08 Momentum Spread E ects in Bending and Focusing*

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Outline

S9: Momentum Spread E ects in Bending and Focusing
S9A: Formulation

Except for brief digressions in, we have concentrated on particle dynamics where all particles have the design longitudinal momentum at a value of *s* in the lattice:

$$p_s = m\gamma_b\beta_b c = \text{same for every particle}$$

Realistically, there will always be a nite spread of particle momentum within a beam slice, so we take:

$$p_s = p_0 + \delta p$$
$$p_0 \equiv m\gamma_b\beta_b c = \text{Design Momentum}$$
$$\delta p \equiv \text{Off Momentum}$$

Typical values of momentum spread in a beam with a single species of particles with conventional sources and accelerating structures:

$$\frac{|\delta p|}{p_0} \sim 10^{-2} \to 10^{-6}$$

The spread of particle momentum can modify particle orbits, particularly when dipole bends are present since the bend radius depends strongly on the particle momentum

The o momentum results in a change in particle rigidity impacting the coupling of the particle to applied elds:

$$[B\rho] \equiv \frac{p}{q} = \left(\frac{p_0}{q}\right)\left(\frac{p}{p_0}\right) = [B\rho]_0\left(\frac{p}{p_0}\right) \qquad [B\rho]_0 = \frac{p_0}{q} = \text{Design Rigidity}$$

- Particles with higher/lower *p* than design will have higher/lower rigidity $[B\rho]$ with weaker/stronger coupling to the applied elds

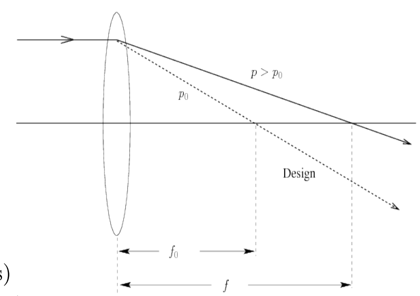
Focusing (thin lens illustration)

$$\frac{1}{f} \simeq \kappa\ell = \frac{G\ell}{[B\rho]}$$

$$\Longrightarrow f \simeq \frac{[B\rho]}{G\ell} = \frac{[B\rho]_0}{G\ell}\left(\frac{p}{p_0}\right) = f_0\left(\frac{p}{p_0}\right)$$

$$f_0 \equiv \frac{[B\rho]_0}{G\ell} = \text{Design Focus}$$

$$\begin{cases} f > f_0 & \text{when } p > p_0 \text{ (weaker focus)} \\ f < f_0 & \text{when } p < p_0 \text{ (stronger focus)} \end{cases}$$



$$[B\rho] \equiv \frac{p}{q} = \left(\frac{p_0}{q}\right)\left(\frac{p}{p_0}\right)$$

$$= [B\rho]_0 \left(\frac{p}{p_0}\right) \qquad [B\rho]_0 = \frac{p_0}{q} = \text{Design Rigidity}$$

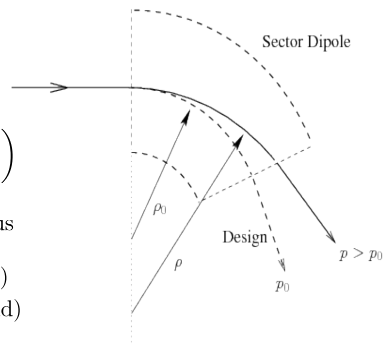
Bending (sector bend illustration)

$$\frac{1}{\rho} = \frac{B_y(0)}{[B\rho]}$$

$$\Longrightarrow \rho = \frac{[B\rho]}{B_y(0)} = \frac{[B\rho]_0}{B_y(0)}\left(\frac{p}{p_0}\right) = \rho_0\left(\frac{p}{p_0}\right)$$

$$\rho_0 \equiv \frac{[B\rho]_0}{B_y(0)} = \text{Design Radius}$$

$$\begin{cases} \rho > \rho_0 & \text{when } p > p_0 \text{ (weaker bend)} \\ \rho < \rho_0 & \text{when } p < p_0 \text{ (stronger bend)} \end{cases}$$



Systematic analysis of o -momentum for magnetic focusing and bending

To derive relevant single-particle equations of motion for o -momentum, revisit analysis of design momentum trajectory in a bent coordinate system

- Consider transverse magnetic eld only (bending + focusing) for simplicity
  - Can put in electric bends and focus paralleling analysis

$$\hat{\mathbf{x}}: \quad x'' - \frac{(\rho_0 + x)}{\rho_0^2} = -\frac{B_y}{[B\rho]}\left(1 + \frac{x}{\rho_0}\right)^2$$

$$\hat{\mathbf{y}}: \quad y'' = \frac{B_y}{[B\rho]}\left(1 + \frac{x}{\rho_0}\right)^2$$

Here we express equations for:

- Transverse magnetic eld components $B_x, \ B_y$

$$B_x = G \cdot y$$
$$B_y = B_y(0) + G \cdot x$$

Design Bend: $\dfrac{1}{\rho_0} = \dfrac{B_y(0)}{[B\rho]_0}$

Quad Gradient: $G = \left.\dfrac{B_x}{\partial y}\right|_0 = \left.\dfrac{B_y}{\partial x}\right|_0$

- Rigidity $[B\rho]$:

$$[B\rho] = \frac{p}{q} = \frac{p_0}{q}\frac{p}{p_0} = [B\rho]_0 \frac{p}{p_0}$$

Design Rigidity: $[B\rho]_0 \equiv \dfrac{p_0}{q}$

Inserting these expressions in the equations of motion:

$$x'' - \frac{(\rho_0 + x)}{\rho_0^2} = -\frac{[B_y(0) + Gx]}{[B\rho]_0}\frac{p_0}{p}\left(1 + \frac{x}{\rho_0}\right)^2$$

$$y'' = \frac{Gy}{[B\rho]_0}\frac{p_0}{p}\left(1 + \frac{x}{\rho_0}\right)^2$$

Expand to leading order in *x* and *y* in rhs terms and rearrange:

$$x'' + \left[-\frac{1}{\rho_0^2} + \frac{2B_y(0)}{\rho_0[B\rho]_0}\left(\frac{p_0}{p}\right) + \frac{G}{[B\rho]_0}\left(\frac{p_0}{p}\right)\right]x = \frac{1}{\rho_0} - \frac{B_y(0)}{[B\rho]_0}\left(\frac{p_0}{p}\right)$$

$$y'' - \frac{G}{[B\rho]_0}\left(\frac{p_0}{p}\right)y = 0$$

Denote:

$$\kappa = \frac{G}{[B\rho]_0}$$ Quadrupole focus (design momentum)

And Apply:

$$\frac{1}{\rho_0} = \frac{B_y(0)}{[B\rho]_0}$$ Bend Radius (design momentum)

And the equations of motion become:

$$x'' + \left[\frac{1}{\rho_0^2}\left(-1 + 2\frac{p_0}{p}\right) + \frac{\kappa}{(p/p_0)}\right]x = \frac{1}{\rho_0}\left(1 - \frac{p_0}{p}\right)$$

$$y'' - \frac{\kappa}{(p/p_0)}y = 0$$

Use in this expression:

$$\frac{p}{p_0} = \frac{p_0 + \delta p}{p_0} = 1 + \frac{\delta p}{p_0} \equiv 1 + \delta$$

$$\delta \equiv \frac{\delta p}{p_0}$$ Fractional Momentum Error

Then:

$$-1 + 2\left(\frac{p_0}{p}\right) = \frac{-p + 2p_0}{p} = \frac{-(p_0 + \delta p) + 2p_0}{p_0 + \delta p} = \frac{p_0 - \delta p}{p_0 + \delta p} = \frac{1 - \delta}{1 + \delta}$$

$$1 - \frac{p_0}{p} = \frac{p - p_0}{p} = \frac{p_0 + \delta p - p_0}{p_0 + \delta p} = \frac{\delta p}{p_0 + \delta p} = \frac{\delta}{1 + \delta}$$

and the equations of motion become :

$$x'' + \left[\frac{1}{\rho_0^2}\frac{1 - \delta}{1 + \delta} + \frac{\kappa}{1 + \delta}\right]x = \frac{\delta}{1 + \delta}\frac{1}{\rho_0}$$

$$y'' - \frac{\kappa}{1 + \delta}y = 0$$

Notion:

- Typically drop "0" subscripts from: $[B\rho]_0, \ \ \rho_0$
  - Understood to be design values

$$\frac{1}{\rho} \equiv \frac{B_y(0)}{[B\rho]} \qquad [B\rho] \equiv \frac{p_0}{q}$$

Can derive analogous equations for:

- Electric focusing and bends
- Magnetic solenoids (straight lattice)
- See also Sec. 1 H which summarizes equations of motion for 3D elds with o -momentum
  - Results obtainable by placing linear eld components in equations summarized

We will summarize equations of motion for these cases in one combined form.

Single particle equations of motion for a particle with momentum spread in linear applied elds

$$x''(s) + \left[\frac{1}{\rho^2(s)}\frac{1-\delta}{1+\delta} + \frac{\kappa_x(s)}{(1+\delta)^n}\right]x(s) = \frac{\delta}{1+\delta}\frac{1}{\rho(s)}$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n}y(s) = 0$$

$\rho(s) =$ Local Bend Radius for design momentum p_0 ($\rho \to \infty$ in straight sections)

Magnetic Dipole Bend

$$\frac{1}{\rho(s)} = \frac{B_y^a|_{\text{dipole}}}{[B\rho]}$$

$$[B\rho] = \frac{p_0}{q}$$

$\delta \equiv \dfrac{\delta p}{p_0}$ $\kappa_{x,y} =$ Focusing Functions (using design momentum)

$$n = \begin{cases} 1, & \text{Magnetic Quadrupoles} \\ 2, & \text{Solenoids, Electric Quadrupoles} \end{cases}$$

Neglects:

- Space-charge: $\phi \to 0$
- Nonlinear applied focusing: $\mathbf{E}^a$, $\mathbf{B}^a$ contain only linear focus terms
- Acceleration: $p_0 = mc\gamma_b\beta_b =$ const

In the equations of motion, it is important to understand that B_y^a of the magnetic bends are set from the radius ρ required by the design particle orbit (see: S1 for details)

- Equation relating ρ to elds must be modi ed for electric bends (see S1)
- y-plane bends also require modi cation of eqns (analogous to x-plane case)

The focusing strengths are de ned with respect to the design momentum:

$$\kappa_x = \begin{cases} -\kappa_y = \frac{G}{\beta_b c[B\rho]}, & G = -\partial E_x^a/\partial x = \partial E_y^a/\partial y = \text{Electric Quad.} \\ -\kappa_y = \frac{G}{[B\rho]}, & G = \partial B_x^a/\partial y = \partial B_y^a/\partial x = \text{Magnetic Quad.} \\ \kappa_y = \left(\frac{B_{z0}}{2[B\rho]}\right)^2, & B_{z0} = \text{Solenoidal Magnetic Field} \end{cases}$$

γ_b, β_b calculated from q, m and $[B\rho]$

Comments:

- Electric and magnetic quadrupoles have di erent variation on δ due to the di erent axial velocity dependance in the coupling to the elds
- Included solenoid case to illustrate focusing dispersion but this would rely on the Larmor transform and that does not make sense in a bent coordinate system

$$x''(s) + \left[\frac{1}{\rho^2(s)}\frac{1-\delta}{1+\delta} + \frac{\kappa_x(s)}{(1+\delta)^n}\right]x(s) = \frac{\delta}{1+\delta}\frac{1}{\rho(s)}$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n}y(s) = 0$$

Terms in the equations of motion associated with momentum spread (δ) can be lumped into two classes:

S.9B: Dispersive -- Associated with Dipole Bends

S.9C: Chromatic -- Associated with Applied Focusing (κ)

S9B: Dispersive E ects

Present only in the *x*-equation of motion and result from bending. Neglecting chromatic terms:

$$x''(s) + \left[\frac{1}{\rho^2(s)}\frac{1-\delta}{1+\delta} + \kappa_x(s)\right] x(s) = \frac{\delta}{1+\delta}\frac{1}{\rho(s)}$$

Term 1 (bracketed term on the left); Term 2 (right-hand side)

Particles are bent at di erent radii when the momentum deviates from the design value ($\delta \neq 0$) leading to changes in the particle orbit

- Dispersive terms contain the bend radius ρ

Generally, the bend radii *R* are large and δ is small, and we can take to leading order:

Term 1: $\left[\frac{1}{\rho^2}\frac{1-\delta}{1+\delta} + \kappa_x\right] x \simeq \left[\frac{1}{\rho^2} + \kappa_x\right] x \quad + \mathcal{O}\left(\frac{\delta}{\rho^2}, \frac{\delta^2}{\rho^2}\right)$

Term 2: $\frac{\delta}{1+\delta}\frac{1}{\rho} \simeq \frac{\delta}{\rho} + \mathcal{O}\left(\frac{\delta^2}{\rho}\right)$

$[\cdots] \equiv \kappa_x$ (Rede ne to incorporate)

The equations of motion then become:

$$x''(s) + \kappa_x(s)x(s) = \frac{\delta}{\rho(s)}$$
$$y''(s) + \kappa_y(s)y(s) = 0$$

- The y-equation is not changed from the usual Hill's Equation

The *x*-equation is typically solved for *periodic* ring lattices by exploiting the linear structure of the equation and linearly resolving:

$$x(s) = x_h(s) + x_p(s)$$
$$x_h \equiv \text{Homogeneous Solution}$$
$$x_p \equiv \text{Particular Solution}$$

where x_h is the *general* solution to the Hill's Equation:

$$x_h''(s) + \kappa_x(s)x_h(s) = 0$$

and x_p is the *periodic* solution to:

$$x_p = \delta \cdot D \qquad D''(s) + \kappa_x(s)D(s) = \frac{1}{\rho(s)}$$
$$D \equiv \text{Dispersion Function} \qquad D(s + L_p) = D(s)$$

This convenient resolution of the orbit *x*(*s*) can *always* be made because the homogeneous solution will be adjusted to match any initial condition

Note that x_p provides a measure of the o set of the particle orbit relative to the design orbit resulting from a small deviation of momentum (δ)

x(*s*) = 0 de nes the design orbit

$[[D]] = \text{meters}$

$\delta \cdot D = \text{Dispersion induced orbit offset in meters}$

Comments:

- It can be shown (see Appendix B) that *D* is unique given a focusing function κ_x for a periodic lattice provided that $\frac{\sigma_{0x}}{2\pi} \neq \text{integer}$
  - In this context *D* is interpreted as a Lattice Function similarly to the betatron function
  - δD gives the closed orbit of an o -momentum particle in a ring due to dispersive e ects
- The case of how to interpret and solve for *D* in a non-periodic lattice (transfer line) will be covered later
  - In this case initial conditions of *D* will matter

Extended 3x3 Transfer Matrix Form for Dispersion Function

Can solve *D* in

$$D'' + \kappa_x D = \frac{1}{\rho}$$

by taking

$$D = D_h + D_p$$

$D_h = \text{Homogeneous Solution}$

$D_p = \text{Particular Solution}$

Homogeneous solution is the general solution to

$$D_h'' + \kappa_x D_h = 0$$

- Usual Hill's equation with solution expressed in terms of principle functions in 2x2 matrix form

$$\begin{bmatrix} D_h \\ D_h' \end{bmatrix}_s = \mathbf{M}(s|s_i) \cdot \begin{bmatrix} D_h \\ D_h' \end{bmatrix}_{s_i} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} D_h \\ D_h' \end{bmatrix}_{s_i}$$

Particular solution take to be the zero initial condition solution to

- Homogeneous part used to adjust for general initial conditions: always integrate from zero initial value and angle

$$D_p'' + \kappa_x D_p = \frac{1}{\rho}$$
$$D_p(s_i) = 0 = D_p'(s_i)$$

Denote solution as from zero initial value and angle at $s = s_i$ as $D_p(s) \equiv D_p(s|s_i)$

Can superimpose the homogeneous and particular solutions to form a generalized 3x3 transfer matrix for the Dispersion function D as:

- Initial condition absorbed on homogeneous solution

$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_s = \begin{bmatrix} C(s|s_i) & S(s|s_i) & D_p(s|s_i) \\ C'(s|s_i) & S'(s|s_i) & D_p'(s|s_i) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s_i}$$
$$= \begin{bmatrix} [\mathbf{M}(s|s_i)] & D_p(s|s_i) \\ & D_p'(s|s_i) \\ 0 \quad 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s_i} \equiv \mathbf{M}_3(s|s_i) \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s_i}$$

For a periodic solution:

$$D(s_i + L_p) = D(s_i)$$
$$D'(s_i + L_p) = D'(s_i)$$

This gives two constraints to determine the needed initial condition for periodicity

- Third row trivial

$$D(s_i) - C(s_i + L_p|s_i)D(s_i) - S(s_i + L_p|s_i)D'(s_i) = D_p(s_i + L_p|s_i)$$
$$D'(s_i) - C'(s_i + L_p|s_i)D(s_i) - S'(s_i + L_p|s_i)D'(s_i) = D_p'(s_i + L_p|s_i)$$

Solving this using matrix methods (inverse by minor) and simplifying the result with the Wronskian invariant (S5C)

$$W = C(s|s_i)S'(s|s_i) - S(s|s_i)C'(s|s_i) = 1$$

and the de nition of phase advance in the periodic lattice (S6G)

$$\cos\sigma_{0x} = \frac{1}{2}\text{Tr}\ \mathbf{M}(s_i + L_p|s_i) = \frac{1}{2}[C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)]$$

Yields:

$$\begin{bmatrix} D \\ D' \end{bmatrix}_{s_i} = \frac{1}{2(1-\cos\sigma_{0x})} \begin{bmatrix} 1 - S'(s_i + L_p|s_i) & S(s_i + L_p|s_i) \\ C'(s_i + L_p|s_i) & 1 - C(s_i + L_p|s_i) \end{bmatrix} \cdot \begin{bmatrix} D_p(s_i + L_p|s_i) \\ D_p'(s_i + L_p|s_i) \end{bmatrix}$$

- Resulting solution for D from this initial condition will have the periodicity of the lattice. These values always exist for real σ_{0x} $(\sigma_{0x} < 180^\circ)$
- Values of $D(s_i),\ D'(s_i)$ depend on location of choice of s_i in lattice period
- Can use 3x3 transfer matrix to nd D anywhere in the lattice
- Formulation assumes that the underlying lattice is stable with $\sigma_{0x} < 180^\circ$

Alternatively, take $s_i = s$ to obtain

$$D(s) = \frac{[1 - S'(s + L_p|s)]\, D_p(s + L_p|s) + S(s + L_p|s)D_p'(s + L_p|s)}{2(1 - \cos\sigma_{0x})}$$
$$D'(s) = \frac{C'(s + L_p|s)D_p(s + L_p|s) + [1 - C(s + L_p|s)]\, D_p'(s + L_p|s)}{2(1 - \cos\sigma_{0x})}$$

Particular Solution for the Dispersion Function in a Periodic Lattice

To solve the particular function of the dispersion from a zero initial condition,

$$D_p'' + \kappa_x D_p = \frac{1}{\rho} \qquad\qquad D_p(s_i) = 0 = D_p'(s_i)$$

A Green's function method can be applied (see Appendix A) to express the solution in terms of projection on the principal orbits of Hill's equation as:

$$D_p(s) = \int_{s_i}^{s} d\tilde{s}\ \frac{1}{\rho(\tilde{s})} G(s, \tilde{s})$$
$$G(s, \tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$
$$\mathcal{C}(s|s_i) = \text{Cosine-like Principal Trajectory}$$
$$\mathcal{S}(s|s_i) = \text{Sine-like Principal Trajectory}$$

Cosine-Like Solution

$$\mathcal{C}''(s|s_i) + \kappa_x(s)\mathcal{C}(s|s_i) = 0$$
$$\mathcal{C}(s_i|s_i) = 1$$
$$\mathcal{C}'(s_i|s_i) = 0$$

Sine-Like Solution

$$\mathcal{S}''(s|s_i) + \kappa_x(s)\mathcal{S}(s|s_i) = 0$$
$$\mathcal{S}(s_i|s_i) = 0$$
$$\mathcal{S}'(s_i|s_i) = 1$$

Discussion:

- The Green's function solution for D_p, together with the 3x3 transfer matrix can be used to solve explicitly for D from an initial value
- The initial values $D(s_i), \quad D'(s_i)$ found will yield the *unique* solution for D with the periodicity of the lattice

The periodic lattice solution for the dispersion function can be expressed in terms of the betatron function of the periodic lattice as follows:

From S7C:

$$\mathbf{M}(s|s_i) = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \qquad \alpha \equiv -\beta'/2$$

$$= \begin{bmatrix} \sqrt{\frac{\beta(s)}{\beta_i}}[\cos\Delta\psi(s) + \alpha_i \sin\Delta\psi(s)] & \sqrt{\beta_i\beta}\sin\Delta\psi(s) \\ -\frac{\alpha(s)-\alpha_i}{\sqrt{\beta_i\beta(s)}}\cos\Delta\psi(s) - \frac{1+\alpha_i\alpha(s)}{\sqrt{\beta_i\beta(s)}}\sin\Delta\psi(s) & \sqrt{\frac{\beta_i}{\beta(s)}}[\cos\Delta\psi(s) - \alpha\sin\Delta\psi(s)] \end{bmatrix}$$

and using

$$D_p(s) = \int_{s_i}^{s} d\tilde{s}\, \frac{1}{\rho(\tilde{s})} G(s,\tilde{s}) \qquad G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

and the periodicity of the lattice functions $\beta, \quad \alpha = -\beta'/2$

along with considerable algebraic manipulations show that the dispersion function D for the periodic lattice can be expressed as:

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin(\sigma_{0x}/2)} \int_s^{s+L_p} d\tilde{s}\, \frac{\sqrt{\beta(\tilde{s})}}{\rho(\tilde{s})} \cos[\Delta\psi(\tilde{s}) - \Delta\psi(s) - \sigma_{0x}/2]$$

$$D'(s) - \frac{\alpha(s)}{\beta(s)}D(s)$$
$$= \frac{1}{2\sqrt{\beta(s)}\sin(\sigma_{0x}/2)} \int_s^{s+L_p} d\tilde{s}\, \frac{\sqrt{\beta(\tilde{s})}}{\rho(\tilde{s})} \sin[\Delta\psi(\tilde{s}) - \Delta\psi(s) - \sigma_{0x}/2]$$

$$\Delta\psi(s) = \int_{s_i}^{s} \frac{1}{\beta(\tilde{s})} d\tilde{s}$$

- Formulas and related information can be found in SY Lee, *Accelerator Physics* and Conte and MacKay, *Introduction to the Physics of Particle Accelerators*
- Provides periodic dispersion function D expressed as an integral of betatron function describing the linear optics of the lattice
  - Have $\beta(s)$ for a ring lattice, then also have the periodic dispersion function

Full Orbit Resolution in a Periodic Dispersive Lattice

Taking a particle initial condition,

$$x(s = s_i) \equiv x_i \qquad \delta = \frac{\delta p}{p_0}$$
$$x'(s = s_i) \equiv x_i'$$

and using the homogeneous (Hill's Equation Solution) and particular solutions (Dispersion function) of the periodic lattice, the orbit can be resolved as

$$x(s) = C_1 C(s|s_i) + C_2 S(s|s_i) + \delta D(s) \qquad C_1, \quad C_2 = \text{constants}$$

$$\Longrightarrow \quad \begin{aligned} x_i &= C_1 + \delta D_i \\ x_i' &= C_2 + \delta D_i' \end{aligned} \quad \text{Fixes constants} \quad \begin{aligned} C_1 &= x_i - \delta D_i \\ C_2 &= x_i' - \delta D_i' \end{aligned}$$

Giving,

$$x(s) = x_h + x_p = x_i C(s|s_i) + x_i' S(s|s_i) + \delta[D(s) - D_i C(s|s_i) - D_i' S(s|s_i)]$$
$$x'(s) = x_h' + x_p' = x_i C'(s|s_i) + x_i' S'(s|s_i) + \delta[D'(s) - D_i C'(s|s_i) - D_i' S'(s|s_i)]$$

here, $D(s = s_i) \equiv D_i$

$D'(s = s_i) \equiv D_i'$

are initial dispersion values that are uniquely determined in the periodic lattice

- Varies with choice of initial condition $(s = s_i)$ in lattice

3x3 Transfer Matrices for Dispersion Function

In problems, will derive 3x3 transfer matrices:

- Summarize results here for completeness
- Can use Green function results and 2x2 transfer matrices from previous sections to derive
- Can apply to *any* initial conditions $D_i, \quad D_i'$
  - Only speci c initial conditions will yield D periodic with (a periodic) lattice
  - Useful in general form for applications to transfer lines, achromatic bends, etc.

$$D'' + \kappa_x D = \frac{1}{\rho}$$

$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_s \equiv \mathbf{M}_3(s|s_i) \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s_i}$$

Drift: $\kappa_x(s) = 0, \qquad \rho \rightarrow \infty$

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} 1 & (s-s_i) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thin Lens: located at $s = s_i$ with focal strength f (no superimposed bend)

$$\kappa_x(s) = -\frac{1}{f}\delta(s-s_i), \qquad \rho \rightarrow \infty$$

$$\mathbf{M}_3(s_i^+|s_i^-) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Can apply to entry and exit angles with sector bend (next page) for slanted edge corrections to dipole when f is used to express the correct kick correction

Thick Focus Lens: with $\kappa_x = \hat{\kappa} = \text{const} > 0$ (no superimposed bend)

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & 0 \\ -\sqrt{\hat{\kappa}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & \cos[\sqrt{\hat{\kappa}}(s-s_i)] & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thick deFocus Lens: with $\kappa_x = -\hat{\kappa} = \text{const} < 0$ (no superimposed bend)

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & 0 \\ \sqrt{\hat{\kappa}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sector Bend with Focusing: $\rho = \text{const}, \quad \kappa_x = \hat{\kappa} = \text{const} > 0$

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\rho\hat{\kappa}}\left\{1-\cos[\sqrt{\hat{\kappa}}(s-s_i)]\right\} \\ -\sqrt{\hat{\kappa}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\rho\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] \\ 0 & 0 & 1 \end{bmatrix}$$

Sector Bend with deFocusing: $\rho = \text{const}, \quad \kappa_x = -\hat{\kappa} = \text{const} < 0$

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\rho\hat{\kappa}}\left\{-1+\cosh[\sqrt{\hat{\kappa}}(s-s_i)]\right\} \\ \sqrt{\hat{\kappa}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\rho\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] \\ 0 & 0 & 1 \end{bmatrix}$$

For the special case of a sector bend of axial length ℓ the bend with focusing, corresponding to

$$\rho = \text{const}, \qquad \kappa_x = \frac{1}{\rho^2}$$

- Bend provides *x*-plane focusing

this result reduces for transport through the full bend to:

$$\ell = \rho\theta, \quad \theta = \text{Bend Angle}$$

$$\mathbf{M}_3 = \begin{bmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

For a small angle bend with $|\theta| \ll 1$, this further reduces to:

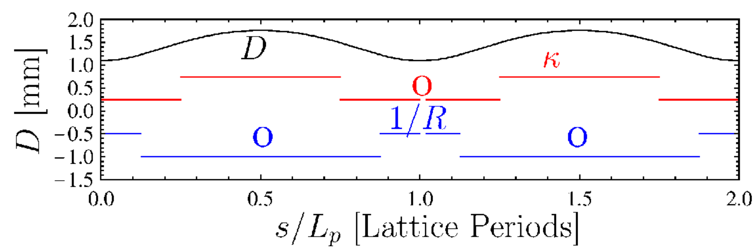
$$\mathbf{M}_3 \simeq \begin{bmatrix} 1 & \ell & \ell\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{bmatrix}$$

// Example: Dispersion function for a simple periodic lattice

For purposes of a simple illustration we here use an imaginary FO (Focus-Drift) piecewise-constant lattice where the *x*-plane focusing is like the focus-plane of a quadrupole with one thick lens focus optic per lattice period and a single drift with the bend in the middle of the drift $\kappa > 0$

- Focus element implemented by *x*-plane quadrupole transfer matrix in S5B.

$$L_p = 0.5 \text{ m} \quad \kappa = 20/\text{m}^2 \text{ in Focusing}$$
$$\eta = 0.5 \quad \rho = R = 15 \text{ m, in bend, 25\% Occupancy in Period}$$



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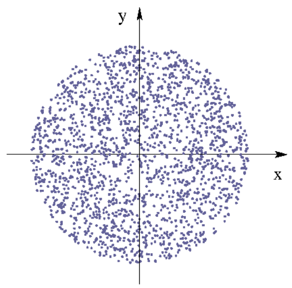
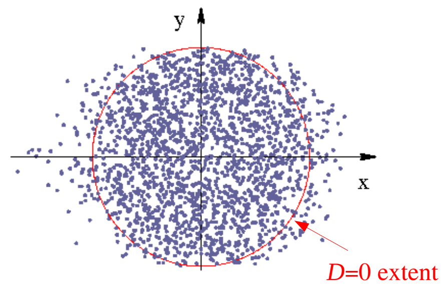
// Example: Dispersion broadens the distribution in x

Uniform Bundle of particles D = 0

Same Bundle of particles D nonzero

- Gaussian distribution of momentum spreads (δ) distorts the *x*-*y* distribution extents in *x* but not in *y*





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// Example: Continuous Focusing in a Continuous Bend

$$\kappa_x(s) = k_{\beta 0}^2 = \text{const}$$
$$\rho(s) = \rho = \text{const}$$

Dispersion equation becomes:

$$D'' + k_{\beta 0}^2 D = \frac{1}{\rho}$$

With constant solution:

$$D = \frac{1}{k_{\beta 0}^2 \rho} = \text{const}$$

From this result we can crudely estimate the average value of the dispersion function in a ring with periodic focusing by taking:

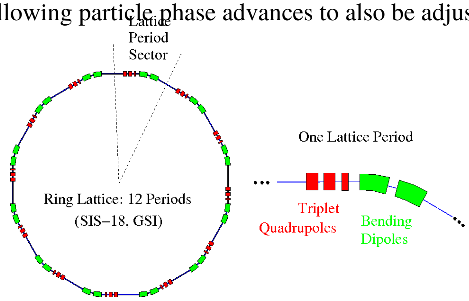
$$\rho = \text{Avg Radius Ring}$$
$$L_p = \text{Lattice Period (Focusing)}$$
$$\sigma_{0x} = x\text{-Plane Phase Advance}$$

$$\Longrightarrow \quad k_{\beta 0} \sim \frac{\sigma_0}{L_p} \qquad \Longrightarrow \quad D \sim \frac{L_p^2}{\sigma_0^2 \rho}$$

//

Many rings are designed to focus the dispersion function *D*(*s*) to small values in straight sections even though the lattice has strong bends

- Desirable since it allows smaller beam extents at locations near where *D* = 0 and these locations can be used to insert and extract (kick) the beam into and out of the ring with minimal losses and/or accelerate the beam
  - Since average value of *D* is dictated by ring size and focusing strength (see example next page) this variation in values can lead to *D* being larger in other parts of the ring
- Quadrupole triplet focusing lattices are often employed in rings since the use of 3 optics per period (vs 2 in doublet) allows more flexibility to tune *D* while simultaneously allowing particle phase advances to also be adjusted



Dispersive Effects in Transfer Lines with Bends

It is common that a beam is transported through a single or series of bends in applications rather than a periodic ring lattice. In such situations, dispersive corrections to the particle orbit are analyzed differently. In this case, the same particular + homogeneous solution decomposition is used as in the ring case with the Dispersion function satisfying:

$$D''(s) + \kappa_x(s)D(s) = \frac{1}{\rho(s)}$$

However, in this case D is solved from an initial condition. Usually (but not always) from a dispersion-free initial condition $s = s_i$ upstream of the bends with:

$$D(s_i) = 0 = D'(s_i)$$

If the bends and focusing elements can be configured such that on transport through the bend $(s = s_d)$ that

$$D(s_d) = 0 = D'(s_d)$$

Then the bend system is first order achromatic meaning there will be no final orbit deviation to 1st order in δ on traversing the system.

This equation has the form of a *Driven Hill's Equation*:

$$x'' + \kappa_x(s)x = p(s) \qquad x \to D \qquad p \to 1/\rho$$

The general solution to this equation can be solved analytically using a Green function method (see Appendix A) based on principle orbits of the homogeneous Hill's equation as:

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \ + \ x'(s_i)\mathcal{S}(s|s_i) \ + \int_{s_i}^{s} d\tilde{s}\ G(s,\tilde{s})p(\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Cosine-Like Solution

$$\mathcal{C}''(s|s_i) + \kappa_x(s)\mathcal{C}(s|s_i) = 0$$
$$\mathcal{C}(s_i|s_i) = 1$$
$$\mathcal{C}'(s_i|s_i) = 0$$

Sine-Like Solution

$$\mathcal{S}''(s|s_i) + \kappa_x(s)\mathcal{S}(s|s_i) = 0$$
$$\mathcal{S}(s_i|s_i) = 0$$
$$\mathcal{S}'(s_i|s_i) = 1$$

$$x(s_i) = \text{Initial value } x$$
$$x'(s_i) = \text{Initial value } x'$$

Green function e ectively casts driven equation in terms of homogeneous solution projections of Hill's equation.

Using this Green function solution from the dispersion-free initial condition gives

$$D(s) = \mathcal{S}(s|s_i)\int_{s_i}^{s} d\tilde{s}\ \frac{1}{\rho(\tilde{s})}\mathcal{C}(\tilde{s}|s_i) \ - \ \mathcal{C}(s|s_i)\int_{s_i}^{s} d\tilde{s}\ \frac{1}{\rho(\tilde{s})}\mathcal{S}(\tilde{s}|s_i)$$

$$\mathcal{C}(s|s_i) = \text{Cosine-like Principal Trajectory}$$
$$\mathcal{S}(s|s_i) = \text{Sine-like Principal Trajectory}$$

- Alternatively, the 3x3 transfer matrices previously derived can also be applied to advance D from a dispersion free point in the the linear lattice

The full particle orbit consistent with dispersive e ects is given by

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \ + \ x'(s_i)\mathcal{S}(s|s_i) \ + \ \delta D(s)$$
$$x'(s) = x(s_i)\mathcal{C}'(s|s_i) \ + \ x'(s_i)\mathcal{S}'(s|s_i) \ + \ \delta D'(s)$$

- Note that $D(s_i) = 0 = D'(s_i)$ in this expansion due to the dispersion free initial condition

For a 1st order achromatic system we requite for no leading-order dispersive corrections to the orbit on transiting the lattice ($s_i \to s_d$). This requires:

$$0 = \int_{s_i}^{s_d} d\tilde{s}\ \frac{1}{\rho(\tilde{s})}\mathcal{C}(\tilde{s}|s_i)$$
$$0 = \int_{s_i}^{s_d} d\tilde{s}\ \frac{1}{\rho(\tilde{s})}\mathcal{S}(\tilde{s}|s_i)$$

Various lattices consisting of regular combinations of bends and focusing optics can be made achromatic to 1st order by meeting these criteria.

- Higher-order achromats also possible under more detailed analysis. See, for examples: Rusthoi and Wadlinger, 1991 PAC, 607

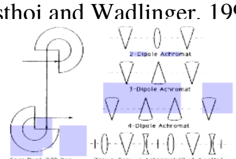


Figure 1. First-order achromats.

Examples are provided in the following slides for achromatic bends as well as bend systems to maximize/manipulate dispersive properties for species separation. Further examples can be found in the literature

Symmetries in Achromatic Lattice Design

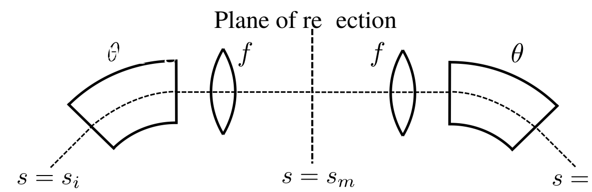
Input from C.Y. Wong, MSU

Symmetries are commonly exploited in the design of achromatic lattices to:

- Simplify the lattice design
- Reproduce (symmetrically) initial beam conditions downstream

Example lattices will be given after discussing general strategies:

Approach 1: beam line with re ection symmetry about its mid-plane



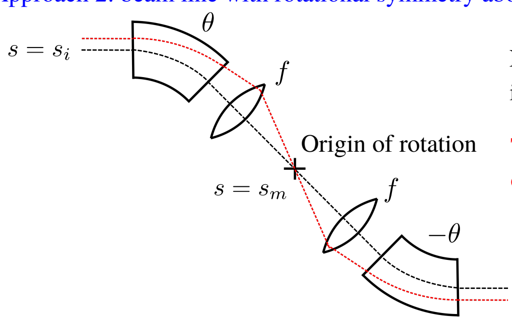
i : initial
m : middle
f : final

If $g'(s_m) = 0$, then $g(s_i) = g(s_f)$, $g'(s_i) = -g'(s_f)$

where g can be β_x, β_y or D

After the mid-plane, the beam traverses the same lattice elements in reverse order. So if the lattice function angle (d/ds) vanishes at mid-plane, the lattice function undergoes "time reversal" in the 2nd half of the beam line exiting downstream at the symmetric axial location with the same initial value and opposite initial angle.

Approach 2: beam line with rotational symmetry about the mid-point:



Note that the dipoles bend in *di erent* directions

Trajectory in red: ideal o -momentum particle

$x(s_i) = D(s_i)\delta \neq 0$

$x'(s_i) = D'(s_i)\delta = 0$

Focusing properties of dipoles are independent of bend direction (sign θ).
Same reasoning as Approach 1 gives:

If $\beta'_{x,y}(s_m) = 0$, then $\beta_{x,y}(s_i) = \beta_{x,y}(s_f)$, $\beta'_{x,y}(s_i) = -\beta'_{x,y}(s_f)$

Dispersive properties of dipoles change with bend direction. See Appendix C.

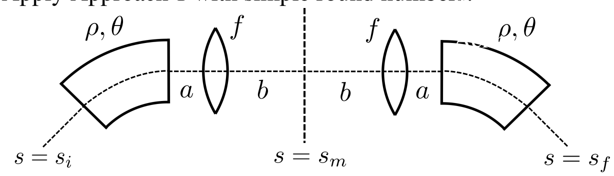
If $D(s_m) = 0$ (instead of D'), then $D(s_i) = -D(s_f)$, $D'(s_i) = D'(s_f)$

If *D* vanishes at mid-plane, the dispersive shift of an o -momentum particle also exhibits rotational symmetry about the mid-point

Example: Achromatic Bend with Thin Lens Focusing

Input from C.Y. Wong, MSU

Apply Approach 1 with simple round numbers:

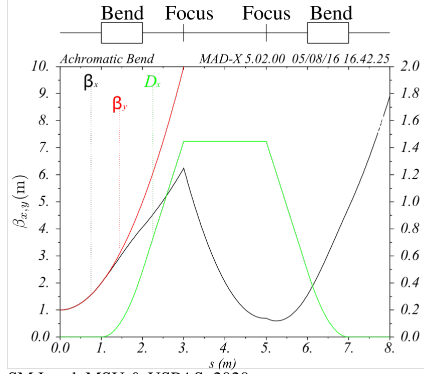


$\theta = \pi/6$

$\rho = 6/\pi$ m

$a = 1$ m

$b = 1$ m



For $D(s_i) = 0 = D'(s_i)$,

$$D'(s_m) = 0 \text{ if } f = \rho \tan\frac{\theta}{2} + a$$

(see next slide)

$$\Rightarrow f = 1.51 \text{ m}$$

The bending system is achromatic, but the betatron functions are asymmetric due to insu cient lattice parameters to tune.

- Add more elements to address

Constraint Derivation

For incident beam with $D(s_i) = 0 = D'(s_i)$, the dispersion function only evolves once the beam enters the dipole

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_m} = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_m - b} = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{M} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_i}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and (dispersion free initial condition)

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_i} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that the drift *b* after the thin lens focus does not a ect D'

$$D'(s_m) = D'(s_m - b) = 0 \text{ if } M_{23} = 0$$

Solution gives:

$$\Longrightarrow f = \rho \tan\frac{\theta}{2} + a \qquad \text{(parameter constraint for Achromat)}$$

Discussion:

- Only have to design half the beam-line by exploiting symmetries:
  - One constraint at mid-point satis es two constraints at the end of the beam line if an asymmetric design approach was taken
  - Symmetric lattice easier to set/tune: strengths in 1st half of the beam line identical to mirror pair in the 2nd half
- It is *possible* to achieve the same nal conditions with an asymmetric beam line, but this is generally not preferred
- There should be more lattice strength parameters that can be turned than constraints – needs more optics elements than this simple example
  - In simple example, dispersion function manipulated as desired but betatron functions behave poorly …. not practical
  - Except in simplest of cases, parameters often found using numerical procedures and optimization criteria

Discussion Continued:

- Usually Approach 1 and Approach 2 are applied for transfer line bends with

$$D(s_i) = 0 = D(s_f), \;\; D'(s_i) = 0 = D'(s_f)$$

  However, this is not necessary
- Common applications with $D(s_i) = 0 = D'(s_i)$ for linacs and transfer lines:
  - Approach 1: fold a linac, or create dispersion at mid-plane to collimate / select species from a multi-species beam
  - Approach 2: translate the beam
- Common applications for rings:
  - Approach 1: Minimize dispersion in straight sections to reduce aberrations in RF cavities, wigglers/undulators, injection/extraction, etc.
- Not only is it desirable to minimize the dispersion at cavities for acceleration purposes for a smaller beam, but an accelerating section has no effect on the dispersion function up to 1st order only if $D = D' = 0$. To see this:
  - Consider an off-momentum particle with $x'_D = \delta D' = 0$, $x_D = \delta D \neq 0$ undergoing a purely longitudinal acceleration
  - δ changes while x_D does not, so that D changes

Example: Simplified Fragment Separator

Input from C.Y. Wong, MSU

Heavy ion beams impinge on a production target to produce isotopes for nuclear physics research. Since many isotopes are produced, a fragment separator is needed downstream to serve two purposes:

- Eliminate unwanted isotopes
- Select and focus isotope of interest onto a transport line to detectors

Different isotopes have different rigidities, which are exploited to achieve isotope selection

Rigidity $[B\rho] = \dfrac{p}{q} = \dfrac{\gamma m v}{q}$ — ref particle (isotope) sets parameters in lattice transfer matrices

$$\delta = \left(\frac{\delta p}{p}\right)_{\rm eff} = \frac{\Delta[B\rho]}{[B\rho]_0}$$

Deviation from the reference rigidity treated as an effective momentum difference

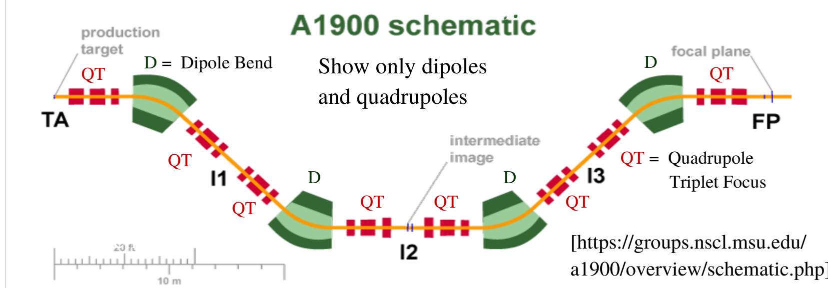
- Applied fields fixed for all species

Dispersion exploited to collimate off-rigidity fragments

Discussion:

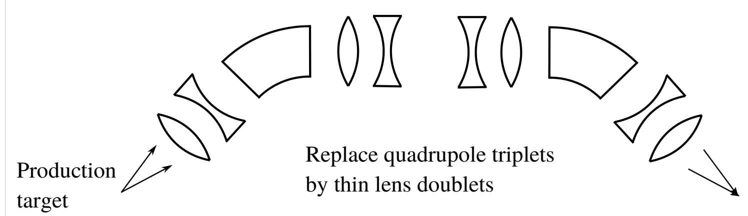
- Only have to design half the beam-line by exploiting symmetries:
  - One constraint at mid-point satisfies two constraints at the end of the beam line if an asymmetric design approach was taken
  - Symmetric lattice easier to set/tune: strengths in 1st half of the beam line identical to mirror pair in the 2nd half
- It is *possible* to achieve the same final conditions with an asymmetric beam line, but this is generally not preferred
- There should be more lattice strength parameters that can be turned than constraints – needs more optics elements than this simple example
  - Except in simplest of cases, parameters often found using numerical procedures and optimization criteria

NSCL A1900 Fragment Separator: Simplified Illustration



[https://groups.nscl.msu.edu/a1900/overview/schematic.php]

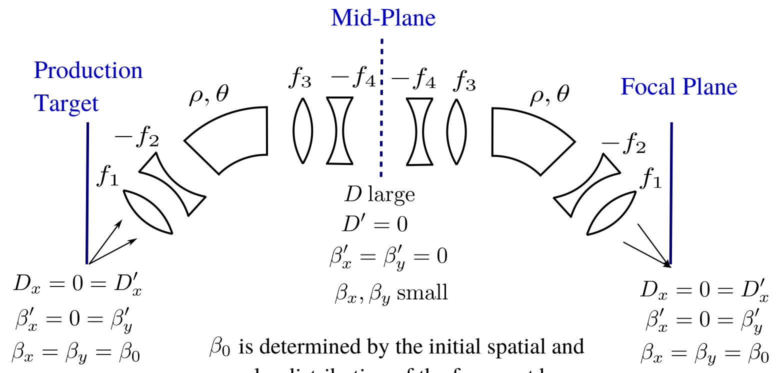
Further Simplified Example: 2 segment version



Design Goals:

- Dipoles set so desired isotope traverses center of all elements
- Dispersion function D is: *large at collimation* for rigidity resolution
  small elsewhere to minimize losses
- β_x β_y should be small at collimation point (compact separated beam) and focal plane

Apply Approach 1 by requiring $D' = \beta_x' = \beta_y' = 0$ at mid-plane



β_0 is determined by the initial spatial and angular distribution of the fragment beam

Supplementary: Parameters for Simplified Fragment Separator

f_1 $-f_2$ ρ, θ f_3 $-f_4$ $-f_4$ f_3 ρ, θ $-f_2$ f_1

0.6m 1m 1m 1.4m 2m 1m 2m 1m 2m 1.4m 1m 1m 0.6m

Desired isotope: $^{31}S^{16+}$ from ^{40}Ar(140 MeV/u) on Be target

Energy: 120 MeV/u

Rigidity: 3.15 Tesla-m

Dipole ρ, θ are fixed
$\rho = 1.78\ \mathrm{m}$ $\theta = \pi/4$

Thus $B_y(0)$ is uniquely determined by $[B\rho]$

$B_y(0) = 1.7$ Tesla

Initial conditions at production target:

$\sqrt{\langle x^2 \rangle} = 1\ \mathrm{mm}$ $\sqrt{\langle x'^2 \rangle} = 10\ \mathrm{mrad}$

$\epsilon_x \sim \sqrt{\langle x^2 \rangle \langle x'^2 \rangle} = 10$ mm-mrad

Impose constraints and solve f's numerically:

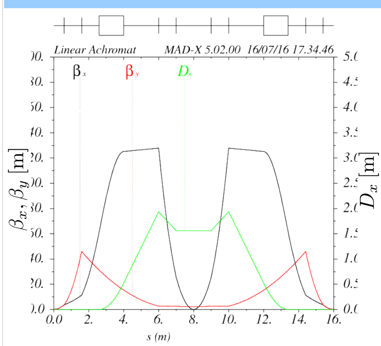
$f_1 = 1.12\ \mathrm{m}$	Quadrupole gradients $\Longleftrightarrow$ for lengths $\ell = 20$ cm	$G_1 = 13.9\ \mathrm{T/m}$
$f_2 = f_1$		$G_2 = 13.9\ \mathrm{T/m}$
$f_3 = 1.79\ \mathrm{m}$		$G_3 = 8.7\ \mathrm{T/m}$
$f_4 = 4.17\ \mathrm{m}$		$G_4 = 3.7\ \mathrm{T/m}$

For other isotopes:

If initial $\langle x^2 \rangle$, $\langle x'^2 \rangle$ are same, scale all fields to match rigidity $[B\rho]$

If not, the f's also have to be re-tuned to meet the constraints

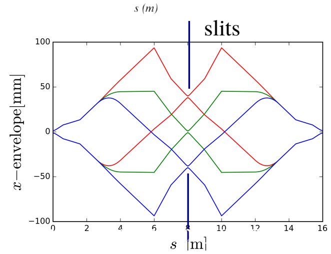
Lattice functions and beam envelope

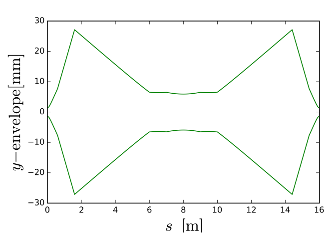


- Slits at mid-plane where dispersion large to collimate unwanted isotopes and discriminate momentum
- x-envelope plotted for 3 momentum values:

$$x_{\mathrm{env}} = \pm\sqrt{\beta_x \epsilon_x} + \delta D$$

- Aperture sizes and D (properties of lattice), determine the angular and momentum acceptance of the fragment separator





$\delta p/p = 2.5\%$

$\delta p/p = 0$

$\delta p/p = -2.5\%$

Comments:

- The real A1900 separator has more stages for improved separation
- At points of high dispersion, a tapered energy degrading wedge is used to increase effective values of δ to further enhance resolution of isotopic components of the beam.
- Sextupoles can be included to correct for chromatic effects in the focusing properties of the lattice
  - See following notes on chromatic effects and correction of chromatic effects

Example: Charge Selection System of the FRIB Front End

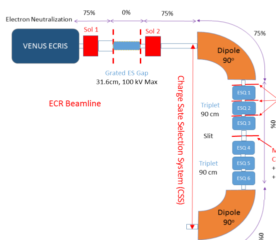
Input from C.Y. Wong, MSU

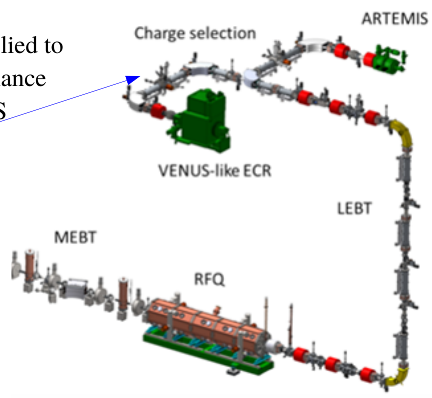
An ECR ion source produces a many-species DC beam

A charge selection system (CSS) is placed shortly downstream of each source to select the desired species for further transport and collimate the rest

- The CSS consists of two quadrupole triplets and two 90-degree sector dipoles
- The dipoles have slanted poles applied to increase *x*-focusing $(\kappa \neq 0)$ to enhance dispersion in the middle of the CSS

FRIB CSS





E ective rigidity of ions emerging from ECR ion source

ECR ion sources typically emits a DC beam with several (many) species of ions with di erent charges (*q*) and masses (*m*) giving di erent rigidities. We can model species deviations with an e ective momentum spread (δ).

- Applied elds xed for all species, so Rigidity measures strength of coupling to the applied elds for all species
- Near source, low energy heavy ions are nonrelativistic

Rigidity $[B\rho] = \frac{p}{q} = \frac{\gamma m v}{q} \simeq \frac{mv}{q}$

In our formulation setup for a *single species* beam of charge *q* and mass *m*, the o momentum parameter δ is de ned by

$$[B\rho] = \frac{p}{q} = \left(\frac{p_0}{q}\right)\left(\frac{p}{p_0}\right) = [B\rho]_0(1+\delta) \qquad 1)$$

"0" $\Rightarrow$ Design Value

$$p = p_0 + \delta p \qquad \delta \equiv \frac{\delta p}{p_0}$$

$$[B\rho]_0 \equiv \frac{p_0}{q} = \text{Design Rigidity}$$

For a ions (species index *j*) of charge and mass $q_j,\ m_j$ accelerated through a common electrostatic source potential V, we have

Energy Conservation: $q_j V = \frac{1}{2} m_j v_j^2$

$$\Longrightarrow \quad [B\rho] = \frac{m_j v_j}{q_j} = \sqrt{2V(m_j/q_j)}$$

Take for the various species:

$$m_j = m_0 + \Delta m \qquad m_0,\ q_0 \ \Rightarrow \text{Design Species}$$
$$q_j = q_0 + \Delta q$$

Giving:

$$[B\rho] = \frac{m_j v_j}{q_j} = \sqrt{2V(m_j/q_j)} = \sqrt{2V(m_0/q_0)}\left(\frac{1+\Delta m/m_0}{1+\Delta q/q_0}\right)^{1/2}$$

$$[B\rho] = [B\rho]_0\left(\frac{1+\Delta m/m_0}{1+\Delta q/q_0}\right)^{1/2} \qquad 2)$$

$$[B\rho]_0 = \sqrt{2V(m_0/q_0)} = \text{Design Rigidity}$$

De ne an e ective o -momentum by the spread in Rigidity from design

$$\delta = \left(\frac{\delta p}{p}\right)_{\text{eff}} = \frac{\Delta[B\rho]}{[B\rho]_0}$$

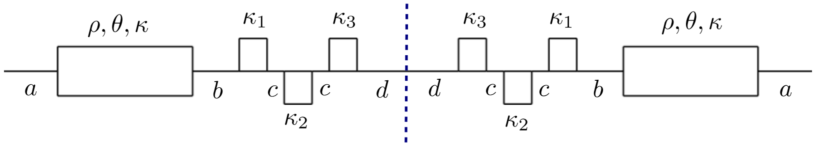
Equating Rigidity expressions for 1) (Single Species) and 2) (multi-Species) identi es the "e ective" momentum spread δ

$$[B\rho]_0(1+\delta) = [B\rho]_0\left(\frac{1+\Delta m/m_0}{1+\Delta q/q_0}\right)^{1/2}$$

$$1 + \delta = \left(\frac{1+\Delta m/m_0}{1+\Delta q/q_0}\right)^{1/2}$$

- Common theme of physics: map new case (multi species) to simpler, familiar case (single species with momentum spread)
- For ECR ion source may have operating cases with $\Delta m = 0$

Parameters for the CSS



Dipole:

$\theta = \pi/2 \quad \rho = 2/\pi$ m

$\kappa_x = 0.1/\rho^2 \quad \kappa_y = 0.9/\rho^2$

where field index $n = 0.9$ from : $x'' + \kappa_x x = x'' + \frac{1-n}{\rho^2} x = 0$

$$y'' + \kappa_y y = y'' + \frac{n}{\rho^2} y = 0$$

Mid-plane conditions:

$\alpha_x(s_m) = \alpha_y(s_m) = D' = 0$

Quadrupoles:

$l_{\text{quad}} = 0.2$ m

$\kappa_{1x} = -\kappa_{1y} = 8.30 \text{ m}^{-2}$

$\kappa_{2x} = -\kappa_{2y} = -15.60 \text{ m}^{-2}$

$\kappa_{3x} = -\kappa_{3y} = 7.51 \text{ m}^{-2}$

Drifts:

$a = 0.4$ m

$b = 0.35$ m

$c = 0.13$ m

$d = 0.19$ m

Initial Conditions:

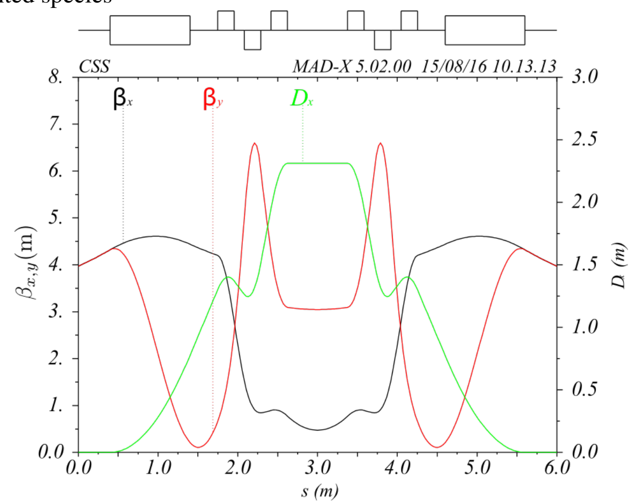
$\beta_x(s_i) = \beta_y(s_i) = 3.971$ m

$\alpha_x(s_i) = \alpha_y(s_i) = -0.380$

$D(s_i) = D'(s_i) = 0$

Lattice Functions of CSS

Large dispersion and small beam size in x at mid-plane facilitates the collimation of unwanted species



S9C: Chromatic E ects

Present in both x- and y-equations of motion and result from applied focusing strength changing with deviations in momentum:

$$x''(s) + \frac{\kappa_x(s)}{(1+\delta)^n} x(s) = 0$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n} y(s) = 0$$

$\rho \to \infty$ to neglect bending terms

$\kappa_{x,y} =$ Focusing Functions with γ_b, β_b calculated from p_0

$$n = \begin{cases} 1, & \text{Magnetic Quadrupoles} \\ 2, & \text{Solenoids, Electric Quadrupoles} \end{cases}$$

- Generally of lesser importance (smaller corrections) relative to dispersive terms (S9C) *except* possibly:
  - In rings where precise control of tunes (betatron oscillations per ring lap) are needed to avoid resonances
  - In nal focus where small focal spots and/or large axial momentum spread (in cases with longitudinal pulse compression) can occur

Can analyze by rede ning kappa function to incorporate o -momentum:

$$\frac{\kappa_x(s)}{(1+\delta)^n} \to \kappa_{x,\text{new}}(s)$$

However, this would require calculating new amplitude/betatron functions for each particle o -momentum value δ in the distribution to describe the evolution of the orbits. That would not be e cient.

Rather, need a *perturbative formula* to calculate the small amplitude correction to the nominal particle orbit with design momentum due to the o -momentum δ.

Either the x- and y-equations of motion can be put in the form:

$$x''(s) + \frac{\kappa(s)}{(1+\delta)^n} x(s) = 0$$

Expand to leading order in δ :

$$x''(s) + \kappa(s)(1 - n\delta)x(s) = 0$$

Set:

$$x(s) = x_0(s) + \eta(s) \qquad \begin{aligned} x_0(s) &= \text{Orbit Solution for } \delta = 0 \\ \eta(s) &= \text{Orbit Correction to } x_0 \text{ for } \delta \neq 0 \end{aligned}$$

Giving:

$$x_0'' + \kappa x_0 = 0 \qquad 1)$$

$$(x_0 + \eta)'' + \kappa(1 - n\delta)(x_0 + \eta) = 0 \qquad 2)$$

Insert Eq. 1) in 2) and neglect the 2nd order term in $\delta \cdot \eta$ to obtain a linear equation for η :

$$\eta'' + \kappa\eta = n\delta\kappa x_0$$

This equation has the form of a *Driven Hill's Equation*:

$$x'' + \kappa(s)x = p(s) \qquad \begin{aligned} x &\rightarrow \eta \\ p &\rightarrow n\delta\kappa x_0 \end{aligned}$$

The general solution to this equation can be solved analytically using a Green function method (see Appendix A) as:

- Same method used in analysis of dispersion function

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \;+\; x'(s_i)\mathcal{S}(s|s_i) \;+\; \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$

$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

<u>Cosine-Like Solution</u>

$$\mathcal{C}''(s|s_i) + \kappa(s)\mathcal{C}(s|s_i) = 0$$
$$\mathcal{C}(s_i|s_i) = 1$$
$$\mathcal{C}'(s_i|s_i) = 0$$

<u>Sine-Like Solution</u>

$$\mathcal{S}''(s|s_i) + \kappa(s)\mathcal{S}(s|s_i) = 0$$
$$\mathcal{S}(s_i|s_i) = 0$$
$$\mathcal{S}'(s_i|s_i) = 1$$

$$x(s_i) = \text{Initial value } x$$
$$x'(s_i) = \text{Initial value } x'$$

Using this result, the general solution for the chromatic correction to the particle orbit can be expressed as:

$$\eta(s) = \eta(s_i)\mathcal{C}(s|s_i) + \eta'(s_i)\mathcal{S}(s|s_i) + n\delta \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})\kappa(\tilde{s})x_0(\tilde{s})$$

$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$
$$\eta(s_i) = \text{Initial value } \eta$$
$$\eta'(s_i) = \text{Initial value } \eta'$$

Chromatic orbit perturbations are typically measured from a point in the lattice where they are initially zero like a drift where the orbit was correct before focusing quadrupoles. In this context, can take:

$$\eta(s_i) = 0 = \eta'(s_i)$$

$$\eta(s) = n\delta \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})\kappa(\tilde{s})x_0(\tilde{s})$$

The Green function can be simpli ed using results from S6F:

$$C(s|s_i) = \frac{w(s)}{w_i}\cos\Delta\psi(s) - w_i' w(s)\sin\Delta\psi(s) \qquad \Delta\psi(s) \equiv \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

$$S(s|s_i) = w_i w(s)\sin\Delta\psi(s) \qquad \begin{aligned} w_i &\equiv w(s = s_i) \\ w_i' &\equiv w'(s = s_i) \end{aligned}$$

Giving after some algebra:

$$\begin{aligned} G(s,\tilde{s}) &= \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i) \\ &= w(s)w(\tilde{s})[\sin\Delta\psi(s)\cos\Delta\psi(\tilde{s}) - \cos\Delta\psi(s)\sin\Delta\psi(\tilde{s})] \\ &= w(s)w(\tilde{s})\sin[\Delta\psi(s) - \Delta\psi(\tilde{s})] \end{aligned}$$

Using this and the phase amplitude form of the orbit:

$$x_0(s) = A_i w(s)\cos[\psi(s)] \qquad \beta(s) = w^2(s)$$
$$= \sqrt{\epsilon} w(s)\cos[\Delta\psi(s) + \psi_i] \qquad = \sqrt{\epsilon\beta(s)}\cos[\Delta\psi(s) + \psi_i]$$

- Initial phase ψ_i implicitly chosen (can always do) for initial amplitude $A_i \geq 0$

the orbit deviation from chromatic e ects can be calculated as:

$$\eta(s) = n\delta \int_{s_i}^{s} d\tilde{s}\, G(s,\tilde{s})\kappa(\tilde{s})x_0(\tilde{s}) \qquad \Delta\psi(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} \quad = \int_{s_i}^{s} \frac{d\tilde{s}}{\beta(\tilde{s})}$$

w-form

$$= n\delta\sqrt{\epsilon} w(s) \int_{s_i}^{s} d\tilde{s}\, \kappa(\tilde{s})w^2(\tilde{s}) \sin[\Delta\psi(s) - \Delta\psi(\tilde{s})]\cos[\Delta\psi(\tilde{s}) + \psi_i]$$

Betatron-form

$$= n\delta\sqrt{\epsilon\beta(s)} \int_{s_i}^{s} d\tilde{s}\, \kappa(\tilde{s})\beta(\tilde{s}) \sin[\Delta\psi(s) - \Delta\psi(\tilde{s})]\cos[\Delta\psi(\tilde{s}) + \psi_i]$$

Formula applicable to all types of focusing lattices:

- Quadrupole: electric and magnetic
- Solenoid (Larmor frame)
- Linac and rings

Comments:

- Perturbative formulas can be derived to calculate the e ect on betatron tunes (particle oscillations per lap) in a ring based on integrals of the unpreturbed betatron function: see Wiedemann, *Particle Accelerator Physics*
- For magnetic quadrupole lattices further detailed analysis (see Ste en, *High Energy Beam Optics*) it can be shown that:
  - Impossible to make an achromatic focus in any quadrupole system. Here achromatic means if

    $$\eta(s_i) = 0 = \eta'(s_i)$$

    that there is an achromatic point $s = s_f$ post optics with

    $$\eta(s_f) = 0 = \eta'(s_f)$$
- More detailed analysis of the chromatic correction to particle orbits in rings show that a properly oriented nonlinear sextupole inserted into the periodic ring lattice with correct azimuthal orientation at a large dispersion points can to leading order compensate for chromatic corrections. We will cover this in the slides that follow.
  - Correction introduces nonlinear terms for large amplitude
  - Correction often distributed around ring for practical reasons

Chromaticity

When a particle has higher/lower momentum (δ), we expect focusing strength to go down/up

- Important for rings since a relatively small shift in tune can drive the particle into a nearby low-order resonance condition resulting in particle losses

Denote:

$$\nu_x = x\text{-Tune including off-momentum } \delta$$
$$= \text{Number } x\text{-Betatron Oscillations in Ring} = \frac{\Delta\psi_x|_{\text{ring}}}{2\pi}$$
$$= \frac{1}{2\pi}\oint \frac{ds}{\beta_x(s)} \qquad \beta_x = \text{Betatron Function } \textit{including } \delta$$
$$\nu_{0x} = \text{Design } x\text{-Tune } (\delta = 0)$$
$$= \frac{1}{2\pi}\oint \frac{ds}{\beta_{0x}(s)} \qquad \beta_{0x} = \text{Design Betatron Function } (\delta = 0)$$

$$\Delta\nu_x = \nu_x - \nu_{0x} = x\text{-Tune Shift}$$

De ne the chromaticity as the change in tune per change in momentum (δ) to measure the chromatic change in focusing strength of the lattice:

- Analogous treatment in *y*-plane

$$\xi_x \equiv \frac{\Delta\nu_x}{\delta p/p_0} = \frac{\Delta\nu_x}{\delta} \qquad x\text{-Chromaticity}$$
$$\xi_y \equiv \frac{\Delta\nu_y}{\delta p/p_0} = \frac{\Delta\nu_y}{\delta} \qquad y\text{-Chromaticity}$$

- Expect $\xi_{x,y} < 0$ for any linear focusing lattice

Go back to the leading order orbit equation of motion describing chromatic e ects:

$$x''(s) + \kappa_x(s)(1 - n\delta)x(s) = 0$$

This is the form of:

$$x'' + \kappa_x x = p_1 x \qquad \text{with} \qquad p_1 = n\delta\kappa_x$$

Which suggests use of a Floquet transformation as in resonance theory:

$$u \equiv \frac{x}{\sqrt{\beta_{0x}}} \qquad \text{Radial coordinate}$$

$$\varphi \equiv \frac{1}{\nu_{0x}} \int_0^s \frac{d\tilde{s}}{\beta_{0x}(\tilde{s})} \qquad \text{Angle advances by } 2\pi \text{ over ring}$$

$$\nu_{0x} = \text{Unperturbed } x\text{-betatron oscillations in ring}$$

Then the same steps in the analysis as employed in our study or resonance e ects shows that

$$x'' + \kappa_x x = p_1 x$$

becomes:

$$\ddot{u} + \nu_{0x}^2 u = \nu_{0x}^2 \beta_{0x}^2 p_1 u$$

$$\dot{} = \frac{d}{d\varphi}$$

On the RHS of this perturbation equation, the coe cient of u is periodic with period 2π in φ , so we can complex Fourier expand it as

- Analogous to steps used to analyzed perturbations in resonances

$$\nu_{0x}^2 \beta_{0x}^2 p_1 = \sum_{k=-\infty}^{\infty} C_k e^{ik\varphi} \qquad i \equiv \sqrt{-1} \qquad C_k = \text{Complex Constants}$$

$$C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \; \nu_{0x}^2 \beta_{0x}^2 p_1 e^{-ik\varphi} = \frac{1}{2\pi} \oint_{\text{ring}} ds \; \nu_{0x} \beta_{0x} p_1 e^{-ik\varphi}$$

$$\varphi = \frac{1}{\nu_{0x}} \int_0^s \frac{d\tilde{s}}{\beta_{0x}(\tilde{s})} \implies d\varphi = \frac{ds}{\nu_{0x}\beta_x}$$

Insert the expansion:

$$C_k = \frac{1}{2\pi} \oint_{\text{ring}} ds \; \nu_{0x} \beta_{0x} p_1 e^{-ik\varphi}$$

$$\ddot{u} + \nu_{0x}^2 u = \nu_{0x}^2 \beta_{0x}^2 p_1 u = \left[\sum_{k=-\infty}^{\infty} C_k e^{ik\varphi} \right] u$$

Isolate the constant k= 0 value in sum and move to LHS, then all terms on RHS have variation in φ :

Tune-Shift Perturbation

$$\ddot{u} + [\nu_{0x}^2 - C_0] u = \left[\sum_{k=-\infty, k\neq 0}^{\infty} C_k e^{ik\varphi} \right] u$$

$$C_0 = \frac{\nu_{0x}}{2\pi} \oint_{\text{ring}} ds \; \beta_{0x} p_1$$

The homogeneous part of this equation has the form:

$$\ddot{u}_h + \nu_x^2 u_h = 0 \qquad \nu_x = x\text{-Tune (shifted)}$$

with:

$$\nu_x^2 \equiv \nu_{0x}^2 - C_0 = \nu_{0x}^2 - \frac{\nu_{0x}}{2\pi} \oint_{\text{ring}} ds \; \beta_{0x} p_1$$

- ν_x measures the x-tune shift due to o momentum δ contained in p_1

The **tune-shift** $\Delta\nu_x$ **due to o -momentum** δ can now be evaluated:

$$\nu_x = \nu_{0x} + \Delta\nu_x$$

with

Previous Analysis

$$\nu_x^2 = (\nu_{0x} + \Delta\nu_x)^2 = \nu_{0x}^2 - C_0 = \nu_{0x}^2 - \frac{\nu_{0x}}{2\pi} \oint_{\text{ring}} ds \; \beta_{0x} p_1$$

$$\simeq \nu_{0x}^2 + 2\nu_{0x}\Delta\nu_x \qquad \text{(Leading Order)}$$

$$\implies \qquad 2\nu_{0x}\Delta\nu_x = -\frac{\nu_{0x}}{2\pi} \oint_{\text{ring}} ds \; \beta_{0x} p_1$$

Identi es:

$$\Delta\nu_x = -\frac{1}{4\pi} \oint_{\text{ring}} ds \; \beta_{0x} p_1 = -\frac{n\delta}{4\pi} \oint_{\text{ring}} ds \; \beta_{0x} \kappa_x \qquad p_1 = n\delta\kappa_x$$

Giving the chromaticity as:

$$\xi_x \equiv \frac{\Delta\nu_x}{\delta p/p_0} = \frac{\Delta\nu_x}{\delta} = -\frac{n}{4\pi} \oint_{\text{ring}} ds \; \beta_{0x} \kappa_x$$

Summary of results with an analogous y-plane derivation:

Tunes

$$\Delta\nu_x = -\frac{1}{4\pi}\oint_{\text{ring}} ds\ \beta_{0x} p_{1x} = -\frac{n\delta}{4\pi}\oint_{\text{ring}} ds\ \beta_{0x}\kappa_x$$

$$\Delta\nu_y = -\frac{1}{4\pi}\oint_{\text{ring}} ds\ \beta_{0y} p_{1y} = -\frac{n\delta}{4\pi}\oint_{\text{ring}} ds\ \beta_{0y}\kappa_y$$

Chromaticities

$$\xi_x = \frac{\Delta\nu_x}{\delta} = -\frac{1}{4\pi}\oint_{\text{ring}} ds\ \beta_{0x}\frac{p_{1x}}{\delta} = -\frac{n}{4\pi}\oint_{\text{ring}} ds\ \beta_{0x}\kappa_x$$

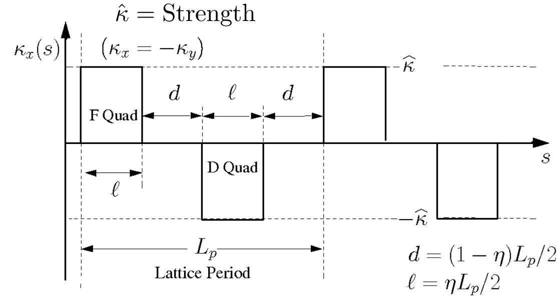
$$\xi_y = \frac{\Delta\nu_y}{\delta} = -\frac{1}{4\pi}\oint_{\text{ring}} ds\ \frac{\beta_{0y}p_{1y}}{\delta} = -\frac{n}{4\pi}\oint_{\text{ring}} ds\ \beta_{0y}\kappa_y$$

- Formulas, as expressed, apply to rings, but can be adapted for linacs
- Chromaticities $\xi_{x,y}$ are *always negative* in any linear focusing lattice
  - Example: see FODO lattice function in following slides
- The same formulas can be derived from an analysis of thin lens transfer matrix corrections used to model o -momentum …. see problems

Reminder: Periodic Quadrupole FODO Lattice

Parameters:
L_p = Lattice Period
$\eta \in (0, 1]$ = Occupancy
$\hat{\kappa}$ = Strength

Characteristics:
$\eta L_p/2 = \ell$ = F/D Len
$(1-\eta)L_p/2 = d$ = Drift Len

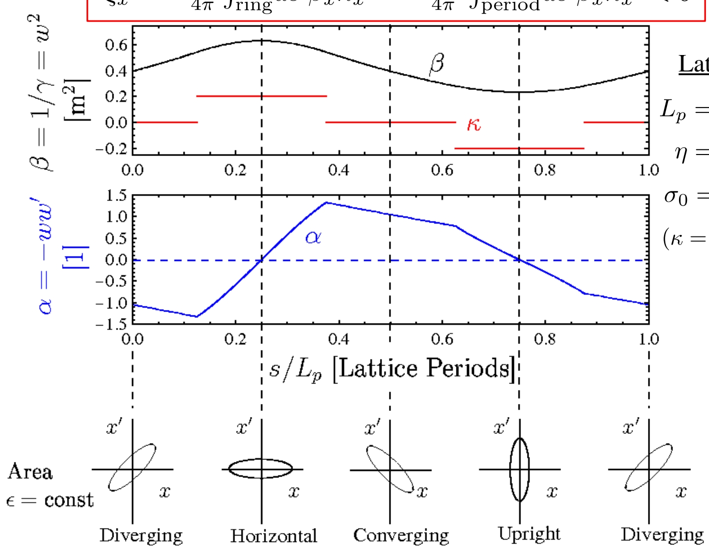


Formula connecting phase advance to eld strength via $\hat{\kappa}$:

$$\cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) - \frac{(1-\eta)^2}{2\eta^2}\Theta^2\sin\Theta\sinh\Theta \qquad \Theta \equiv \frac{\eta}{2}\sqrt{|\hat{\kappa}|}L_p$$

Phase-Space Evolution (see also: S7):

$$\xi_x = -\frac{n}{4\pi}\oint_{\text{ring}} ds\ \beta_x\kappa_x = -\frac{n\mathcal{N}}{4\pi}\int_{\text{period}} ds\ \beta_x\kappa_x \ < 0$$



Lattice
$L_p = 0.5$ m
$\eta = 0.5$
$\sigma_0 = \pi/3 = 60°$
$(\kappa = 39.24\ \text{m}^{-2})$

Chromaticity correction in a magnetic quadrupole focusing lattice with Sextupoles

To leading order, we will nd that nonlinear focusing Sextupole optics can introduce the correct form of perturbation to compensate for chromatic aberrations in a quadrupole focusing lattice

- Important to do with limited amplitude since a large sextupole can also drive nonlinear resonances

Particle equations of motion in this context for a transverse magnetic eld are:

$$x'' = -\frac{q}{m\gamma_b\beta_b c}B_y^a = -\frac{B_y^a}{[B\rho]}$$

$$y'' = \frac{q}{m\gamma_b\beta_b c}B_x^a = \frac{B_x^a}{[B\rho]} \qquad [B\rho] = [B\rho]_0(1+\delta)$$

Expand to leading order in δ :

$$x'' \simeq -\frac{B_y^a}{[B\rho]_0}(1-\delta)$$

$$y'' \simeq \frac{B_x^a}{[B\rho]_0}(1-\delta)$$

Review: Symmetries of applied eld components

Within a 2D transverse model it was shown that transverse applied magnetic eld components entering the equations of motion can be expanded as:

- See: S3, Transverse Particle Dynamics: 2D components axial integral 3D components
- Applied electric elds can be analogously expanded

$$\underline{B}^*(\underline{z}) = B^a_x(x,y) - iB^a_y(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}$$

$$\underline{b}_n = \text{const (complex)} \equiv \mathcal{A}_n - i\mathcal{B}_n \qquad \underline{z} = x + iy \qquad i = \sqrt{-1}$$

$$n = \text{Multipole Index} \qquad r_p = \text{Aperture "Pipe" Radius}$$

$$\mathcal{B}_n \Longrightarrow \text{"Normal" Multipoles}$$

$$\mathcal{A}_n \Longrightarrow \text{"Skew" Multipoles}$$

Cartesian projections: $\overline{B_x} - i\overline{B_y} = (\mathcal{A}_n - i\mathcal{B}_n)(x+iy)^{n-1}/r_p^{n-1}$

Index n	Name	Normal ($\mathcal{A}_n = 0$) $B_x r_p^{n-1}/\mathcal{B}_n$	Normal ($\mathcal{A}_n = 0$) $B_y r_p^{n-1}/\mathcal{B}_n$	Skew ($\mathcal{B}_n = 0$) $B_x r_p^{n-1}/\mathcal{A}_n$	Skew ($\mathcal{B}_n = 0$) $B_y r_p^{n-1}/\mathcal{A}_n$
1	Dipole	0	1	1	
2	Quadrupole	y	x	x	$-y$
3	Sextupole	$2xy$	x^2-y^2	x^2-y^2	$-2xy$
4	Octupole	$3x^2y-y^3$	x^3-3xy^2	x^3-3xy^2	$-3x^2y+y^3$
5	Decapole	$4x^3y-4xy^3$	$x^4-6x^2y^2+y^4$	$x^4-6x^2y^2+y^4$	$-4x^3y+4xy^3$

Applied Quadrupole Field Component: linear focusing, normal orientation

$$B^a_x = Gy = \kappa[B\rho]_0 y \qquad \kappa(s) = \frac{G(s)}{[B\rho]_0} \qquad G = \text{Mag. Field Gradient}$$
$$B^a_y = Gx = \kappa[B\rho]_0 x$$

Applied Sextupole Field Component: nonlinear focusing, normal orientation

$$B^a_x = 2\mathcal{S}xy \qquad \mathcal{S}(s) = \frac{B_p}{r_p^2} = \text{Sextupole Field Amplitude}$$
$$B^a_y = \mathcal{S}(x^2 - y^2)$$

Superimpose quadrupole and sextupole eld components (outside dipole bend):

$$B^a_x = \kappa[B\rho]_0 y + 2\mathcal{S}xy$$
$$B^a_y = \kappa[B\rho]_0 x + \mathcal{S}(x^2 - y^2)$$

Insert in equations of motion:

$$x'' \simeq -\frac{B^a_y}{[B\rho]_0}(1-\delta) = -\kappa(1-\delta)x - \frac{\mathcal{S}}{[B\rho]_0}(1-\delta)(x^2-y^2)$$
$$y'' \simeq \frac{B^a_x}{[B\rho]_0}(1-\delta) = \kappa(1-\delta)y + 2\frac{\mathcal{S}}{[B\rho]_0}(1-\delta)xy$$

Taking the sextupole amplitude small so that $\mathcal{S}(1-\delta) \simeq \mathcal{S}$ and rearranging

$$x'' + \kappa x \simeq \delta\kappa x - \frac{\mathcal{S}}{[B\rho]_0}(x^2 - y^2)$$
$$y'' - \kappa y \simeq -\delta\kappa y + 2\frac{\mathcal{S}}{[B\rho]_0}xy$$

Former (Quadrupole) New (Sextupole)

Set, and consider only x-plane dispersion, and resolve the particle orbit as:

$$x(s) = x_\beta(s) + \delta \cdot D(s) \qquad x_\beta(s),\ y_\beta(s) = \text{Linear betatron motion}$$
$$y(s) = y_\beta(s) \qquad D(s) = \text{Dispersion Function (Periodic Ring Lattice)}$$

- Here we bring the periodic dispersion component back for the ring lattice though we are analyzing the evolution outside a bend

Insert these into the equations of motion, and neglect nonlinear amplitude terms considering the orbit amplitudes $x_\beta,\ y_\beta$ small and the momentum spread δ to be small.

$$x(s) = x_\beta(s) + \delta \cdot D(s) \qquad x'' + \kappa x \simeq \delta \cdot \kappa x - \frac{\mathcal{S}}{[B\rho]_0}(x^2 - y^2)$$
$$y(s) = y_\beta(s) \qquad y'' - \kappa y \simeq -\delta \cdot \kappa y + 2\frac{\mathcal{S}}{[B\rho]_0}xy$$

Using:

$$D'' + \kappa D = 0$$

- Equations are applied outside of bend where $\rho \rightarrow \infty$

approximating

$$\delta \cdot \kappa x = \delta \cdot \kappa x_\beta + \delta^2 \cdot \kappa D \simeq \delta \cdot \kappa x_\beta$$
$$\delta \cdot \kappa y = \delta \cdot \kappa y_\beta$$

And isolating the *linear* betatron amplitude component of the sextupole terms

$$\frac{\mathcal{S}}{[B\rho]_0}(x^2 - y^2) = \frac{\mathcal{S}}{[B\rho]_0}(x_\beta^2 + 2\delta D x_\beta + \delta^2 D^2 - y_\beta^2) \simeq 2\frac{\mathcal{S}}{[B\rho]_0}\delta D x_\beta$$
$$2\frac{\mathcal{S}}{[B\rho]_0}xy = 2\frac{\mathcal{S}}{[B\rho]_0}(x_\beta y_\beta + \delta D y_\beta) \simeq 2\frac{\mathcal{S}}{[B\rho]_0}\delta D y_\beta$$

- Requires small particle oscillation amplitudes $x_\beta,\ y_\beta$ and small δ
- Validity will typically need to be veri ed numerically
  - Becomes questionable for larger $x_\beta,\ y_\beta$ and δ

Gives:

$$x''_\beta + \kappa x_\beta \simeq \delta \left(\kappa - 2\frac{\mathcal{S}}{[B\rho]_0} D \right) x_\beta$$
$$y''_\beta - \kappa y_\beta \simeq -\delta \left(\kappa - 2\frac{\mathcal{S}}{[B\rho]_0} D \right) y_\beta$$

In the previous section we showed that the betatron tune shift in the equations

$$x'' + \kappa_x x = p_{1x} x \qquad y'' + \kappa_y y = p_{1y} y$$

is given by

$$\Delta\nu_x = -\frac{1}{4\pi} \oint_{\text{ring}} ds\ \beta_{0x} p_{1x}$$
$$\Delta\nu_y = -\frac{1}{4\pi} \oint_{\text{ring}} ds\ \beta_{0y} p_{1y}$$

with chromaticities

$$\xi_x = \frac{\Delta\nu_x}{\delta} = -\frac{1}{4\pi} \oint_{\text{ring}} ds\ \beta_{0x} \frac{p_{1x}}{\delta}$$
$$\xi_y = \frac{\Delta\nu_y}{\delta} = -\frac{1}{4\pi} \oint_{\text{ring}} ds\ \frac{\beta_{0y} p_{1y}}{\delta}$$

Identify:

$$\frac{p_{1x,1y}}{\delta} = \kappa - 2\frac{\mathcal{S}}{[B\rho]_0} D$$

This gives for the chormaticities **including the sextupole applied eld** to leading order:

$$\xi_x = -\frac{1}{4\pi} \oint_{\text{ring}} ds\ \beta_{0x} \left(\kappa - 2\frac{\mathcal{S}}{[B\rho]_0} D \right)$$
$$\xi_y = \frac{1}{4\pi} \oint_{\text{ring}} ds\ \beta_{0y} \left(\kappa - 2\frac{\mathcal{S}}{[B\rho]_0} D \right)$$

- 1st term previous result, also called the *natural chromaticity* due to linear focus
- 2nd term leading-order shifted chromaticity due to sextupole optic $\mathcal{S} \neq 0$

Result shows that if you place a normal orientation sextupole optic at a point of nonzero dispersion $(D \neq 0)$, then you can adjust the amplitude $\mathcal{S}$ to null the chromatic shift in focusing strength to leading order.

- Correction *independent of* δ to leading order
- Want to place also where both betatron amplitudes $\beta_{x,y}$ and Dispersion D are large to limit setupole amplitudes $\mathcal{S}$
- Need min of 2 sextupoles to correct both *x*- and *y*-chromaticities
- Typically want more sextupoles in ring for exibility and to keep amplitudes limited to maintain validity of ordering assumptions made
  - Sextupoles also drive nonlinear resonances so large amplitudes problematic

- Will generally also have $\beta_x \gg \beta_y$ at one setupole and $\beta_y \gg \beta_x$ at the other sextupole (min 2 for correction in each plane simultaneously)
  - Design lattice to take advantage so correction amplitudes do not " ght"
- Formulation applicable to bends in linacs also
  - Can apply to Fragment Separators, LINAC folding sections (FRIB), ….

Problem assigned to illustrate chromatic corrections more

Sextupole correction of chromaticities one example of numerous creative optical corrections exploiting properties of nonlinear focusing magnets:

- Creativity and may years of thinking / experience
- Speci c to application and needs
- Electron microscope optics provides examples of nonlinear optics used to correct higher order aberrations

Appendix A: Green Function for Driven Hill's Equation

Following Wiedemann (Particle Accelerator Physics, 1993, pp 106) rst, consider more general *Driven Hill's Equation*

$$x'' + \kappa(s)x = p(s)$$

The corresponding homogeneous equation:

$$x'' + \kappa(s)x = 0$$

has principal solutions

$$x(s) = C_1\mathcal{C}(s|s_i) + C_2\mathcal{S}(s|s_i) \qquad C_1, C_2 = \text{constants}$$

where

Cosine-Like Solution	Sine-Like Solution
$\mathcal{C}'' + \kappa(s)\mathcal{C} = 0$	$\mathcal{S}'' + \kappa(s)\mathcal{S} = 0$
$\mathcal{C}(s = s_i) = 1$	$\mathcal{S}(s = s_i) = 0$
$\mathcal{C}'(s = s_i) = 0$	$\mathcal{S}'(s = s_i) = 1$

Recall that the homogeneous solutions have the Wronskian symmetry:

- See S5C

$$W(s) = \mathcal{C}(s)\mathcal{S}'(s) - \mathcal{C}'(s)\mathcal{S}(s) = 1 \qquad \mathcal{C}(s) \equiv \mathcal{C}(s|s_i) \quad \text{etc.}$$

A particular solution to the *Driven Hill's Equation* can be constructed using a Greens' function method:

$$x(s) = \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Demonstrate this works by rst taking derivatives: $\mathcal{C}(s) \equiv \mathcal{C}(s|s_i)$, etc.

$$x = \mathcal{S}(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{S}(\tilde{s})p(\tilde{s})$$

$$x' = \mathcal{S}'(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}'(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{S}(\tilde{s})p(\tilde{s})$$
$$+\,p(s)\,[\mathcal{S}(s)\mathcal{C}(s) - \mathcal{S}(s)\mathcal{C}(s)] \quad (\to 0)$$
$$= \mathcal{S}'(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}'(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{S}(\tilde{s})p(\tilde{s})$$

$$x'' = \mathcal{S}''(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}''(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{S}(\tilde{s})p(\tilde{s})$$
$$+\,p(s)\,[\mathcal{S}'(s)\mathcal{C}(s) - \mathcal{C}'(s)\mathcal{S}(s)] \quad (\to 1)$$
$$= p(s) \;+\; \mathcal{S}''(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}''(s)\int_{s_i}^{s} d\tilde{s}\;\mathcal{S}(\tilde{s})p(\tilde{s})$$

<= From Wronskian

Insert these results for $x,\; x''$ in the *Driven Hill's Equation*:

From De nition of Principal Orbit Functions

$$x'' + \kappa(s)x = p(s) \;+\; [\mathcal{S}'' + \kappa\mathcal{S}]\int_{s_i}^{s} d\tilde{s}\;\mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; [\mathcal{C}'' + \kappa\mathcal{C}]\int_{s_i}^{s} d\tilde{s}\;\mathcal{S}(\tilde{s})p(\tilde{s})$$
$$= p(s)$$

(each bracket $\to 0$)

Thereby proving we have a valid particular solution. The general solution to the *Driven Hill's Equation* is then:

$$x(s) = C_1\mathcal{C}(s|s_i) \;+\; C_2\mathcal{S}(s|s_i) \;+\; \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$

$$\begin{aligned} x(s=s_i) &= x(s_i) \\ x'(s=s_i) &= x'(s_i) \end{aligned} \quad\Longrightarrow\quad \begin{aligned} C_1 &= x(s_i) \\ C_2 &= x'(s_i) \end{aligned}$$

- Choose constants $C_1,\; C_2$ consistent with particle initial conditions at $s = s_i$

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \;+\; x'(s_i)\mathcal{S}(s|s_i) \;+\; \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Appendix B: Uniqueness of the Dispersion Function in a Periodic (Ring) Lattice

Consider the equation for the dispersion function in a periodic lattice

$$D'' + \kappa_x D = \frac{1}{\rho} \qquad \begin{aligned} \kappa_x(s+L_p) &= \kappa_x(s) \\ R(s+L_p) &= R(s) \end{aligned}$$

It is required that the solution for a periodic (ring) lattice has the periodicity of the lattice:

$$\mathrm{D}(\mathrm{s}+\mathrm{L}_p) = D(s)$$

Assume that there are two unique solutions to *D* and label them as D_j. Each must satisfy:

$$D_j'' + \kappa_x D_j = \frac{1}{\rho} \qquad \mathrm{D}_j(s+L_p) = D_j(s) \qquad \mathrm{j} = 1,\;\; 2$$

Subtracting the two equations shows that $D_1 - D_2$ satis es Hill's equation:

$$(D_1 - D_2)'' + \kappa_x(D_1 - D_2) = 0$$

The solution can be expressed in terms of the usual principal orbit functions of Hill's Equation in matrix form as:

$$\begin{bmatrix} D_1 - D_2 \\ (D_1 - D_2)' \end{bmatrix}_s = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} D_1 - D_2 \\ (D_1 - D_2)' \end{bmatrix}_{s_i}$$

Because C and S do not, in general, have the periodicity of the lattice, we must have for consistency with periodicity of $D_j(s+L_p) = D_j(s)$:

$$D_1(s_i) = D_2(s_i)$$
$$D_1'(s_i) = D_2'(s_i)$$

which implies a zero solution for $D_1 - D_2$ and:

$$D_1(s) = D_2(s) \quad \Longrightarrow D \text{ us unique for a periodic lattice}$$

The proof fails for $\sigma_{0x}/(2\pi) = \text{integer}$ however, this exceptional case should never correspond to a lattice choice because it would result in operation beyond the 1st stability boundary and/or with unstable particle orbits.

An alternative proof based on the eigenvalue structure of the 3x3 transfer matrices for *D* can be found in "Accelerator Physics" by SY Lee.

- Proof helps further clarify the structure of *D*

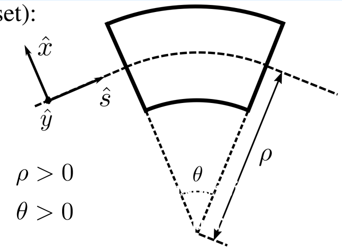
Appendix C: Transfer Matrix of a Negative Bend

Input from C.Y. Wong, MSU

For a clockwise bend (derived in the problem set):

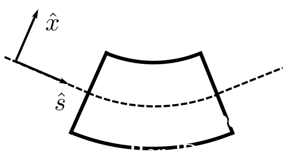
$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_f = \mathbf{M}_B \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_i$$

$$\mathbf{M}_B = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \rho > 0 \\ \theta > 0 \end{array}$$



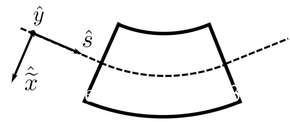
This de nition of the x,y,s coordinates is right-handed

The transfer matrix for a negative (anti-clockwise) bend is obtained by making the transformation $\rho \rightarrow -\rho, \ \theta \rightarrow -\theta$



$$\mathbf{M}_{-B} = \begin{pmatrix} \cos|\theta| & |\rho|\sin|\theta| & -|\rho|\,(1-\cos|\theta|) \\ -\frac{\sin|\theta|}{|\rho|} & \cos|\theta| & -\sin|\theta| \\ 0 & 0 & 1 \end{pmatrix}$$

If one nds the result counterintuitive, it can be derived as follows:



De ne $\tilde{x} = -x$

(The new set of coordinates is not right-handed, but this does not a ect the reasoning)

The dispersion functions in the two coordinate systems are related by

$$\begin{pmatrix} \tilde{D} \\ \tilde{D}' \\ 1 \end{pmatrix} = \mathbf{R} \begin{pmatrix} D \\ D \\ 1 \end{pmatrix} \qquad \text{where} \qquad \mathbf{R} = \mathbf{R}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The anti-clockwise bend is e ectively clockwise in the primed coordinate system:

$$\begin{pmatrix} \tilde{D} \\ \tilde{D}' \\ 1 \end{pmatrix}_f = \mathbf{M}_B \begin{pmatrix} \tilde{D} \\ \tilde{D}' \\ 1 \end{pmatrix}_i \quad \Longrightarrow \quad \mathbf{R} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_f = \mathbf{M}_B \mathbf{R} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{\mathbf{i}}$$

Transfer matrix of anti-clockwise bend in normal coordinates:

$$\mathbf{M}_{-B} = \mathbf{R}^{-1}\mathbf{M}_B\mathbf{R} = \begin{pmatrix} \cos|\theta| & |\rho|\sin|\theta| & -|\rho|\,(1-\cos|\theta|) \\ -\frac{\sin|\theta|}{|\rho|} & \cos|\theta| & -\sin|\theta| \\ 0 & 0 & 1 \end{pmatrix}$$

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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