07 Transverse Nonlinear Resonances with Application to Circular Accelerators*

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Transverse Particle Resonances: Outline

Transverse Particle Resonances: Detailed Outline

Section headings include embedded links that when clicked on will direct you to the section

Transverse Particle Resonances: Detailed Outline - 2

S1: Overview

In our treatment of transverse single particle orbits of lattices with s-varying focusing, we found that Hill's Equation describes the orbits to leading-order approximation:

$$x''(s) + \kappa_x(s)x(s) = 0$$
$$y''(s) + \kappa_y(s)y(s) = 0$$

where $\kappa_x(s)$, $\kappa_y(s)$ are functions that describe linear applied focusing forces of the lattice

- Focusing functions can also incorporate linear space-charge forces
  - Self-consistent for special case of a KV distribution

In analyzing Hill's equations we employed phase-amplitude methods

- See: S.M. Lund lectures on Transverse Particle Dynamics, S8, on the betatron form of the solution

$$x(s) = A_{xi}\sqrt{\beta_x(s)}\cos\psi_x(s) \qquad A_{xi} = \text{const}$$
$$\frac{1}{2}\beta_x(s)\beta_x''(s) - \frac{1}{4}\beta_x'^2(s) + \kappa_x(s)\beta_x^2(s) = 1 \qquad \psi_x(s) = \psi_{xi} + \int_{s_i}^{s} \frac{d\bar{s}}{\beta_x(\bar{s})}$$
$$\beta_x(s+L_p) = \beta_x(s) \qquad \beta_x(s) > 0$$

This formulation simpli ed identi cation of the Courant-Snyder invariant:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_x^2 \equiv \epsilon_x = \text{const}$$
$$\frac{1+\beta_x'^2/4}{\beta_x}x^2 - \beta_x\beta_x' x x' + \beta_x x'^2 = A_x^2 = \epsilon_x \qquad w_x = \sqrt{\beta_x}$$
$$\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 =$$

which helped to interpret the dynamics.

We will now exploit this formulation to better (analytically!) understand resonant instabilities in periodic focusing lattices. This is done by choosing coordinates such that *stable* unperturbed orbits described by Hill's equation:

$$x''(s) + \kappa_x(s)x(s) = 0$$

are mapped to a continuous oscillator

$$\tilde{x}''(\tilde{s}) + \tilde{k}_{\beta 0}^2\tilde{x}(\tilde{s}) = 0 \qquad \tilde{\cdot\cdot} = \text{Transformed Coordinate}$$
$$\tilde{k}_{\beta 0}^2 = \text{const} > 0$$

- Because the linear lattice is designed for single particle stability this transformation can be e ected for any practical machine operating point

These transforms will help us more simply understand the action of perturbations (from applied eld nonlinearities, ...) acting on the particle orbits:

$$x''(s) + \kappa_x(s)x(s) = \mathcal{P}_x(s;\mathbf{x}_\perp,\mathbf{x}_\perp',\vec{\delta})$$
$$y''(s) + \kappa_y(s)y(s) = \mathcal{P}_y(s;\mathbf{x}_\perp,\mathbf{x}_\perp',\vec{\delta})$$
$$\mathcal{P}_x,\ \mathcal{P}_y = \text{Perturbations}$$
$$\vec{\delta} = \text{Extra Coupling Variables}$$

For simplicity, we restrict analysis to:

$\gamma_b\beta_b = \text{const}$ No Acceleration

$\delta = 0$ No Axial Momentum Spread

$\phi = 0$ Neglect Space-Charge

Weak acceleration can be incorporated using transformations

We also take the applied focusing lattice to be periodic with:

$$\kappa_x(s+L_p) = \kappa_x(s) \qquad L_p = \text{Lattice Period}$$
$$\kappa_y(s+L_p) = \kappa_y(s)$$

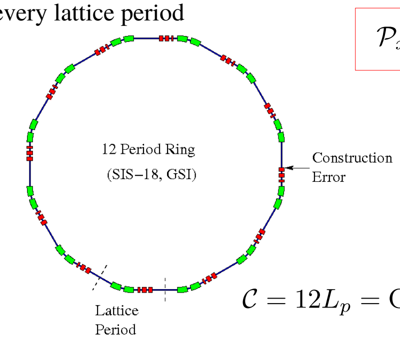
For a ring we also always have the superperiodicity condition:

$$\mathcal{P}_x(s+\mathcal{C};\mathbf{x}_\perp,\mathbf{x}_\perp',\vec{\delta}) = \mathcal{P}_x(s;\mathbf{x}_\perp,\mathbf{x}_\perp',\vec{\delta})$$
$$\mathcal{P}_y(s+\mathcal{C};\mathbf{x}_\perp,\mathbf{x}_\perp',\vec{\delta}) = \mathcal{P}_y(s;\mathbf{x}_\perp,\mathbf{x}_\perp',\vec{\delta})$$
$$\mathcal{C} = \mathcal{N}L_p = \text{Circumference Ring}$$
$$\mathcal{N} \equiv \text{Superperiodicity}$$

Perturbations can be Random and/or Systematic:

Random Errors in a ring will be felt once per particle lap in the ring rather than every lattice period

$$\mathcal{P}_{x,y}(\cdots, s+\mathcal{N}L_p) = \mathcal{P}_{x,y}(\cdots, s)$$



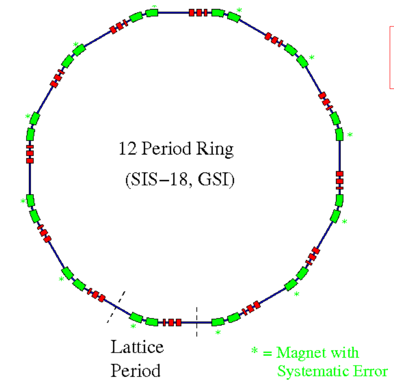
Random Error Sources:

- Fabrication
- Assembly/Construction
- Material Defects
- ....

$$\mathcal{C} = 12L_p = \text{Circumference of Ring}$$

Systematic Errors can occur in both linear machines and rings and affect *every* lattice period in the same manner.

Example: FODO Lattice with the same error in each dipole of pair



$$\mathcal{P}_{x,y}(\cdots, s + L_p) = \mathcal{P}_{x,y}(\cdots, s)$$

Systematic Error Sources:

- Design Idealization (e.g., truncated pole)
- Repeated Construction or Material Error
- ....

We will find that perturbations arising from both random and systematic error can drive resonance phenomena that destabilize particle orbits and limit machine performance

S2: Floquet Coordinates and Hill's Equation

Define for a *stable* solution to Hill's Equation

- Drop *x* subscripts and only analyze *x*-orbit for now to simplify analysis
- Later will summarize results from coupled *x-y* orbit analysis

"Radial" Coordinate: $u \equiv \dfrac{x}{\sqrt{\beta}}$

"Angle" Coordinate: $\varphi \equiv \dfrac{1}{\nu_0} \displaystyle\int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \equiv \frac{\Delta\psi(s)}{\nu_0}$

(dimensionless, normalized)

where: $\varphi(s = s_i) = 0$ reference choice

$\beta = w^2 =$ Betatron Amplitude Function

$\nu_0 \equiv \dfrac{\Delta\psi(\mathcal{N} L_p)}{2\pi} = \dfrac{\mathcal{N}\sigma_0}{2\pi} =$ Number undepressed *x*-betatron oscillations in ring

$\psi =$ Phase of x-orbit

$\Delta\psi(s) = \psi(s) - \psi(s_i)$

Can also take $\mathcal{N} = 1$ and then ν_0 is the number (usually fraction thereof) of undepressed particle oscillations in *one* lattice period

Comment:

φ can be interpreted as a normalized angle measured in the particle betatron phase advance:

Ring: ($\mathcal{N} =$ Superperiod #) $\Longrightarrow$ φ advances by 2π on one transit around ring for analysis of Random Errors

Linac or Ring: ($\mathcal{N} = 1$) $\Longrightarrow$ φ advances by 2π on transit through one lattice period for analysis of Systematic Errors in a ring *or* linac

Take φ as the independent coordinate:

$$\boxed{u = u(\varphi)}$$

and define a new "momentum" phase-space coordinate

$$\boxed{\dot{u} \equiv \frac{du}{d\varphi}} \qquad \boxed{\dot{} \equiv \frac{d}{d\varphi}}$$

These new variables will be applied to express the unpreturbed Hill's equation in a simpler (continuously focused oscillator) form

/// Aside: Comment on use of φ as an independent coordinate

To use this formulation explicitly, locations of perturbations need to be cast in terms of φ rather than the reference particle axial coordinate *s*:

- Will find that we do not need to explicitly carry this out to identify parameters leading to resonances
- However, to analyze resonant growth characteristics or particular orbit phases it is necessary to calculate $s(\varphi)$ to explicitly specify amplitudes and phases of driving perturbation terms

The needed transform is obtained by integration and (in principle) inversion

- In most cases of non-continuous focusing lattices, this will need to be carried out numerically

$$\varphi(s) = \frac{1}{\nu_0}\int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})}$$

$$\varphi(s) \quad \Longrightarrow \quad s(\varphi)$$

$$\varphi(s) \equiv \frac{1}{\nu_0}\int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \quad\Longrightarrow\quad \frac{d\varphi}{ds} = \frac{1}{\nu_0 \beta}$$

Rate of change in s not constant
except for continuous focusing lattices

Continuous Focusing: Simplest case

$$\kappa_x = k_{\beta 0}^2 = \text{const}$$

$$\frac{d\varphi}{ds} = \frac{2\pi}{\mathcal{C}} = \text{const} \quad\Longrightarrow\quad \varphi(s) = \frac{2\pi}{\mathcal{C}}(s - s_i)$$

Periodic Focusing: Simple FODO lattice to illustrate

Add numerical example/plot
in future version of notes.

///

From the de nition

$$u \equiv \frac{x}{\sqrt{\beta}}$$

Rearranging this and using the chain rule with $u = u(\varphi), \quad \beta = \beta(s)$

$$x = \sqrt{\beta} u$$

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \sqrt{\beta} \frac{du}{d\varphi}\frac{d\varphi}{ds} \qquad\qquad \frac{d}{ds} = \frac{d\varphi}{ds}\frac{d}{d\varphi}$$

From:

$$\varphi \equiv \frac{1}{\nu_0}\int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \quad\Longrightarrow\quad \boxed{\frac{d\varphi}{ds} = \frac{1}{\nu_0 \beta}}$$

we obtain

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0\sqrt{\beta}}\dot{u}$$

cancels

$$x'' = \frac{d}{ds}x' = \frac{\beta''}{2\sqrt{\beta}} u - \frac{\beta'^2}{4\beta^{3/2}} u + \cancel{\frac{\beta'}{2\nu_0\beta^{3/2}}\dot{u}} - \cancel{\frac{\beta'}{2\nu_0\beta^{3/2}}\dot{u}} + \frac{1}{\nu_0^2\beta^{3/2}}\ddot{u}$$

Summary:

$$\boxed{\begin{aligned} x' &= \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0\sqrt{\beta}}\dot{u} \\ x'' &= \frac{\beta''}{2\sqrt{\beta}} u - \frac{\beta'^2}{4\beta^{3/2}} u + \frac{1}{\nu_0^2\beta^{3/2}}\ddot{u} \end{aligned}}$$

Using these results, Hill's equation:

$$x''(s) + \kappa_x(s) x(s) = 0$$

becomes

$$\ddot{u} + \nu_0^2 \left[\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \kappa\beta^2 \right] u = 0$$

But the betatron amplitude equation satisfies:

$$\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \kappa\beta^2 = 1 \qquad\qquad \beta(s + L_p) = \beta(s)$$

Thus the terms in [...] = 1 and Hill's equation reduces to simple harmonic oscillator form:

$$\boxed{\ddot{u} + \nu_0^2 u = 0 \qquad\qquad \nu_0^2 = \text{const} > 0}$$

Transform has mapped a stable, time dependent solution to Hill's equation to a simple harmonic oscillator!

The general solution to the unperturbed simple harmonic oscillator equation can be expressed as:

$$u(\varphi) = u_i \cos(\nu_0\varphi) + \frac{\dot{u}_i}{\nu_0}\sin(\nu_0\varphi)$$

$$\frac{\dot{u}(\varphi)}{\nu_0} = -u_i \sin(\nu_0\varphi) + \frac{\dot{u}_i}{\nu_0}\cos(\nu_0\varphi)$$

$$u(\varphi = 0) = u_i = \text{const}$$

$$\dot{u}(\varphi = 0) = \dot{u}_i = \text{const}$$

u_i and $\dot{u}_i$ set by x, x' initial conditions at $s = s_i$ (phase choice $\varphi = 0$ at $s = s_i$)

Floquet representation simpli es interpretation of the Courant-Snyder invariant:

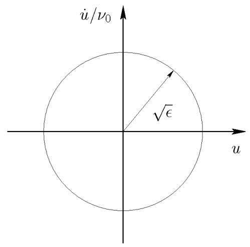
$$u^2 + \left(\frac{\dot{u}}{\nu_0}\right)^2 = u_i^2 \underbrace{\left[\sin^2(\nu_0\varphi) + \cos^2(\nu_0\varphi)\right]}_{1} + \left(\frac{\dot{u}_i}{\nu_0}\right)^2 \underbrace{\left[\sin^2(\nu_0\varphi) + \cos^2(\nu_0\varphi)\right]}_{1}$$

$$+ u_i \frac{\dot{u}_i}{\nu_0} \underbrace{\left[\sin(\nu_0\varphi)\cos(\nu_0\varphi) - \sin(\nu_0\varphi)\cos(\nu_0\varphi)\right]}_{0}$$

$$\Longrightarrow \quad \boxed{u^2 + \left(\frac{\dot{u}}{\nu_0}\right)^2 = u_i^2 + \left(\frac{\dot{u}_i}{\nu_0}\right)^2 \equiv \epsilon = \text{const}}$$

- Unperturbed phase-space in $u - \dot{u}/\nu_0$ variables is a circle of area $\pi\epsilon$!
- Relate this area to *x*-*x*' phase-space area shortly
  - Preview: areas are equal due to the transform being symplectic
  - Same symbols used for area as in Transverse Particle Dynamics is on purpose

Unperturbed phase-space ellipse:



This simple structure will also allow more simple visualization of perturbations as distortions on a unit circle, thereby clarifying symmetries:

(Picture to be replaced ... had poor schematic example)

The $u - \dot{u}/\nu_0$ variables also preserve phase-space area

- Feature of the transform being symplectic (Hamiltonian Dynamics)

From previous results:

$$x = \sqrt{\beta} u \qquad \frac{d\varphi}{ds} = \frac{1}{\nu_0 \beta}$$

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \sqrt{\beta} \frac{d\varphi}{ds} \dot{u} = \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0 \sqrt{\beta}} \dot{u}$$

Transform area elements by calculating the Jacobian:

$$dx \otimes dx' = |J| du \otimes d\dot{u}$$

$$J = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \dot{u}} \\ \frac{\partial x'}{\partial u} & \frac{\partial x'}{\partial \dot{u}} \end{vmatrix} = \det \begin{vmatrix} \sqrt{\beta} & 0 \\ \frac{\beta'}{2\sqrt{\beta}} & \frac{1}{\nu_0 \sqrt{\beta}} \end{vmatrix} = \frac{1}{\nu_0}$$

$$\boxed{dx \otimes dx' = du \otimes \frac{d\dot{u}}{\nu_0}}$$

Thus the Courant-Snyder invariant ϵ is the usual single particle emittance in *x*-*x*' phase-space

S3: Perturbed Hill's Equation in Floquet Coordinates

Return to the perturbed Hill's equation in S1:

$$\boxed{\begin{aligned} &x''(s) + \kappa_x(s) x(s) = \mathcal{P}_x(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta}) \\ &y''(s) + \kappa_y(s) y(s) = \mathcal{P}_y(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta}) \\ &\qquad \mathcal{P}_x,\ \mathcal{P}_y = \text{Perturbations} \\ &\qquad \vec{\delta} = \text{Extra Coupling Variables} \end{aligned}}$$

Drop the extra coupling variables and apply the Floquet transform in S2 and consider only transverse multipole magnetic eld perturbations

- Examine only *x*-equation, *y*-equation analogous
- From S4 in Transverse Particle Dynamics terms $B_x,\ B_y$ only have variation in *x*,*y*. If solenoid magnetic eld errors are put in, terms with $x',\ y'$ dependence will also be needed
- Drop *x*-subscript in $\mathcal{P}_x$ to simplify notation

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \beta^{3/2} \mathcal{P}} \qquad \mathcal{P} = \mathcal{P}(s(\varphi), \sqrt{\beta} u, y, \vec{\delta})$$

Transform *y* similarly to *x*
If analyzing general orbit with *x* and *y* motion

Expand the perturbation in a power series:

- Can be done for *all* physical applied eld perturbations
- Multipole symmetries can be applied to restrict the form of the perturbations
- Perturbations can be random (once per lap; in ring) or systematic (every lattice period; in ring or in linac)

$$\boxed{\begin{aligned} \mathcal{P}(x, y, s) &= \mathcal{P}_0(y, s) + \mathcal{P}_1(y, s) x + \mathcal{P}_2(y, s) x^2 + \cdots \\ &= \sum_{n=0}^{\infty} \mathcal{P}_n(y, s) x^n \end{aligned}}$$

Take: $x = \sqrt{\beta} u$

to obtain:

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} \mathcal{P}_n(y, s) u^n}$$

A similar equation applies in the *y*-plane.

S4: Sources of and Forms of Perturbation Terms

Within a 2D transverse model it was shown that transverse applied magnetic eld components entering the equations of motion can be expanded as:

- See: S3, Transverse Particle Dynamics: 2D components axial integral 3D components
- Applied electric elds can be analogously expanded

$$\underline{B}^*(\underline{z}) = B_x^a(x,y) - iB_y^a(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}$$

$$\underline{b}_n = \text{const (complex)} \equiv \mathcal{A}_n - i\mathcal{B}_n \qquad \underline{z} = x + iy \qquad i = \sqrt{-1}$$

$$n = \text{Multipole Index} \qquad r_p = \text{Aperture "Pipe" Radius}$$

$$\mathcal{B}_n \Longrightarrow \text{"Normal" Multipoles}$$

$$\mathcal{A}_n \Longrightarrow \text{"Skew" Multipoles}$$

Cartesian projections: $\overline{B_x} - i\overline{B_y} = (\mathcal{A}_n - i\mathcal{B}_n)(x+iy)^{n-1}/r_p^{n-1}$

Index	Name	Normal ($\mathcal{A}_n = 0$)		Skew ($\mathcal{B}_n = 0$)	
n		$B_x r_p^{n-1}/\mathcal{B}_n$	$B_y r_p^{n-1}/\mathcal{B}_n$	$B_x r_p^{n-1}/\mathcal{A}_n$	$B_y r_p^{n-1}/\mathcal{A}_n$
1	Dipole	0	1	1	
2	Quadrupole	y	x	x	$-y$
3	Sextupole	$2xy$	$x^2 - y^2$	$x^2 - y^2$	$-2xy$
4	Octupole	$3x^2y - y^3$	$x^3 - 3xy^2$	$x^3 - 3xy^2$	$-3x^2y + y^3$
5	Decapole	$4x^3y - 4xy^3$	$x^4 - 6x^2y^2 + y^4$	$x^4 - 6x^2y^2 + y^4$	$-4x^3y + 4xy^3$

Trace back how the applied magnetic eld terms enter the x-plane equation of motion:

- See: S2, Transverse Particle Dynamics and reminder on next page
- Apply equation in S2 with: $\beta_b = \text{const}, \ \phi \simeq \text{const}, \ E_x^a \simeq 0, \ B_z^a \simeq 0$
- To include axial ($B_z^a \neq 0$) eld errors, follow a similar pattern to generalize

$$x'' = -\frac{q}{m\gamma_b\beta_b c} B_y^a$$

Express this equation as:

$$x'' + \kappa_x(s)x = -\frac{q}{m\gamma_b\beta_b c}\left[B_y^a(x,y,s) - B_y^a(x,y,s)\big|_{\text{lin } x\text{-foc}}\right]$$

Nonlinear focusing terms only in []

- "Normal" part of linear applied magnetic eld contained in focus function κ_x

Compare to the form of the perturbed Hill's equation:

$$x'' + \kappa_x x = \mathcal{P}_x = \sum_{n=0}^{\infty} \mathcal{P}_n(y,s)x^n$$

// Reminder: Particle equations of motion from Transverse Particle Dynamics lecture notes

Transverse particle equations of motion in explicit component form:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' = \frac{q}{m\gamma_b\beta_b^2c^2}E_x^a - \frac{q}{m\gamma_b\beta_b c}B_y^a + \frac{q}{m\gamma_b\beta_b c}B_z^a y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}E_y^a + \frac{q}{m\gamma_b\beta_b c}B_x^a - \frac{q}{m\gamma_b\beta_b c}B_z^a x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

Equations previously derived under assumptions:

- No bends (xed x-y-z coordinate system with no local bends)
- Paraxial equations ($x'^2, y'^2 \ll 1$)
- No dispersive e ects (β_b same all particles), acceleration allowed ($\beta_b \neq \text{const}$)
- Electrostatic and leading-order (in β_b) self-magnetic interactions

Reduce the x-plane equation to our situation:

0 No accel 0 No E-Focus 0 No B_z

$$x'' + \cancel{\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x'} = \cancel{\frac{q}{m\gamma_b\beta_b^2c^2}E_x^a} - \frac{q}{m\gamma_b\beta_b c}B_y^a + \cancel{\frac{q}{m\gamma_b\beta_b c}B_z^a y'}$$

0 No Space-Charge

$$\cancel{- \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}}$$

Giving the equation we are analyzing:

$$\Longrightarrow \quad x'' = -\frac{q}{m\gamma_b\beta_b c}B_y^a$$

//

Gives:

$$\Longrightarrow \boxed{\mathcal{P}_x = -\frac{q}{m\gamma_b\beta_b c}\left[B_y^a - B_y^a|_{\text{lin } x\text{-foc}}\right]}$$

where the *y*- eld components can be obtained from the multipole expansion as:

$$B_y^a = -\text{Im}[\underline{B}^*] \qquad \underline{B}^* = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{x+iy}{r_p}\right)^{n-1}$$

$$B_y^a|_{\text{lin } x\text{-focus}} = -\text{Im}[\underline{B}^*|_{n=1 \text{ term}}]$$

- Use multipole eld components of magnets to obtain explicit form of eld component perturbations consistent with the Maxwell equations
- Need to subtract o design component of linear led from $\mathcal{P}_x$ perturbation term since it is included in κ_x
- Similar steps employed to identify *y*-plane perturbation terms, perturbations from axial eld components, and perturbations for applied electric eld components

Caution: Multipole index *n* and power series index *n* in $\mathcal{P}_x$ expansion not the same (notational overuse: wanted analogous symbol)

- <u>Multipole Expansion</u> for $B_x^a,\ B_y^a$:

n = 1	Dipole	*n* = 3	Sextupole
n = 2	Quadrupole	*n* = ...	

- <u>Power Series</u> Expansion for $\mathcal{P}_x$:

x-plane Motion (*y=0*)		*x-y* plane motion
n = 0	Dipole	Depends on form of y-coupling
n = 1	Quadrupole	
n = 2	Sextupole	
....		

S5: Solution of the Perturbed Hill's Equation: Resonances

Analyze the solution of the perturbed orbit equation:

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} \mathcal{P}_n(y,s) u^n}$$

derived in S4.

To more simply illustrate resonances, we analyze motion in the *x*-plane with:

$$\boxed{y(s) \equiv 0}$$

- Essential character of general analysis illustrated most simply in one plane
- Can generalize by expanding $\mathcal{P}_n(y,s)$ in a power series in *y* and generalizing notation to distinguish between Floquet coordinates in the *x*- and *y*-planes
  - Results in coupled *x*- and *y*-equations of motion

Each *n*-labeled perturbation expansion coe cient is periodic with period of the ring circumference (random perturbations) or lattice period (systematic):

$$L_p = \text{Lattice Period}$$

$$\mathcal{C} = \mathcal{N} L_p = \text{Ring Circumference}$$

$$\beta(s + L_p) = \beta(s)$$

$$\beta(s + \mathcal{N} L_p) = \beta(s)$$

<u>Random Perturbation:</u>

$$\mathcal{P}_n(y, s + \mathcal{N}L_p) = \mathcal{P}_n(y,s)$$

$$\Longrightarrow \boxed{\beta^{\frac{n+3}{2}}(s + \mathcal{N}L_p)\mathcal{P}_n(y, s+\mathcal{N}L_p) = \beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y,s)}$$

<u>Systematic Perturbation:</u>

$$\mathcal{P}_n(y, s + L_p) = \mathcal{P}_n(y,s)$$

$$\Longrightarrow \boxed{\beta^{\frac{n+3}{2}}(s + L_p)\mathcal{P}_n(y, s+L_p) = \beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y,s)}$$

Expand each n-labeled perturbation expansion coe cient in a Fourier series as:

$$\beta^{\frac{n+3}{2}}\mathcal{P}_n(y=0,s) = \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ikp\varphi}$$

$$i \equiv \sqrt{-1} \qquad p \equiv \begin{cases} 1, & \text{Random perturbation (once per lap in ring)} \\ \mathcal{N}, & \text{Systematic perturbation (every lattice period)} \end{cases}$$

$$C_{n,k} = \int_{-\pi/p}^{\pi/p} \frac{d\varphi}{2\pi/p}\, e^{-ikp\varphi}\beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y=0,s) = \text{const}$$

(complex-valued)

$$s = s(\varphi) \qquad \varphi = \int_{s_0}^{s} \frac{1}{\nu_0}\frac{d\tilde{s}}{\beta(\tilde{s})}$$

- Can apply to Rings for random perturbations (with $p=1$) or systematic perturbations (with $p=\mathcal{N}$)
- Can apply to linacs for periodic perturbations (every lattice period) with $p=1$
- Does not apply to random perturbations in a linac
  - In linac random perturbations will vary every lattice period and drive random walk type e ects but not resonances

The perturbed equation of motion becomes:

$$\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty}\sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ikp\varphi} u^n$$

Expand the solution as:

$$u = u_0 + \delta u \qquad \begin{aligned} u_0 &= \text{unperturbed solution} \\ \delta u &= \text{perturbation due to errors}\end{aligned}$$

where u_0 is the solution to the *simple harmonic oscillator* equation in the absence of perturbations:

$$\ddot{u}_0 + \nu_0^2 u_0 = 0$$

Unperturbed equation of motion

Assume small-amplitude perturbations so that

$$|u_0| \gg |\delta u|$$

Then to leading order, the equation of motion for δu is:

$$\ddot{\delta u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty}\sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ipk\varphi} u_0^n$$

Perturbed equation of motion

To obtain the perturbed equation of motion, the unperturbed solution u_0 is inserted on the RHS terms

- Gives simple harmonic oscillator equation with driving terms

Solution of the unperturbed orbit is simply expressed as:

$$u_0 = u_{0i}\cos(\nu_0\varphi + \varphi_i) = u_{0i}\frac{e^{i(\nu_0\varphi+\varphi_i)} + e^{-i(\nu_0\varphi+\varphi_i)}}{2}$$

$$\left.\begin{aligned} u_{0i} &= \text{const} \\ \varphi_i &= \text{const}\end{aligned}\right\} \quad \text{Set by particle initial conditions: } x(s_i) = x_i, \quad x'(s_i) = x'_i$$

Then binomial expand:

$$\begin{aligned} u_0^n &= u_{0i}^n \left(\frac{e^{i(\nu_0\varphi+\varphi_i)} + e^{-i(\nu_0\varphi+\varphi_i)}}{2}\right)^n \\ &= \frac{u_{oi}^n}{2^n}\sum_{m=0}^{n}\binom{n}{m} e^{i(n-m)(\nu_0\varphi+\varphi_i)} e^{-im(\nu_0\varphi+\varphi_i)} \\ &= \frac{u_{oi}^n}{2^n}\sum_{m=0}^{n}\binom{n}{m} e^{i(n-2m)\nu_0\varphi} e^{i(n-2m)\varphi_i}\end{aligned}$$

where $\binom{n}{m} \equiv \frac{n!}{m!(n-m)!}$ is a binomial coe cient

Using this expansion the linearized perturbed equation of motion becomes:

$$\ddot{\delta u} + \nu_0^2\delta u \simeq \nu_0^2 \sum_{n=0}^{\infty}\sum_{k=-\infty}^{k=\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{C_{n,k}}{2^n} e^{i[(n-2m)\nu_0+pk]\varphi} e^{i(n-2m)\varphi_i}$$

The solution for δu can be expanded as:

$$\delta u = \delta u_h + \delta u_p$$

$$\delta u_h = \text{homogenous solution}$$

General solution to: $\ddot{\delta u}_h + \nu_0^2 \delta u_h = 0$

$$\delta u_p = \text{particular solution}$$

Any solution with: $\delta u \to \delta u_p$

- **Can drop homogeneous solution** because it can be absorbed in unperturbed solution u_0
  - Exception: some classes of linear amplitude errors in adjusting magnets
- **Only a particular solution need be found**, take:

$$\delta u = \delta u_p$$

$$\delta\ddot{u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{C_{n,k}}{2^n} e^{i[(n-2m)\nu_0 + pk]\varphi} e^{i(n-2m)\varphi_i}$$

Equation describes a driven simple harmonic oscillator with periodic driving terms on the RHS:

- Homework problem reviews that solution of such an equation will be unstable when the driving term has a frequency component equal to the restoring term
  - Resonant exchange and amplitude grows *linearly* (not exponential!) in φ
  - Parameters meeting resonance condition will lead to instabilities with particle oscillation amplitude growing in $\varphi(s)$

Resonances occur when:

$$(n-2m)\nu_0 + pk = \pm\nu_0$$

is satisfied for the operating tune ν_0 and some values of:

$$n = 0,\ 1,\ 2,\ \cdots \qquad m = 0,\ 1,\ 2,\ \cdots,\ n$$

$$k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty$$

$$p \equiv \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

If growth rate is sufficiently large, machine operating points satisfying the resonance condition will be problematic since particles will be lost (scraped) by the machine aperture due to increasing oscillation amplitude:

- Machine operating tune (ν_0) can be adjusted to avoid
- Perturbation can be actively corrected to reduce amplitude of driving term

Low order resonance terms with smaller *n, k, m* magnitudes are expected to be more dangerous because:

- Less likely to be washed out by effects not included in model
- Amplitude coefficients expected to be stronger

More detailed theories consider coherence length, finite amplitude, and nonlinear term effects. Such treatments and numerical analysis concretely motivate importance/strength of terms. A standard reference on analytic theory is:

- Kolomenskii and Lebedev, *Theory of Circular Accelerators*, North-Holland (1966)

We only consider lowest order effects in these notes.

In the next section we will examine how resonances restrict possible machine operating parameters.

S6: Machine Operating Points: Tune Restrictions Resulting from Resonances

Examine situations where the *x*-plane motion resonance condition:

$$(n-2m)\nu_0 + pk = \pm\nu_0$$

is satisfied for the operating tune ν_0 and some values of:

Multipole Order Index:

$$n = 0,\ 1,\ 2,\ \cdots \quad \Longleftarrow$$

Specify to understand class of perturbations

Particle Binomial Expansion Index:

$$m = 0,\ 1,\ 2,\ \cdots,\ n$$

Linear superposition for multiple perturbations

Periodicity Fourier Series Expansion Index:

$$k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty$$

Perturbation Symmetry Factor:

$$p \equiv \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

Resonances can be analyzed one at a time using linear superposition

- Analysis valid for small-amplitudes

Analyze resonance possibilities starting with index *n* <==> Multipole Order

n = 0, Dipole Perturbations: $(n-2m)\nu_0 + pk = \pm\nu_0$

$$n = 0, \quad \Longrightarrow \quad m = 0$$

and the resonance condition gives a single constraint:

$$\nu_0 = \pm pk \qquad pk = \text{integer} \qquad k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty$$

$$p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

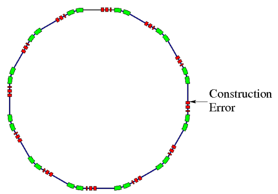
Therefore, to avoid dipole resonances *integer tunes* operating points not allowed:

$p = 1$	Random Perturbation	$\nu_0 \neq 1,\ 2,\ 3,\ \cdots$
$p = \mathcal{N}$	Systematic Perturbation	$\nu_0 \neq \mathcal{N},\ 2\mathcal{N},\ 3\mathcal{N},\ \cdots$

- Systematic errors are significantly *less restrictive* on machine operating points for large $\mathcal{N}$
  - Illustrates why high symmetry is desirable
  - Racetracks with $\mathcal{N} = 2$ can be problem
- Multiply random perturbation tune restrictions by $\mathcal{N}$ to obtain the systematic perturbation case

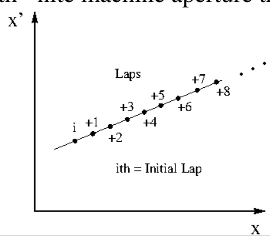
Interpretation of result:

Consider a ring with a single (random) dipole error along the reference path of the ring:



If the particle is oscillating with integer tune, then the particle experiences the dipole error on each lap in the same oscillation phase and the trajectory will "walk-o " on a lap-to-lap basis in phase-space:

- With nite machine aperture the particle will be scraped/lost



n = 1, Quadrupole Perturbations: $(n-2m)\nu_0 + pk = \pm\nu_0$

$$n = 1, \implies m = 0,\ 1$$

and the resonance conditions give:

$$n = 1,\ m = 0: \qquad \nu_0 + pk = \pm\nu_0$$

$$n = 1,\ m = 1: \qquad -\nu_0 + pk = \pm\nu_0$$

Give two cases:

$$\implies pk = 0,\ \nu_0 = \pm\frac{pk}{2}$$

Implications of two cases:

$$1)\ pk = 0 \ \Rightarrow k = 0$$

Can be treated by "renormalizing" oscillator focusing strength: need not be considered

$$\ddot{u} + \nu_0^2 u = \nu_0^2 C_{1,0} u$$

$$2)\ \nu_0 = \pm\frac{pk}{2} \ \Rightarrow \nu_0 = \frac{|pk|}{2}$$

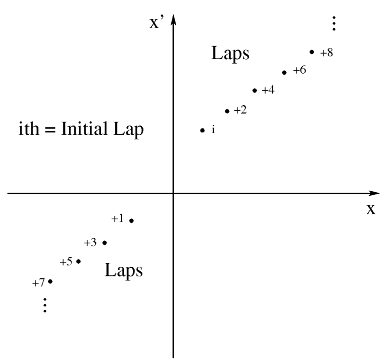
Therefore, to avoid quadrupole resonances, the following tune operating points are not allowed:

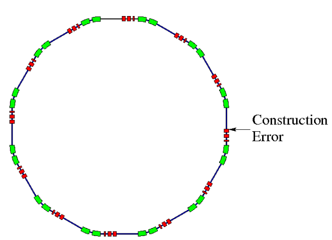
$$\boxed{\nu_0 \neq \frac{|pk|}{2} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases} \qquad k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty}$$

- New restriction: tunes cannot be half-integer values
- Integers also restricted for p = 1 random, but redundant with dipole case
- Some large integers restricted for $p = \mathcal{N}$ systematic perturbations

Interpretation of result (new restrictions):

For a single (random) quadrupole error along the azimuth of a ring, a similar qualitative argument as presented in the dipole resonance case leads one to conclude that if a particle oscillates with ½ integer tune, then the orbit can "walk-o " on a lap-to-lap basis in phase-space:





n = 2, Sextupole Perturbations: $(n-2m)\nu_0 + pk = \pm\nu_0$

$$n = 2, \implies m = 0,\ 1,\ 2$$

and the resonance conditions give the three constraints below:

$$n = 2,\ m = 0: \qquad 2\nu_0 + pk = \pm\nu_0$$

$$n = 2,\ m = 1: \qquad pk = \pm\nu_0$$

$$n = 2,\ m = 2: \qquad -2\nu_0 + pk = \pm\nu_0$$

Therefore, to avoid sextupole resonances, the following tunes are not allowed:

$$\boxed{\nu_0 \neq \begin{cases} |pk| & \text{integer} \\ |pk|/3 & \text{third-integer} \end{cases} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases} \qquad k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty}$$

- Integer restrictions already obtained for dipole perturbations
- 1/3-integer restriction new

n = 3, Octupole Perturbations: $(n-2m)\nu_0 + pk = \pm\nu_0$

$$n = 3, \implies m = 0,\ 1,\ 2,\ 3$$

and the resonance conditions give the three constraints below:

$$n=3,\ m=0: \quad 3\nu_0 + pk = \pm\nu_0$$

$$n=3,\ m=1: \quad \nu_0 + pk = \pm\nu_0$$

$$n=3,\ m=2: \quad -\nu_0 + pk = \pm\nu_0$$

$$n=3,\ m=3: \quad -3\nu_0 + pk = \pm\nu_0$$

Therefore, to avoid octupole resonances, the following tunes are not allowed:

$$\boxed{\nu_0 \neq \begin{cases} |pk|/2 & \text{half-integer} \\ |pk|/4 & \text{quarter-integer} \end{cases} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases} \qquad k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty}$$

- ½ - integer restrictions already obtained for quadrupole perturbations
- 1/4-integer restriction new

Higher-order (n > 3) cases analyzed analogously as n increases

- Resonances expected to be weaker as order increases

Higher-order (n > 2) cases analyzed analogously: $(n-2m)\nu_0 + pk = \pm\nu_0$

- Produce more constraints but expected to be weaker as order increases
- Will always generate a new constraint $\nu_0 \neq \dfrac{|pk|}{n+1}$

General form of resonance condition

The general resonance condition (all n-values) for x-plane motion can be summarized as:

$$\boxed{M\nu_0 = N \qquad \begin{array}{l} M,\ N = \text{Integers of same sign} \\ |M| = \text{"Order" of resonance} \end{array}}$$

- Higher order numbers M are typically less dangerous
  - Longer coherence length for validity of theory: e ects not included can "wash-out" the resonance
  - Coe cients generally smaller

Particle motion is not (measure zero) really restricted to the x-plane, and a more complete analysis taking into account coupled x- and y-plane motion shows that the generalized resonance condition is:

- Place unperturbed y-orbit in rhs perturbation term, then leading-order expand analogously to x-case to obtain additional driving terms

$$\boxed{\begin{array}{ll} M_x\nu_{0x} + M_y\nu_{0y} = N & M_x,\ M_y,\ N = \text{Integers of same sign} \\ \nu_{0x} = x\text{-plane tune} & |M_x| + |M_y| = \text{"Order" of resonance} \\ \nu_{0y} = y\text{-plane tune} & \end{array}}$$

- Lower order resonances are more dangerous analogously to x-case

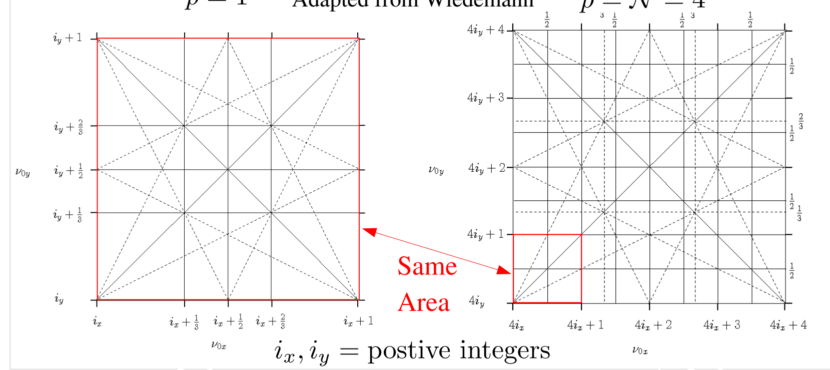
Restrictions on machine operating points

Tune restrictions are typically plotted in $\nu_{0x} - \nu_{0y}$ space order-by-order up to a max order value to nd allowed tunes where the machine can safely operate

- Often 3rd order is chosen as a maximum to avoid
- Cases for random ($p = 1$) and systematic ($p = \mathcal{N}$) perturbations considered

Machine operating points chosen as far as possible from low order resonance lines

Random Perturbations $p = 1$ Adapted from Wiedemann Systematic Perturbations $p = \mathcal{N} = 4$



Discussion: Restrictions on machine operating points

Random Errors:

- Errors always present and give low-order resonances
- Usually have weak amplitude coefficients
  - Can be corrected/compensated to reduce effects

Systematic Errors:

- Lead to higher-order resonances for large $\mathcal{N}$ and a lower density of resonance lines (see plots on previous slide comparing the equal boxed red areas)
  - Large symmetric rings with high $\mathcal{N}$ values have less operating restrictions from systematic errors
  - Practical issues such as construction cost and getting the beam into and out of the ring can lead to smaller $\mathcal{N}$ values (racetrack lattice)
- BUT systematic error Amplitude coefficients can be large
  - Systematic effects accumulate in amplitude period by period

Resonances beyond 3rd order rarely need be considered

- Effects outside of model assumed tend to wash-out higher order resonances

More detailed treatments calculate amplitudes/strengths of resonant terms

- See accelerator physics references:

Further info: Wiedemann, *Particle Accelerator Physics* (2007)

Amplitudes/Strengths: Kolomenskii and Lebedev, *Theory of Circular Accel* (1966)

Notation/Nomenclature: Laslett Limits

Ring operating points are chosen to be far from low-order particle resonance lines in *x-y* tune space. Processes that act to shift particle resonances closer towards the low-order lines can prove problematic:

- Oscillation amplitudes increase (spoiling beam quality and control)
- Particles can be lost

Tune shift limits of machine operation are often named "Laslett Limits" in honor of Jackson Laslett who first calculated tune shift limits for various processes:

- Image charges
- Image currents
- Internal beam self-fields
- ...

Processes shifting resonances can be grouped into two broad categories:

Coherent — Same for every particle in distribution

- Usually most dangerous: full beam resonant

Incoherent — Different for particles in separate parts of the distribution

- Usually less dangerous: only effects part of beam

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/msu/phy905_2020/

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