattice (symmetric quadrupole doublet)



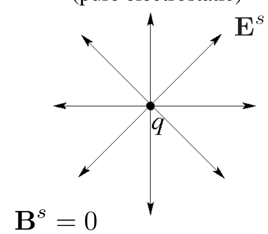
S1D: Self   elds

Self-   elds are generated by the distribution of beam particles:

Charges

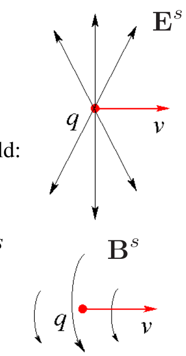
Currents

Particle at Rest (pure electrostatic)



Particle in Motion

Obtain from Lorentz boost of rest-frame   eld: see Jackson, *Classical Electrodynamics*



- Superimpose for all particles in the beam distribution
- Accelerating particles also radiate

We neglect self-   elds

- Possible to include at various levels of approx. Will be touched on later in the course.

Neglect

	Total		Applied		Self
Electric Field:	$\mathbf{E}(\mathbf{x},t)$	$=$	$\mathbf{E}^a(\mathbf{x},t)$	$+$	$\mathbf{E}^s(\mathbf{x},t)$
Magnetic Field:	$\mathbf{B}(\mathbf{x},t)$	$=$	$\mathbf{B}^a(\mathbf{x},t)$	$+$	$\mathbf{B}^s(\mathbf{x},t)$

$$\mathbf{E}(\mathbf{x},t) \simeq \mathbf{E}^a(\mathbf{x},t)$$

$$\mathbf{B}(\mathbf{x},t) \simeq \mathbf{B}^a(\mathbf{x},t)$$

- Applied   elds must obey the Maxwell Equations
- Expansions and idealized forms of the   elds are often used
  - Example: Linear applied focusing   elds

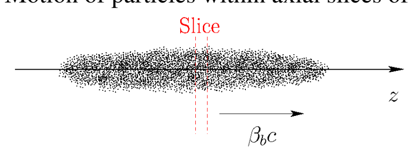
In accelerators, typically there is ideally a single species of particle:

$$q_i \rightarrow q \qquad m_i \rightarrow m$$

Large Simpli cation!

Multi-species results in more complex collective e ects

Motion of particles within axial slices of the "bunch" are highly directed:



$$\beta_b(z)c \equiv \frac{1}{N'}\sum_{i=1}^{N'} \mathbf{v}_i \cdot \hat{\mathbf{z}}$$

$=$ Mean axial velocity of N' particles in beam slice

$$\frac{d}{dt}\mathbf{x}_i(t) = \mathbf{v}_i(t) = \hat{\mathbf{z}}\beta_b(z)c + \delta\mathbf{v}_i$$

$|\delta\mathbf{v}_i| \ll |\beta_b|c$ Paraxial Approximation

Force:

$$\mathbf{F}_i^a = q\mathbf{E}_i^a + q\mathbf{v}_i \times \mathbf{B}_i^a \qquad \mathbf{E}^a(\mathbf{x}_i,t) \equiv \mathbf{E}_i^a \ \text{ etc.}$$

The particle equations of motion in $\mathbf{x}_i - \mathbf{v}_i$ phase-space variables become:

- Separate parts of $q\mathbf{E}_i^a + q\mathbf{v}_i \times \mathbf{B}_i^a$ into transverse and longitudinal comp

Transverse

$$\frac{d}{dt}\mathbf{x}_{\perp i} = \mathbf{v}_{\perp i}$$
$$\frac{d}{dt}(m\gamma_i \mathbf{v}_{\perp i}) \simeq \quad q\mathbf{E}_{\perp i}^a + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}_{\perp i}^a + qB_{zi}^a \mathbf{v}_{\perp i} \times \hat{\mathbf{z}}$$

Longitudinal

$$\frac{d}{dt}z_i = v_{zi}$$
$$\frac{d}{dt}(m\gamma_i v_{zi}) \simeq \quad qE_{zi}^a - q(v_{xi}B_{yi}^a - v_{yi}B_{xi}^a)$$

In the remainder of this (and most other) lectures, we analyze Transverse Dynamics. Longitudinal Dynamics will covered in later lectures

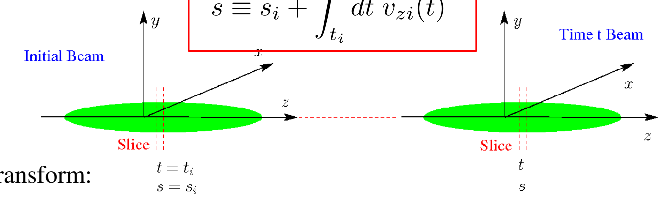
- Except near injector, acceleration is typically slow
  - Fractional change in γ_b, β_b small over characteristic transverse dynamical scales such as lattice period and betatron oscillation periods
- Regard γ_b, β_b as speci ed functions given by the "acceleration schedule"

S1E: Equations of Motion in s and the Paraxial Approximation

In transverse accelerator dynamics, it is convenient to employ the axial coordinate (s) of a particle in the accelerator as the independent variable:

- Need elds at lattice location of particle to integrate equations for particle trajectories

$$s \equiv s_i + \int_{t_i}^{t} d\tilde{t}\; v_{zi}(\tilde{t})$$



Transform:

$$v_{zi} = \frac{ds}{dt} \quad \Longrightarrow \quad v_{xi} = \frac{dx_i}{dt} = \frac{ds}{dt}\frac{dx_i}{ds} = v_{zi}\frac{dx_i}{ds} = (\beta_b c + \delta v_{zi})\frac{dx_i}{ds}$$
$$\simeq \beta_b c \frac{dx_i}{ds}$$

Neglect

Denote:

$$' \equiv \frac{d}{ds} \qquad v_{xi} = \frac{dx_i}{dt} \simeq \beta_b c x_i' \qquad v_{yi} = \frac{dy_i}{dt} \simeq \beta_b c y_i'$$

Neglecting term consistent with assumption of small longitudinal momentum spread (paraxial approximation)

- Procedure becomes more complicated when bends present

In the paraxial approximation, x' and y' can be interpreted as the (small magnitude) angles that the particles make with the longitudinal-axis:

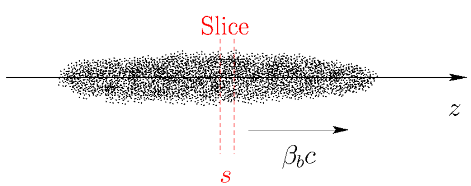
$$x-\text{angle} = \frac{v_{xi}}{v_{zi}} \simeq \frac{v_{xi}}{\beta_b c} = x_i'$$
$$y-\text{angle} = \frac{v_{yi}}{v_{zi}} \simeq \frac{v_{yi}}{\beta_b c} = y_i'$$

Typical accel lattice values:
$|x'| < 50$ mrad

The angles will be *small* in the paraxial approximation:

$$v_{xi}^2,\ v_{yi}^2 \ll \beta_b^2 c^2 \quad \Longrightarrow \quad x_i'^2,\ y_i'^2 \ll 1$$

Since the spread of axial momentum/velocities is small in the paraxial approximation, a thin axial slice of the beam maps to a thin axial slice and s can also be thought of as the axial coordinate of the slice in the accelerator lattice



$$\beta_b \equiv \sum_{i=1}^{N'} \frac{v_{zi}}{c}$$
slice

$$s \simeq s_i + \int_{t_i}^{t} d\tilde{t}\; \beta_b(\tilde{t})$$

$$s \simeq s_i + \int_{t_i}^{t} d\tilde{t}\; \beta_b(\tilde{t})$$

The coordinate s can alternatively be interpreted as the axial coordinate of a reference (design) particle moving in the lattice

- Design particle has no momentum spread

It is often desirable to express the particle equations of motion in terms of s rather than the time t

- Makes it clear where you are in the lattice of the machine
- Sometimes easier to use t in codes when including many e ects to high order

Transform transverse particle equations of motion to *s* rather than *t* derivatives

$$\frac{d}{dt}(m\gamma_i \mathbf{v}_{\perp i}) \simeq q\mathbf{E}^a_{\perp i} + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} + qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}}$$

Term 1 (left boxed) — Term 2 (right boxed)

Transform Terms 1 and 2 in the particle equation of motion:

$$\frac{d}{dt} = v_{zi}\frac{d}{ds}$$

Term 1:
$$\frac{d}{dt}\left(m\gamma_i \frac{d\mathbf{x}_{\perp i}}{dt}\right) = mv_{zi}\frac{d}{ds}\left(\gamma_i v_{zi}\frac{d}{ds}\mathbf{x}_{\perp i}\right)$$
$$= m\gamma_i v_{zi}^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i} + mv_{zi}\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_i v_{zi})$$

Term 1A — Term 1B

Approximate:

Term 1A: $m\gamma_i v_{zi}^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i} \simeq m\gamma_b\beta_b^2 c^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i} = m\gamma_b\beta_b^2 c^2 \mathbf{x}''_{\perp i}$

Term 1B: $mv_{zi}\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_i v_{zi}) \simeq m\beta_b c\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_b\beta_b c)$

$$\simeq m\beta_b c^2(\gamma_b\beta_b)'\mathbf{x}'_{\perp i}$$

Using the approximations 1A and 1B gives for Term 1:

$$m\frac{d}{dt}\left(\gamma_i\frac{d\mathbf{x}_{\perp i}}{dt}\right) \simeq m\gamma_b\beta_b^2c^2\left[\mathbf{x}''_{\perp i} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_{\perp i}\right]$$

Similarly we approximate in Term 2:

$$qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}} \simeq qB^a_{zi}\beta_b c\mathbf{x}'_{\perp i} \times \hat{\mathbf{z}}$$

Using the simplified expressions for Terms 1 and 2 obtain the reduced transverse equation of motion:

$$\mathbf{x}''_{\perp i} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_{\perp i} = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_{\perp i} + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} + \frac{qB^a_{zi}}{m\gamma_b\beta_b c}\mathbf{x}'_{\perp i} \times \hat{\mathbf{z}}$$

Write out transverse particle equations of motion in explicit component form:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' = \frac{q}{m\gamma_b\beta_b^2c^2}E^a_x - \frac{q}{m\gamma_b\beta_b c}B^a_y + \frac{q}{m\gamma_b\beta_b c}B^a_z y'$$
$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}E^a_y + \frac{q}{m\gamma_b\beta_b c}B^a_x - \frac{q}{m\gamma_b\beta_b c}B^a_z x'$$

For linear fields without skew coupling, incorporate in lattice functions $\kappa_x, \quad \kappa_y$

Equations previously derived under assumptions:

- No bends (fixed *x*-*y*-*z* coordinate system with no local bends)
- Paraxial equations ($x'^2, y'^2 \ll 1$)
- No dispersive effects (β_b same all particles), acceleration allowed ($\beta_b \neq$ const)
- Self-field interactions neglected

Summary of Transverse Particle Equations of Motion

In linear applied focusing channels, without momentum spread or radiation, and space-charge effects, the particle equations of motion in both the *x*- and *y*-planes expressed as:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x = 0$$
$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y = 0$$

$\kappa_x(s) = x\text{-focusing function of lattice}$

$\kappa_y(s) = y\text{-focusing function of lattice}$

Common focusing functions:

Continuous:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Quadrupole (Electric or Magnetic):

$$\kappa_x(s) = -\kappa_y(s) = \kappa(s)$$

Solenoidal (equations must be interpreted in rotating Larmor Frame):

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

Although the equations have the same form, the couplings to the fields are different which leads to different regimes of applicability for the various focusing technologies with their associated technology limits:

Focusing:

Continuous:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Good qualitative guide
BUT not physically realizable

Quadrupole:

$$\kappa_x(s) = -\kappa_y(s) = \begin{cases} \frac{G(s)}{\beta_b c[B\rho]}, & \text{Electric} \\ \frac{G(s)}{c[B\rho]}, & \text{Magnetic} \end{cases} \qquad [B\rho] = \frac{m\gamma_b\beta_b c}{q}$$

G is the field gradient which for linear applied fields is:

$$G(s) = \begin{cases} -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2}, & \text{Electric} \\ \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \frac{B_p}{r_p}, & \text{Magnetic} \end{cases}$$

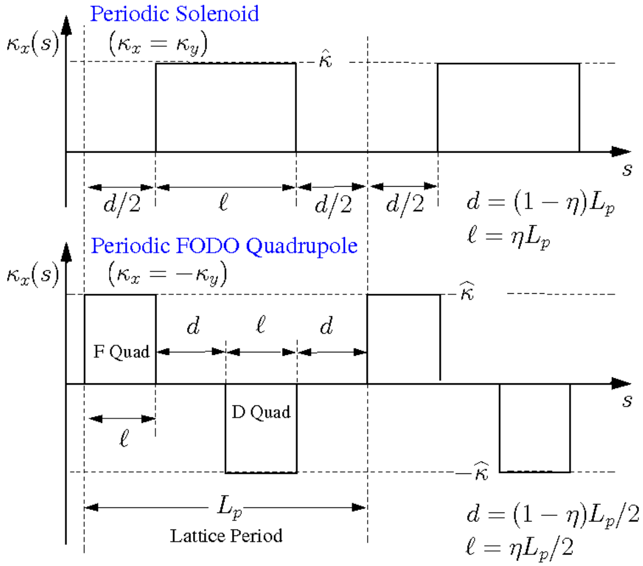
Solenoid:

$$\kappa_x(s) = \kappa_y(s) = k_L^2(s) = \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2 \qquad \omega_c(s) = \frac{qB_{z0}(s)}{m}$$

In many cases beam transport lattices are designed where the applied focusing functions are periodic:

$$\kappa_x(s + L_p) = \kappa_x(s) \qquad L_p = \text{Lattice Period}$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

Common, simple examples of periodic lattices:



However, the focusing functions need not be periodic:

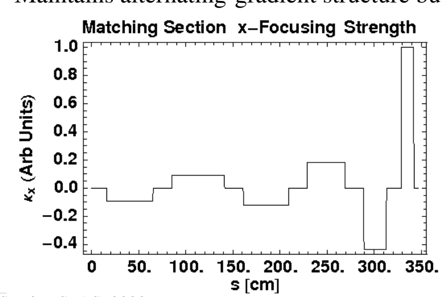
- Often take periodic or continuous in this class for simplicity of interpretation

Focusing functions can vary strongly in many common situations:

- Matching and transition sections
- Strong acceleration
- Signi cantly di erent elements can occur within periods of lattices in rings
  - "Panofsky" type (wide aperture along one plane) quadrupoles for beam insertion and extraction in a ring

Example of Non-Periodic Focusing Functions: Beam Matching Section

Maintains alternating-gradient structure but not quasi-periodic



Example corresponds to High Current Experiment Matching Section (hard edge equivalent) at LBNL (2002)

S10: Acceleration and Normalized Emittance

S10A: Introduction

If the beam is accelerated longitudinally in a linear focusing channel, the *x*-particle equation of motion is:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = 0$$

Analogous equation holds in *y*

Neglects:

- Nonlinear applied focusing elds
- Momentum spread e ects

Comments:

- $\gamma_b, \ \beta_b$ are regarded as prescribed functions of s set by the acceleration schedule of the machine/lattice
- Variations in $\gamma_b, \ \beta_b$ due to acceleration must be included in and/or compensated by adjusting the strength of the optics via optical parameters contained in $\kappa_x, \ \kappa_y$ to maintain lattice quasi-periodicity
  - Example: for quadrupole focusing adjust eld gradients (see: S2)

Acceleration Factor: Characteristics of

Relativistic Factor

$$\gamma_b\beta_b \simeq \begin{cases} \gamma_b, & \text{Ultra Relativistic Limit} \\ \beta_b, & \text{Nonrelativistic Limit} \end{cases} \qquad \gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$

Beam/Particle Kinetic Energy:

$$\mathcal{E}_b(s) = (\gamma_b - 1)mc^2 = \text{Beam Kinetic Energy}$$

- Function of s speci ed by Acceleration schedule for transverse dynamics
- See S11 for calculation of $\mathcal{E}_b$ and $\gamma_b\beta_b$ from longitudinal dynamics and later lectures on Longitudinal Dynamics

Approximate energy gain from average gradient:

$$\mathcal{E}_b \simeq \mathcal{E}_i + G(s - s_i)$$

$\mathcal{E}_i = \text{const} = \text{Initial Energy}$

$G = \text{const} = \text{Average Gradient}$

- Real energy gain will be rapid when going through discreet acceleration gaps

$$\mathcal{E}_b \simeq \begin{cases} \gamma_b mc^2, & \text{Ultra Relativistic Limit}, \gamma_b \gg 1 \\ \frac{1}{2}m\beta_b^2c^2, & \text{Nonrelativistic Limit}, |\beta_b| \ll 1 \end{cases}$$

Comments Continued:

- In typical accelerating systems, changes in $\gamma_b\beta_b$ are slow and the fractional changes in the orbit induced by acceleration are small
  - Exception near an injector since the beam is often not yet energetic
- The acceleration term:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} > 0$$

will act to **damp particle oscillations** (see following slides for motivation)

Even with acceleration, we will nd that there is a Courant-Snyder invariant (normalized emittance) that is valid in an analogous context as in the case without acceleration provided phase-space coordinates are chosen to compensate for the damping of particle oscillations

Identify relativistic factor with average gradient energy gain:

Ultra Relativistic Limit: $\gamma_b \gg 1, \quad \beta_b \simeq 1$

$$\gamma_b \simeq \frac{\mathcal{E}_b}{mc^2} = \frac{\mathcal{E}_i}{mc^2} + \frac{G}{mc^2}(s - s_i)$$

$$\Longrightarrow \quad \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} \simeq \frac{1}{\frac{\mathcal{E}_i}{G} + (s - s_i)} \sim \frac{1}{s - s_i}$$

Nonrelativistic Limit: $|\beta_b| \ll 1, \quad \gamma_b \simeq 1$

$$\beta_b \simeq \sqrt{2\frac{\mathcal{E}_b}{mc^2}} = \sqrt{2\frac{\mathcal{E}_i}{mc^2} + 2\frac{G}{mc^2}(s - s_i)}$$

$$\Longrightarrow \quad \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\beta_b'}{\beta_b} = \frac{1/2}{\frac{\mathcal{E}_i}{G} + (s - s_i)} \sim \frac{1}{2(s - s_i)}$$

- Expect Relativistic and Nonrelativistic motion to have similar solutions
  - Parameters for each case will be quite di erent

/// Aside: Acceleration and Continuous Focusing Orbits with $\kappa_x = k_{\beta 0}^2 = \text{const}$

Assume relativistic motion and negligible space-charge:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} = \frac{1}{(\frac{\mathcal{E}_i}{G} - s_i) + s}$$

Then the equation of motion $x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = 0$ reduces to:

$$x'' + \frac{1}{(\frac{\mathcal{E}_i}{G} - s_i) + s}x' + k_{\beta 0}^2 x = 0$$

This equation is the equation of a Bessel Function of order zero:

$$\frac{d^2x}{d\xi^2} + \frac{1}{\xi}\frac{dx}{d\xi} + x = 0 \qquad \xi = k_{\beta 0}s + k_{\beta 0}\left(\frac{\mathcal{E}_i}{G} - s_i\right)$$

$$\xi' = k_{\beta 0}$$

$$x = C_1 J_0(\xi) + C_2 Y_0(\xi)$$

$$x' = -C_1 k_{\beta 0} J_1(\xi) - C_2 k_{\beta 0} Y_1(\xi)$$

$C_1 = \text{const}$ $\quad C_2 = \text{const}$

$J_n =$ Order n Bessel Func (1st kind)

$Y_n =$ Order n Bessel Func (2nd kind)

$dJ_0(x)/dx = -J_1(x)$ and same for Y_0

Solving for the constants in terms of the particle initial conditions:

$$\begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} J_0(\xi_i) & Y_0(\xi_i) \\ -k_{\beta 0}J_1(\xi_i) & -k_{\beta 0}Y_1(\xi_i) \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$x_i \equiv x(s = s_i) \qquad \xi_i \equiv k_{\beta 0}\frac{\mathcal{E}_i}{G} = \xi(s = s_i)$$

$$x_i' \equiv x'(s = s_i)$$

Invert matrix to solve for constants in terms of initial conditions:

$$\Longrightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -k_{\beta 0}Y_1(\xi_i) & -Y_0(\xi_i) \\ k_{\beta 0}J_1(\xi_i) & J_0(\xi_i) \end{bmatrix} \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

$$\Delta \equiv k_{\beta 0}[Y_0(\xi_i)J_1(\xi_i) - J_0(\xi_i)Y_1(\xi_i)]$$

Comments:

- Bessel functions behave like *damped harmonic oscillators*
  - See texts on Mathematical Physics or Applied Mathematics
- Nonrelativistic limit solution is *not* described by a Bessel Function solution
  - The coe cient in the damping term $\propto x'$ has a factor of 2 di erence, preventing exact Bessel function form
  - Properties of solution will be similar though (similar special function)

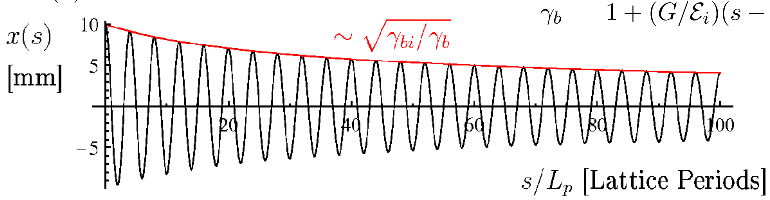
Using this solution, plot the orbit for (contrived parameters for illustration only):

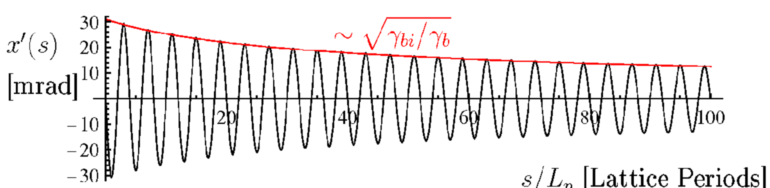
$$k_{\beta 0} = \frac{\sigma_0}{L_p} \qquad \sigma_0 = 90^\circ/\text{Period} \qquad \mathcal{E}_i = 1000 \text{ MeV}$$

$$L_p = 0.5 \text{ m} \qquad G = 100 \text{ MeV/m}$$

$$x(0) = 10 \text{ mm} \qquad s_i = 0$$

$$x'(0) = 0 \text{ mrad} \qquad \frac{\gamma_{bi}}{\gamma_b} = \frac{1}{1 + (G/\mathcal{E}_i)(s - s_i)}$$





- Solution shows damping: phase volume scaling $\sim 1/(\gamma_b\beta_b) \simeq 1/\gamma_b$ ///

S10B: Transformation to Normal Form

"Guess" transformation to apply motivated by conjugate variable arguments

$$\tilde{x} \equiv \sqrt{\gamma_b\beta_b}\,x$$

Here we reuse tilde variables to denote a transformed quantity we choose to look like something familiar from simpler contexts

Then:

$$x = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}$$

$$x' = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}' - \frac{1}{2}\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)^{3/2}}\tilde{x}$$

$$x'' = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}'' - \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)^{3/2}}\tilde{x}' + \left[\frac{3}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^{5/2}} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)^{3/2}}\right]\tilde{x}$$

The inverse phase-space transforms will also be useful later:

$$\tilde{x} = \sqrt{\gamma_b\beta_b}\,x$$

$$\tilde{x}' = \sqrt{\gamma_b\beta_b}\,x' + \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{(\gamma_b\beta_b)}}x$$

Applying these results, the particle x- equation of motion with acceleration becomes:

$$\tilde{x}'' + \left[\kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}\right]\tilde{x} = 0$$

An additional step can be taken to further stress the correspondence between the transformed system with acceleration and the untransformed system in the absence of acceleration.

Denote an e ective focusing strength:

$$\tilde{\kappa}_x \equiv \kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}$$

$\tilde{\kappa}_x$ incorporates acceleration terms beyond γ_b, β_b factors already included in the de nition of κ_x (see: S2):

$$\kappa_x = \begin{cases} \frac{qG}{m\gamma_b\beta_b^2c^2}, & G = -\partial E_x^a/\partial x = \partial E_y^a/\partial y = \text{Electric Quad. Grad.} \\ \frac{qG}{m\gamma_b\beta_b c}, & G = \partial B_x^a/\partial y = \partial B_y^a/\partial x = \text{Magnetic Quad. Grad.} \\ \frac{qB_{z0}}{4m\gamma_b^2\beta_b^2c^2}, & B_{z0} = \text{Solenoidal Magnetic Field} \end{cases}$$

The transformed equation of motion with acceleration then becomes:

$$\tilde{x}'' + \tilde{\kappa}_x\tilde{x} = 0$$

The transformed equation with acceleration has the same form as the equation in the absence of acceleration. If space-charge is negligible ($\partial\phi/\partial\mathbf{x}_\perp \simeq 0$) we have:

Accelerating System		Non-Accelerating System
$\tilde{x}'' + \tilde{\kappa}_x\tilde{x} = 0$	$\Longrightarrow$	$x'' + \kappa_x x = 0$

Therefore, *all previous analysis* on phase-amplitude methods and Courant-Snyder invariants associated with Hill's equation in x-x' phase-space can be immediately applied to $\tilde{x} - \tilde{x}'$ phase-space for an accelerating beam

$$\left(\frac{\tilde{x}}{\tilde{w}_x}\right)^2 + (\tilde{w}_x\tilde{x}' - \tilde{w}_x'\tilde{x})^2 = \tilde{\epsilon} = \text{const}$$

$$\tilde{w}_x'' + \tilde{\kappa}_x\tilde{w}_x - \frac{1}{\tilde{w}_x^3} = 0$$

$$\tilde{w}_x(s + L_p) = \tilde{w}_x(s)$$

$$\pi\tilde{\epsilon} = \text{Area traced by orbit} = \text{const in } \tilde{x}\text{-}\tilde{x}' \text{ phase-space}$$

- Focusing eld strengths need to be adjusted to maintain periodicity of $\tilde{\kappa}_x$ in the presence of acceleration
  - Not possible to do exactly, but can be approximate for weak acceleration

S10C: Phase Space Relation Between Transformed and UnTransformed Systems

It is instructive to relate the transformed phase-space area in tilde variables to the usual x-x' phase area:

$$d\tilde{x} \otimes d\tilde{x}' = |J| dx \otimes dx'$$

where J is the Jacobian:

$$J \equiv \det\begin{bmatrix} \frac{\partial\tilde{x}}{\partial x} & \frac{\partial\tilde{x}}{\partial x'} \\ \frac{\partial\tilde{x}'}{\partial x} & \frac{\partial\tilde{x}'}{\partial x'} \end{bmatrix} = \det\begin{bmatrix} \sqrt{\gamma_b\beta_b} & 0 \\ \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{\gamma_b\beta_b}} & \sqrt{\gamma_b\beta_b} \end{bmatrix} = \gamma_b\beta_b$$

Inverse transforms derived in S10B:

$$\tilde{x} = \sqrt{\gamma_b\beta_b}\,x$$

$$\tilde{x}' = \sqrt{\gamma_b\beta_b}\,x' + \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{\gamma_b\beta_b}}x$$

Thus:

$$d\tilde{x} \otimes d\tilde{x}' = \gamma_b\beta_b \, dx \otimes dx'$$

Based on this area transform, if we de ne the (instantaneous) phase space area of the orbit trance in *x-x'* to be $\pi\epsilon_x$ "regular emittance", then this emittance is related to the "normalized emittance" $\tilde{\epsilon}_x$ in $\tilde{x} - \tilde{x}'$ phase-space by:

$$\tilde{\epsilon}_x = \gamma_b \beta_b \epsilon_x$$
$$\equiv \text{Normalized Emittance} \equiv \epsilon_{nx}$$

- Factor $\gamma_b \beta_b$ compensates for acceleration induced damping in particle orbits
- Normalized emittance is very important in design of lattices to transport accelerating beams
  - Designs usually made assuming conservation of normalized emittance

S11: Calculation of Acceleration Induced Changes in gamma and beta

S11A: Introduction

The transverse particle equation of motion with acceleration was derived in a Cartesian system by approximating (see: S1):

$$\frac{d}{dt}\left(m\gamma\frac{d\mathbf{x}_\perp}{dt}\right) \simeq q\mathbf{E}^a_\perp + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}^a_\perp + qB^a_z\mathbf{v}_\perp \times \hat{\mathbf{z}} - q\frac{1}{\gamma_b^2}\frac{\partial \phi}{\mathbf{x}_\perp}$$

using

$$m\frac{d}{dt}\left(\gamma\frac{d\mathbf{x}_\perp}{dt}\right) \simeq m\gamma_b\beta_b^2c^2\left[\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp\right]$$

to obtain:

$$\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_\perp + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}^a_\perp + \frac{qB^a_z}{m\gamma_b\beta_b c}\mathbf{x}'_\perp \times \hat{\mathbf{z}} - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial \mathbf{x}_\perp}\phi$$

To integrate this equation, we need the variation of β_b and $\gamma_b = 1/\sqrt{1-\beta_b^2}$ as a function of s. For completeness here, we brie y outline how this can be done by analyzing longitudinal equations of motion. More details can be found in lectures to follow on Longitudinal Dynamics.

S11B: Solution of Longitudinal Equation of Motion

Changes in $\gamma_b\beta_b$ are calculated from the longitudinal particle equation of motion:

- See equation at end of S1D

$$\frac{d}{dt}\left(m\gamma\frac{dz}{dt}\right) \simeq qE^a_z - q(v_xB^a_y - v_yB^a_x) - q\frac{\partial\phi}{\partial z}$$

Term 1 Term 2 Term 3 Neglect Rel to Term 2

Using steps similar to those in S1, we approximate terms:

Term 1: $\frac{d}{dt}\left(\gamma\frac{dz}{dt}\right) \simeq c^2\beta_b(\gamma_b\beta_b)'$ $\frac{dz}{dt} = v_z \simeq \beta_b c$ $\gamma \simeq \gamma_b$

$\frac{d}{dt} \simeq \beta_b c\frac{d}{ds}$

Term 2: $\frac{q}{m}E^a_z \simeq -\frac{q}{m}\frac{\partial\phi^a}{\partial s}\Big|_{x=y=0}$

ϕ^a is a quasi-static approximation accelerating potential (see next pages)

Term 3: $-q(v_xB^a_y - v_yB^a_x) = -q\left(\frac{dx}{dt}B^a_y - \frac{dy}{dt}B^a_x\right) \simeq 0$

- Transverse magnetic elds typically only weakly change particle energy and terms can typically be neglected relative to others

The longitudinal particle equation of motion for $\gamma_b,\ \beta_b$ then reduces to:

$$\beta_b(\gamma_b\beta_b)' \simeq -\frac{q}{mc^2}\frac{\partial\phi^a}{\partial s}\Big|_{x=y=0}$$

Some algebra shows:

$$\gamma_b' = \left(\frac{1}{\sqrt{1-\beta_b^2}}\right)' = \frac{\beta_b\beta_b'}{(1-\beta_b^2)^{3/2}} = \gamma_b^3\beta_b\beta_b'$$

First apply chain rule, then use the result above twice to simplify results:

$$\Longrightarrow \quad \beta_b(\gamma_b\beta_b)' = \beta_b^2\gamma_b' + \gamma_b\beta_b\beta_b'$$
$$= \beta_b^3\gamma_b^3\beta_b' + \gamma_b\beta_b\beta_b' = (1+\gamma_b^2\beta_b^2)\gamma_b\beta_b\beta_b' = \gamma_b^3\beta_b\beta_b'$$
$$= \gamma_b'$$

Giving:

$$\gamma_b' = -\frac{q}{mc^2}\frac{\partial\phi^a}{\partial s}\Big|_{x=y=0}$$

Which can then be integrated to obtain:

$$\gamma_b = -\frac{q}{mc^2}\phi^a(r=0, z=s) + \text{const}$$

We denote the on-axis accelerating potential as:

$$V(s) \equiv \phi^a(x=y=0, z=s)$$

- Can represent RF or induction accelerating gap elds
  See: Longitudinal Dynamics lectures for more details

Using this and setting $\gamma_b(s=s_i) = \gamma_{bi}$ gives for the gain in axial kinetic energy $\mathcal{E}_b$ and corresponding changes in $\gamma_b,\ \beta_b$ factors:

$$\mathcal{E}_b = (\gamma_b - 1)mc^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi}$$
$$\gamma_b = 1 + \mathcal{E}_{bi}/(mc^2) \qquad \mathcal{E}_{bi} = (\gamma_{bi}-1)mc^2$$
$$\beta_b = \sqrt{1 - 1/\gamma_b^2}$$

These equations can be solved for the consistent variation of $\gamma_b(s),\ \beta_b(s)$ to integrate the transverse equations of motion:

$$\mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}_\perp^a + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}_\perp^a + \frac{qB_z^a}{m\gamma_b\beta_b c}\mathbf{x}_\perp'\times\hat{\mathbf{z}}$$
$$-\frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi$$

Nonrelativistic limit results

In the nonrelativistic limit:

$$\gamma_b \simeq 1 + \frac{1}{2}\beta_b^2 \qquad \beta_b^2 \ll 1 \qquad \mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \frac{1}{2}m\beta_b^2c^2$$

and the previous (relativistic valid) energy gain formulas reduce to:

$$\mathcal{E}_b \simeq \frac{1}{2}m\beta_b^2c^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi}$$
$$\gamma_b \simeq 1 \qquad \mathcal{E}_{bi} = \frac{1}{2}m\beta_b^2c^2$$
$$\beta_b = \sqrt{\frac{2\mathcal{E}_b}{mc^2}}$$

Using this result, in the nonrelativistic limit we can take in the transverse particle equation of motion:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\beta_b'}{\beta_b} = \frac{1}{2}\frac{\mathcal{E}_b'}{\mathcal{E}_b} = -\frac{1}{2}\frac{qV'(s)}{q[V(s_i) - V(s)] + \mathcal{E}_{bi}}$$

Ultra-relativistic limit results

In the ultra-relativistic limit:

$$\gamma_b \gg 1 \qquad \beta_b \simeq 1 \qquad \mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \gamma_b mc^2$$

and the previous (relativistic valid) energy gain formulas reduce to:

$$\mathcal{E}_b \simeq \gamma_b mc^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi}$$
$$\beta_b \simeq 1$$

Using this result, in the ultra-relativistic limit we can take in the transverse particle equation of motion:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} = \frac{\mathcal{E}_b'}{\mathcal{E}_b} = -\frac{qV'(s)}{q[V(s_i) - V(s)] + \mathcal{E}_{bi}}$$

- Same form as NR limit expression with only a factor of ½ di erence; see also S10A

S11C: Longitudinal Solution via Energy Gain

An alternative analysis of the particle energy gain carried out in S11B can be illuminating. In this case we start from the exact Lorentz force equation with time as the independent variable for a particle moving in the full electromagnetic field:

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\vec{\beta}c \times \mathbf{B}$$

$$\mathbf{p} \equiv \gamma m \vec{\beta} c \qquad \gamma \equiv 1/\sqrt{1 - \vec{\beta}\cdot\vec{\beta}}$$

Comments:
- Formulation exact in context of classical electrodynamics
- $\gamma, \ \vec{\beta}$ not expanded
- $\mathbf{E}, \ \mathbf{B}$ electromagnetic

Dotting $mc\vec{\beta}$ into this equation:

$$mc\vec{\beta}\cdot\frac{d}{dt}(c\gamma\vec{\beta}) = qc\vec{\beta}\cdot\mathbf{E} + qc\vec{\beta}\cdot[c\vec{\beta}\times\mathbf{B}] \quad (\to 0)$$

[1] [2]

$$\left[\vec{\beta}\cdot\vec{\beta}\dot{\gamma}\right] + \left[\gamma\vec{\beta}\cdot\dot{\vec{\beta}}\right] = \frac{q}{mc}\vec{\beta}\cdot\mathbf{E}$$

Then

$$\gamma \equiv (1 - \vec{\beta}\cdot\vec{\beta})^{-1/2}$$

Gives:

[1]: $\left[\vec{\beta}\cdot\vec{\beta}\right] = 1 - 1/\gamma^2$ [2]: $\left[\vec{\beta}\cdot\dot{\vec{\beta}}\right] = \dot{\gamma}/\gamma^3$

Inserting these factors:

$$(1 - 1/\gamma^2)\dot{\gamma} + \dot{\gamma}/\gamma^2 = \frac{q}{mc^2}\vec{\beta}\cdot\mathbf{E}$$

or:

$$\dot{\gamma} = \frac{q}{mc}\vec{\beta}\cdot\mathbf{E}$$

Equivalently: $\mathcal{E} = (\gamma - 1)mc^2$

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\left[(\gamma-1)mc^2\right] = qc\vec{\beta}\cdot\mathbf{E}$$

- Only the electric field changes the kinetic energy of a particle
- No approximations made to this point within the context of classical electrodynamics: valid for evolving $\mathbf{E}, \ \mathbf{B}$ consistent with the Maxwell equations.

Now approximating to our slowly varying and paraxial formulation:

$$\frac{d}{dt} = c\beta_z\frac{d}{ds} \qquad \beta_z \simeq \beta \simeq \beta_b \qquad \gamma \simeq \gamma_b \qquad \mathcal{E} \simeq \mathcal{E}_b = (\gamma_b - 1)mc^2$$

and approximating the axial electric field by the applied component then obtains

$$\frac{d}{ds}\mathcal{E}_b \simeq \frac{dt}{ds}\frac{d}{dt}\left[(\gamma-1)mc^2\right] \simeq qE_z^a$$

which is the longitudinal equation of motion analyzed in S11B.

S11D: Quasistatic Potential Expansion

In the quasistatic approximation, the accelerating potential can be expanded in the axisymmetric limit as:

- See: USPAS, *Beam Physics with Intense Space-Charge*; and Reiser, *Theory and Design of Charged Particle Beams*, (1994, 2008) Sec. 3.3.
- See also: S2, Appendix D

We take:

$$\mathbf{E}^a = -\frac{\phi^a}{\partial\mathbf{x}}$$

and apply the results of S2, Appendix D to expand ϕ^a in terms of the on-axis potential in an axisymmetric (acceleration gap) system:

$$\phi^a(r,z) = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(\nu!)^2}\frac{\partial^{2\nu}\phi^a(r=0,z)}{\partial z^{2\nu}}\left(\frac{r}{2}\right)^{2\nu}$$

Denote for the on-axis potential

$$\phi^a(r=0,z) \equiv V(z)$$

$$\Longrightarrow \phi^a = V(z) - \frac{1}{4}\frac{\partial^2}{\partial z^2}V(z)(x^2+y^2) + \frac{1}{64}\frac{\partial^4}{\partial z^4}V(z)(x^2+y^2)^2 + \cdots$$

The longitudinal acceleration also result in a transverse focusing field

$$\mathbf{E}_\perp^a = \mathbf{E}_\perp^a|_{\text{foc}} - \frac{\partial\phi^a}{\partial\mathbf{x}_\perp}$$

$$\mathbf{E}_\perp^a|_{\text{foc}} = \text{Fields from Any Applied Focusing Optics}$$

$$-\frac{\partial\phi^a}{\partial\mathbf{x}_\perp} \simeq \frac{1}{2}\frac{\partial^2}{\partial z^2}V(z)\mathbf{x}_\perp = \text{Focusing Field from Acceleration}$$

- Results can be used to cast acceleration terms in more convenient forms. See USPAS, Beam Physics with Intense Space-Charge for more details
- RF defocusing in the quasistatic approximation can be analyzed using this formulation
- Einzel lens focusing exploits accel/de-acell cycle to make AG focusing

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

Prof. Steven M. Lund
Facility for Rare Isotope Beams
Michigan State University
640 South Shaw Lane
East Lansing, MI 48824

lund@frib.msu.edu
(517) 908 – 7291 o   ce
(510) 459 - 4045 mobile

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