Particle Equations of Motion
(Magnetic Optics and Bends)

Consider a long magnet where we can approximate the fields as 2D transverse within the magnet:

$$\boxed{\vec{B} = B_x(x,y)\,\hat{x} + B_y(x,y)\,\hat{y}}$$

$$\nabla\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ B_x & B_y & 0 \end{vmatrix} = \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}}$$

Taylor expand the field about $x=y=0$ for small x, y:

$$\vec{B} = B_x\,\hat{x} + B_y\,\hat{y}$$

$$\simeq \Big[\underbrace{B_x(0)}_{\substack{\text{Choose symmetry to } 0 \\ \text{(no y-plane bends)}}} + \frac{\partial B_x}{\partial y}\Big|_0 y + \underbrace{\frac{\partial B_x}{\partial x}\Big|_0}_{\substack{\text{Choose symmetry to } 0 \\ \text{(skew term)}}} x + \underbrace{\cdots}_{\text{Neglect}}\Big]\hat{x} + \Big[B_y(0) + \frac{\partial B_y}{\partial x}\Big|_0 x + \underbrace{\frac{\partial B_y}{\partial y}\Big|_0}_{\substack{\text{Choose symmetry to } 0 \\ \text{(skew term)}}} y + \underbrace{\cdots}_{\text{Neglect}}\Big]\hat{y}$$

Denote

$$\boxed{B' = \frac{\partial B_y}{\partial x}\Big|_0 = \frac{\partial B_x}{\partial y} = \text{Quadrupole Gradient} = \text{Const in axial length magnet.}}$$

Then

$$\boxed{\vec{B} \simeq B'y\,\hat{x} + \left[B_y(0) + B'x\right]\hat{y}}$$

$B' \neq 0 \iff$ Quadropole Magnet Focus / Defocus

$B_y(0) \neq 0 \iff$ Dipole x-plane bend of particle trajectory

Lorentz Force Eqn

$$\frac{d\vec{p}}{dt} = q\,\vec{v} \times \vec{B} \qquad \vec{p} = m\gamma\vec{v} = m\gamma\dot{\vec{x}} \qquad \dot{} \equiv \frac{d}{dt}$$

For motion in magnetic field, particle energy does not change (See problem sets)

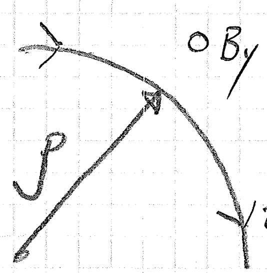
$$\gamma = \text{const} \qquad \frac{d\vec{p}}{dt} = m\gamma\ddot{\vec{x}}$$

$$\boxed{\gamma m \ddot{\vec{x}} = q\,\dot{\vec{x}} \times \vec{B}}$$

Case 1) If $B_y(0) \neq 0$ and $B' = 0$: Pure Dipole Bend.

Particle will bend in a uniform magnetic field on a circular arc.

$\rho = $ const in field



$v_s = |\dot{\vec{x}}| = \text{const}$

speed const along arc. No Accel.

$$\gamma m \ddot{\vec{x}} = q\,\dot{\vec{x}} \times B_y(0)\,\hat{y}$$

$$-\gamma m \frac{v_s^2}{\rho} = -q\,v_s B_y(0)$$

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{1}{\rho} = \dfrac{B_y(0)}{(B\rho)} \quad \text{Bend Radius.} \\[2ex] (B\rho) \equiv \dfrac{p}{q} = \dfrac{\text{momentum}}{\text{charge}} \equiv \text{Rigidity} \end{array}}$$

Dipole Bends used to manipulate "reference" path
- Rings
- Transfer lines

and also, various focusing properties for non-ref path orbit (later).

Rigidity

$$\boxed{(B\rho) = \frac{p}{q} = \frac{\gamma m v}{q} = \frac{mc}{q}\gamma\beta = \frac{\text{momentum}}{\text{charge}} \equiv \text{"Rigidity"}}$$

Read as one symbol "B-rho"!

Is a convenient measure of the coupling strength of particles to applied magnetic fields. Usually measured

$$p = \gamma m v$$

$$p_\nu = \left(\frac{\mathcal{E}}{c}, p\right) \qquad \mathcal{E} = \gamma m c^2 \qquad \text{4-vector} \qquad \mathcal{E} = mc^2 + W \qquad W = \text{kinetic Energy}$$

$$\underbrace{p_\nu p^\nu = \frac{\mathcal{E}^2}{c^2} - p^2}_{\text{4-vector contraction}} = \underbrace{(mc)^2}_{\text{rest frame evaluation of Lorentz Invariant}}$$

$$\rightarrow \quad cp = \sqrt{\mathcal{E}^2 - (mc^2)^2} = \sqrt{W^2 + 2Wmc^2}$$

Alternative derivation

$$W = (\gamma - 1)mc^2 \rightarrow \gamma = 1 + \frac{W}{mc^2}$$

$$\gamma^2 = \frac{1}{1-\beta^2} \rightarrow \beta^2 = 1 - 1/\gamma^2$$

$$\gamma\beta = \sqrt{\gamma^2 - 1}$$

$$\gamma\beta = \sqrt{\left(\frac{W}{mc^2}\right)^2 + 2\left(\frac{W}{mc^2}\right)}$$

$$\boxed{(B\rho) = \frac{p}{q} = \frac{mc}{q}\gamma\beta = \frac{\sqrt{W^2 + 2Wmc^2}}{qc} = \frac{mc}{q}\sqrt{\left(\frac{W}{mc^2}\right)^2 + 2\left(\frac{W}{mc^2}\right)}}$$

Electrons: $mc^2 = 511$ keV $\quad q = -e$

$$\frac{mc}{q} = -\frac{mc}{e} = -1.7045 \; 10^{-3} \text{ Tesla-m} \quad \Rightarrow \quad (B\rho) = 1.7045 \times 10^{-3} \sqrt{\left(\frac{W}{mc^2}\right)^2 + 2\left(\frac{W}{mc^2}\right)} \text{ Tesla-m}$$

Ions: $m = A m_u \qquad m_u c^2 = 931.49$ MeV

$q = Qe$

$$\frac{mc}{q} = \frac{A}{Q}\frac{m_u c}{e} = 3.107\left(\frac{A}{Q}\right) \text{ Tesla-m} \quad \Rightarrow \quad (B\rho) = 3.107\left(\frac{A}{Q}\right)\sqrt{\left(\frac{W}{mc^2}\right)^2 + 2\left(\frac{W}{mc^2}\right)} \text{ Tesla-m}$$

Case 2) If $B_y(0) = 0$ and $B' \neq 0$

Pure Quadrupole Focus In Straight System

Quadrupole magnet with no bend.

$$\gamma m \ddot{\vec{x}} = q \dot{\vec{x}} \times \vec{B} \qquad \text{with} \qquad \vec{B} = B' y \hat{x} + B' x \hat{y}$$

$$\gamma m (\ddot{x}\hat{x} + \ddot{y}\hat{y})_\perp = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B'y & B'x & 0 \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}_\perp = -q \dot{z} B' x \hat{x} + q \dot{z} B' y \hat{y}$$

But

$$\dot{} = \frac{d}{dt} = \frac{dz}{dt}\frac{d}{dz} = v_z \frac{d}{dz} \simeq v \frac{d}{dz}$$

$v \simeq$ const

motion primarily axially directed. (Paraxial Approximation)

Using this our equations become

$$\gamma m v \frac{d}{dz} v \frac{d}{dz} x = -q v B' x$$

$$\gamma m v \frac{d}{dz} v \frac{d}{dz} y = q v B' y$$

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{d^2x}{dz^2} + \frac{B'}{(B\rho)} x &= 0 \\ \frac{d^2y}{dz^2} - \frac{B'}{(B\rho)} y &= 0 \end{aligned}}$$

Recall $(B\rho) = \frac{\gamma m v}{q} = \frac{p}{q}$

$= \text{Rigidity}$

For purposes of later notational uniformity we take

$z = s =$ ref. trajectory (centerline) coordinate

$$\frac{B'}{(B\rho)} = K(s) = \text{Lattice focusing Function}$$

$$[K] = \frac{1}{(\text{length})^2}$$

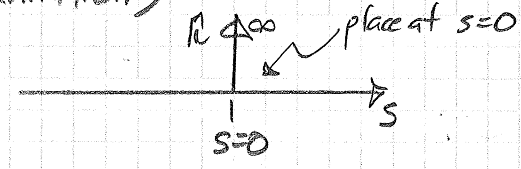
Allow B' to vary in s

Hills' Equation

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{d^2x}{ds^2} + K x &= 0 \\ \frac{d^2y}{ds^2} - K y &= 0 \end{aligned}}$$

Thin Lens Kick of Quadropole

Consider a short quadrupole of axial length $\ell \to 0$
But keep $\mathcal{K}\ell$ finite. (Kick Approximation)



$$\frac{d^2x}{ds^2} + \mathcal{K}x = 0$$

$$\left.\begin{aligned} x(0^-) &= x_0 \\ x'(0^-) &= x_0' \end{aligned}\right\} \text{ initial conditions}$$

$$\int_{0^-}^{0^+} \frac{d^2x}{ds^2}\,ds = -\int_{0^-}^{0^+} \mathcal{K}\,x\,ds$$

$$x'(0^+) - \underbrace{x'(0^-)}_{=x_0'} = -\hat{\mathcal{K}}\ell\, x(0)$$

$$\int_{0^-}^{0^+} \mathcal{K}\,ds = \underbrace{\hat{\mathcal{K}}\ell}_{\text{finite}}$$

Integrate again

$$x(0^-) = x(0^+) = x_0$$

Summarize results in matrix form:

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{0^+} = \begin{bmatrix} 1 & 0 \\ -\hat{\mathcal{K}}\ell & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

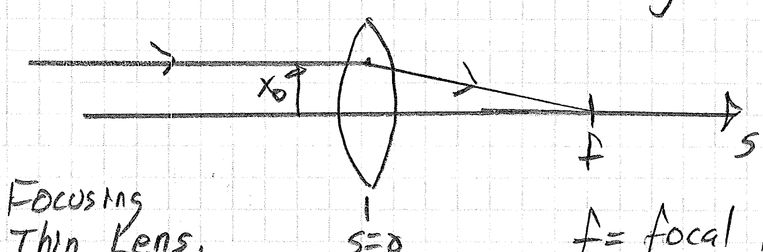
-or-

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{0^+} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{0^-}$$

$\frac{1}{f} \equiv \mathcal{K}\ell \qquad f =$ focal length.

sometimes write as

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{s} = \underbrace{\bar{M}(s|s_i)}_{2\times 2 \text{ Transfer Matrix}} \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i}$$



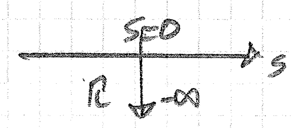
Focusing Thin Lens.

f = focal length

Thin lens kick changes angle but not coordinate. Interpreted analogously to a thin lens in light optics.

Similarly, for the defocus (y) plane. $\ell \to 0$, $\hat{K}\ell$ finite kick approx:

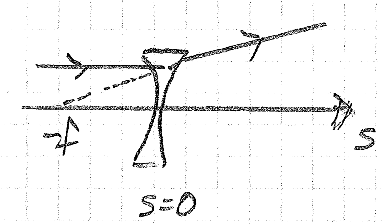
$$\frac{d^2 y}{ds^2} - K y = 0$$



leads to

$$\begin{bmatrix} y \\ y' \end{bmatrix}_{0^+} = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}_{0^-}$$

$$\frac{1}{f} = \hat{K}\ell$$



Defocusing Thin Lens

When the quadrupole polarity is reversed

$$B' \to -B' \qquad K \to -K$$

and the focusing (x) and defocusing (y) plane solutions are exchanged.

In a drift; $K = 0$ (No focus)

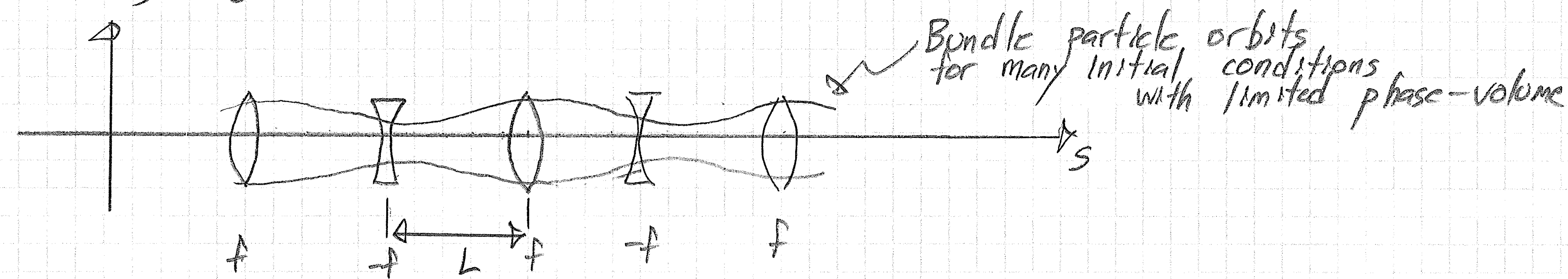
$$\frac{d^2 x}{ds^2} = 0 \quad ; \quad \Rightarrow \quad \begin{aligned} x(s) &= x_0 + x_0'(s - s_i) \\ x'(s) &= x_0' \end{aligned}$$

leads to

$$\begin{bmatrix} x \\ x' \end{bmatrix}_s = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{initial}}$$

$$\begin{bmatrix} y \\ y' \end{bmatrix}_s = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}_{\text{initial}}$$

This gives the tools we need to analyze particles evolving in an alternating-gradient transport lattice.



On average
- Particles farther from axis $x=0$ in focus lens
- " closer to " " in defocus lens

$\Rightarrow$ Net focusing in both planes

Called "Alternating Gradient" Focusing or "Strong Focusing" also.

We are not so far from being able to model a real accelerator now. Will show in homework any linear optic $\overline{M}$ can be replaced by two drifts (d_1, d_2) and between a thin lens kick (f).
So appropriately chosen drifts and thin lens kicks can exactly model a linear beamline.
- Details in-between kicks will differ. – But overall trends, including stability similar.

Now (gulp ....) how do we deal with case of simultaneous focusing ($B' \neq 0$) and bending ($B_y(0) \neq 0$).

- This will also allow us to understand focusing effects from bends when particles do not enter on design (reference) trajectory.

Case 3) $B_y(0) \neq 0$ and $B' \neq 0$

Frenet-Serret Coordinates

Edwards and Syphers

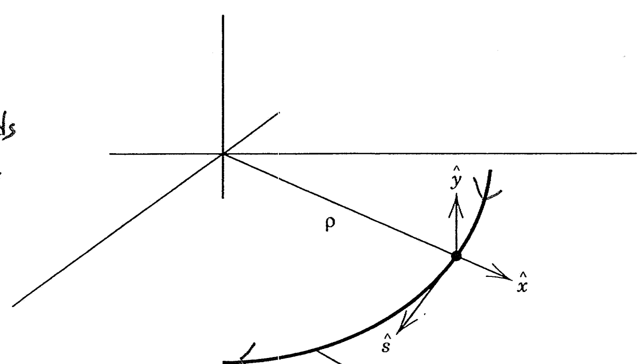


Figure 3.7. Coordinate system for development of equation of motion.

Transverse Particle Coordinate	$\vec{R} = r\hat{x} + y\hat{y}$ $r = \rho + x$

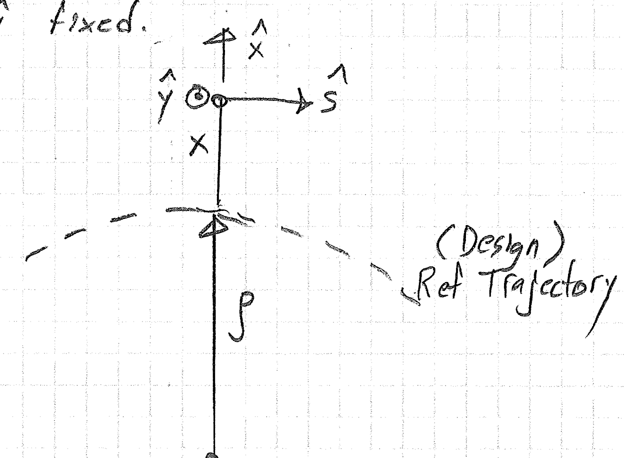
Also called "Reference Orbit"

Design (centerline) trajectory:

- Straight line down center – quadrupoles
- Circular arc segment – dipole. Where $B_y = B_y(0) = \text{const}$.

$\hat{x}, \hat{y}, \hat{s}$ unit vectors form right-hand system

- $\hat{x}, \hat{s}$ change orientation depending on location s on ref. trajectory
- $\hat{y}$ fixed.



Lorentz Force:

$$\frac{d\vec{p}}{dt} = q\,\vec{v}\times\vec{B} \qquad \vec{p} = m\gamma\,\dot{\vec{R}} \qquad \bullet = \frac{d}{dt}$$

$\gamma = \text{const}$ Magnetic field only bends.

$$\boxed{m\gamma\,\ddot{\vec{R}}_{\perp} = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{s} \\ v_x & v_y & v_s \\ B_x & B_y & 0 \end{vmatrix}_{\perp} = -q v_s B_y \hat{x} + q v_s B_x \hat{y}}$$

Must evaluate $\ddot{\vec{R}}$ in coordinates:

$$\vec{R} = r\hat{x} + \cancel{y\hat{y}}$$
$$\dot{\vec{R}} = \dot{r}\hat{x} + r\underbrace{\dot{\hat{x}}}_{\text{unit-vector changes orientation}} + \cancel{\dot{y}\hat{y}}$$

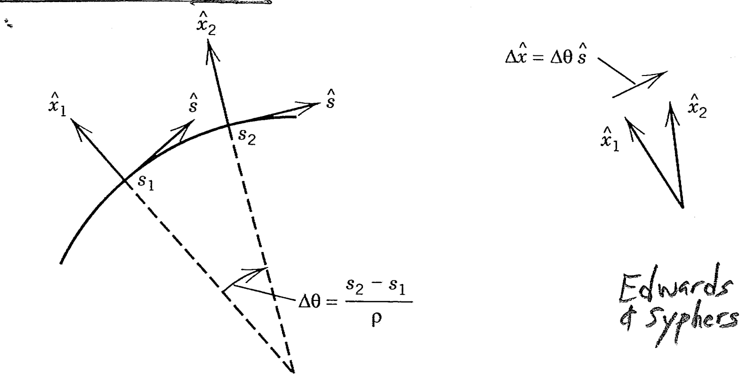


Figure 3.8. Time rate of change of unit vector $\hat{x}$.

Edwards & Syphers

$$\boxed{\begin{aligned} \dot{\hat{x}} &= \dot{\theta}\hat{s} = \frac{v_s}{r}\hat{s} \\ &\text{and for later} \\ \dot{\hat{s}} &= -\dot{\theta}\hat{x} \end{aligned}}$$

$\dot{\theta} = \frac{v_s}{r}$

$\Rightarrow \quad \dot{\hat{x}} = \dot{\theta}\hat{s}$

$\dot{\hat{s}} = -\dot{\theta}\hat{x}$

Analogy: Cylindrical Coords:

$\hat{e}_r = \hat{x}\cos\theta + \hat{y}\sin\theta$

$\hat{e}_\theta = -\hat{x}\sin\theta + \hat{y}\cos\theta$

$\dot{\hat{e}}_r = -\hat{x}\sin\theta\dot{\theta} + \hat{y}\cos\theta\dot{\theta} = \dot{\theta}\hat{e}_\theta$

$\dot{\hat{e}}_\theta = -\hat{x}\cos\theta\dot{\theta} - \hat{y}\sin\theta\dot{\theta} = -\dot{\theta}\hat{e}_r$

Giving

$$\boxed{\dot{\vec{R}} = \dot{r}\hat{x} + \cancel{\dot{y}\hat{y}} + r\dot{\theta}\hat{s}}$$

Differentiate again

$$\ddot{\vec{R}} = \ddot{r}\hat{x} + \dot{r}\dot{\hat{x}} + \cancel{\ddot{y}\hat{y}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + r\dot{\theta}\dot{\hat{s}}$$

$$\boxed{\ddot{\vec{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{x} + \ddot{y}\hat{y} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{s}}$$

Put these results in Lorentz force equation:

$$m\gamma \ddot{\vec{R}}_\perp = -q v_s B_y \hat{x} + q v_s B_x \hat{y}$$

$$B_y = B_y(0) + B'x$$
$$B_x = 0 + B'y$$

$$\boxed{\begin{aligned} \hat{x}: &\quad \ddot{r} - r\dot{\theta}^2 = \frac{-q v_s B_y}{\gamma m} = -\frac{q v_s}{\gamma m}\left[B_y(0) + B'x\right] \\ \hat{y}: &\quad \ddot{y} = \frac{q v_s B_x}{\gamma m} = \frac{-q v_s}{\gamma m} B'y \end{aligned}}$$

Take: $|v_x| \ll v_s$, $|v_y| \ll v_s$ $\Rightarrow$ Motion primarily directed axially in beam
Paraxial Approximation

Then

$$p \simeq \gamma m v_s$$

Change from t (time) to s (ref. trajectory position) as independent variable.

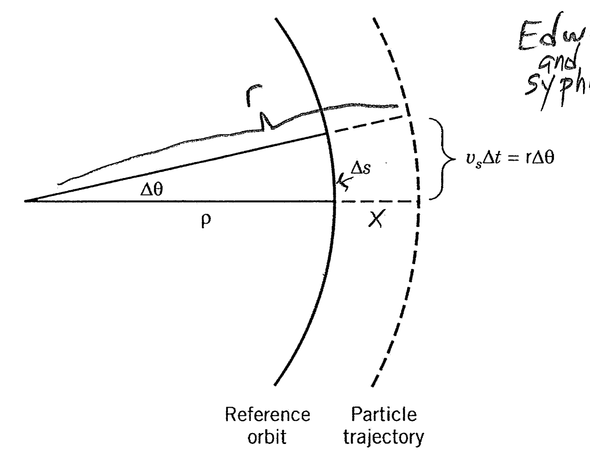
$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds}$$

But from diagram:

$$ds = \rho\, d\theta = \rho \frac{v_s\, dt}{r} \qquad \dot{\theta} = \frac{v_s}{r}$$

$$\Rightarrow \frac{ds}{dt} = v_s \frac{\rho}{r} = \dot{\theta}\rho$$

$$\boxed{\begin{aligned} \frac{d}{dt} &= \left(\frac{ds}{dt}\right)\frac{d}{ds} = \left(\frac{v_s \rho}{r}\right)\frac{d}{ds} \\ \frac{d^2}{dt^2} &= \cancel{\left(\frac{d^2 s}{dt^2}\right)}^{\to 0 \text{ neglect}}\frac{d}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2\frac{d^2}{ds^2} \end{aligned}}$$



Edwards and Syphers

Figure 3.9. Comparison of reference orbit path length ds and particle path length $v_s\, dt$.

Use these results to simplify

$\hat{x}$: $\ddot{r} - r\dot{\theta}^2 = \dfrac{-q v_s}{\gamma m}\left[B_y(0) + B' x\right]$

$\hat{y}$: $\ddot{y} = \dfrac{q v_s}{\gamma m} B' y$

$r = \rho + x$

$\ddot{r} = \ddot{x}$

$\dot{\theta} = \dfrac{v_s}{r}$

$\dfrac{d^2}{dt^2} = \left(\dfrac{v_s \rho}{r}\right)^2 \dfrac{d^2}{ds^2}$

$\Rightarrow \ddot{r} = \ddot{x} = \left(\dfrac{v_s \rho}{r}\right)^2 \dfrac{d^2}{ds^2}$

to obtain

$\hat{x}$: $\left(\dfrac{v_s \rho}{r}\right)^2 \dfrac{d^2 x}{ds^2} - r\left(\dfrac{v_s}{r}\right)^2 = \dfrac{-q v_s}{\gamma m}\left[B_y(0) + B' x\right]$

$\hat{y}$: $\left(\dfrac{v_s \rho}{r}\right)^2 \dfrac{d^2 y}{ds^2} = \dfrac{q v_s}{\gamma m} B' y$

$\dfrac{q v_s}{\gamma m}\left(\dfrac{r}{v_s \rho}\right)^2 = \left(\dfrac{q}{\gamma m v_s}\right)\left(\dfrac{\rho + x}{\rho}\right)^2 = \dfrac{1}{(B\rho)}\left(1 + \dfrac{x}{\rho}\right)^2$

or

$$\boxed{\begin{aligned} \hat{x}:&\quad \frac{d^2 x}{ds^2} - \frac{\rho + x}{\rho^2} = \frac{-1}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2\left[B_y(0) + B' x\right] \\ \hat{y}:&\quad \frac{d^2 y}{ds^2} = \frac{1}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2 B' y \end{aligned}}$$

Rigidity:

$(B\rho) \equiv \dfrac{m\gamma v_s}{q}$

$= \dfrac{p}{q}$

Expand results to leading <u>linear order</u>, neglecting terms $\propto x^2$ and xy

$\hat{x}$: $\dfrac{d^2 x}{ds^2} + \left[-\dfrac{1}{\rho^2} + \dfrac{2 B_y(0)}{\rho (B\rho)} + \dfrac{B'}{(B\rho)}\right] x = \dfrac{1}{\rho} - \dfrac{B_y(0)}{(B\rho)}$

[ ]·x terms $\propto x$

const terms RHS

$\hat{y}$ $\dfrac{d^2 y}{ds^2} - \dfrac{B'}{(B\rho)} y = 0$

only 1 term $\propto y$

Employ bend constraint: $\dfrac{1}{\rho} = \dfrac{B_y(0)}{(B\rho)}$

Dipole field tuned to reference bend

Final result: Hill's equation in both planes:

$$\boxed{\begin{aligned} \hat{x}: \quad & \frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{B'}{(B\rho)}\right] x = 0 \\ \hat{y}: \quad & \frac{d^2y}{ds^2} - \frac{B'}{(B\rho)} y = 0 \end{aligned}}$$

often write as:

$$\frac{d^2x}{ds^2} + K_x(s)\, x = 0$$

$$\frac{d^2y}{ds^2} + K_y(s)\, y = 0$$

Lattice focusing functions:

$$K_x = \frac{1}{\rho^2} + \frac{B'}{(B\rho)}$$

$$K_y = \frac{-B'}{(B\rho)}$$

$$B' = \left.\frac{\partial B_y}{\partial x}\right|_0 = \left.\frac{\partial B_x}{\partial y}\right|_0$$

$$B' = B'(s)$$

Comment:

- x-and y-equations same mathematical form (Hill's equation) to allow general conclusions on orbit properties
- Relativity incorporated in K lattice functions but don't need to think much about relativity when solving.
- Solve for initial conditions

  $x(0) \quad x'(0)$ $\qquad s = 0$ "initial" ref.
  $y(0) \quad y'(0)$

  to understand evolution of bundle of orbits.

- $\rho \to \infty \implies$ straight lattice. reduces to pure quadropole in straight beamline.

Forms of Lattice Functions

Drift

$\kappa_x = 0$
$\kappa_y = 0$

Free expansion, straight-line trajectories

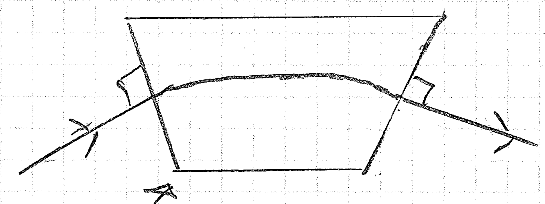
Quadrupole Magnet (no superimposed bend $\rho \to \infty$)

$\kappa_x = \dfrac{B'}{(B\rho)}$ x-plane focus

$\kappa_y = -\dfrac{B'}{(B\rho)}$ y-plane defocus

Sector Dipole Bend

$\kappa_x = \dfrac{1}{\rho^2}$ Focusing: • strong for tight bend (ρ small) • weak for small bend (ρ large)



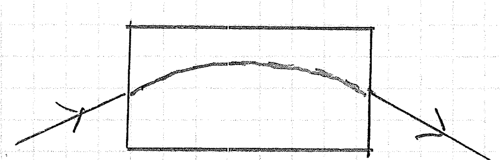
Normal incidence ref trajectory

κ_x independent of x entering/exiting.

$\kappa_y = 0$ Free expansion

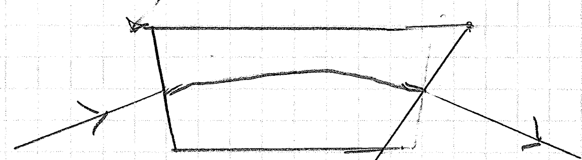
Other Dipoles

Rectangular box



Particles with $\pm x$ incident see less/more bending
Augment formulation

Slanted Edges



These will be worked on the problem set.

More Problems can de mapped to Hill's Equation with κ_x, κ_y functions

Electric Quadropoles
Electric Bends

Similar results
see Lund: USPAS notes for
Beam Physics with Intense Space Charge
Transverse Particle Dynamics

Solenoids — Apply rotating frame "Larmor" transform — Cover Next Week.

Accelerating Beam — Apply normalized coordinates — Cover Next Week.

Try to map new problems to what you know .... linear Oscillator equation.

For more info on these topics see supplemental notes on course web site!

More on this later......

Transfer Matrix

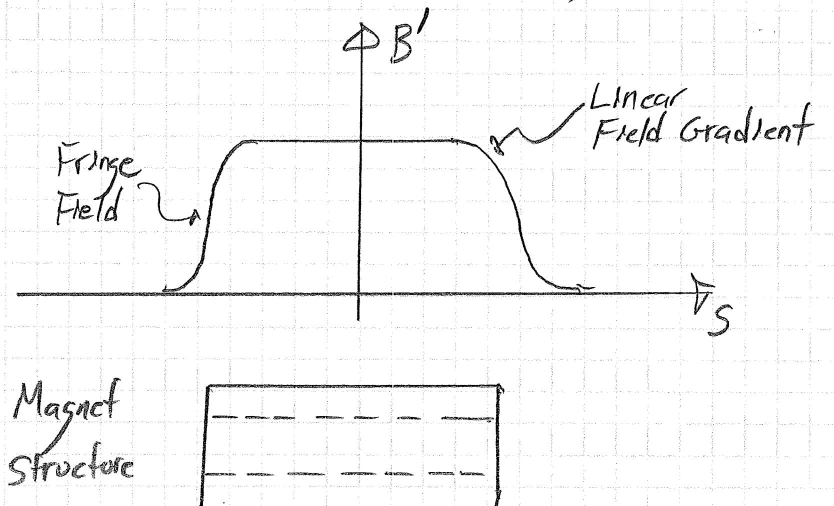
Solutions to the Hill eqns are often expressed in matrix form

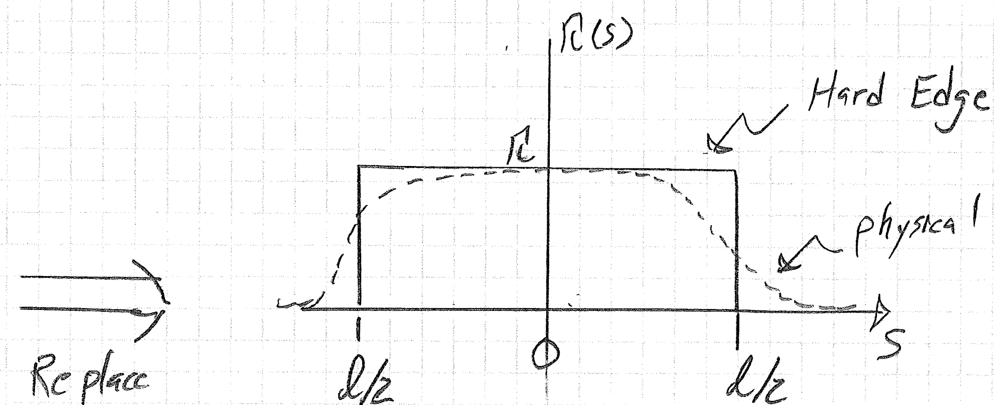
$$\boxed{\begin{bmatrix} x \\ x' \end{bmatrix}_s = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i} \equiv \underbrace{\bar{M}}_{2\times 2 \text{ matrix}}(s|s_i) \cdot \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i}}$$

$s = s_i$ = initial condition

Piecewise Method

Solutions can be constructed using magnet codes to get actual s-variation in $K(s)$ lattice function. This enables more realistic modeling. However, can approximate $K(s) = \text{const}$ in a regions to model the focusing properties of a lattice.





l = Effective Length

K = const Effective Strength

In applying this method need to relate $\mathcal{K}$ value / length to actual magnet properties. Many ways to do this:

Quadrupole

ℓ = Physical length magnet
$\mathcal{K}$ = Set from peak (middle) gradient, B'

Dipole

ℓ = Length ref trajectory in magnet
ρ Set from center (middle) value of $B_y(0)$ from $\frac{1}{\rho} = \frac{B_y(0)}{(B\rho)}$

Augment for non-sector dipoles with thin lens focus corrections for slanted edges.

Orbits within elements in piecewise approximation:

$$\begin{bmatrix} x_0 \\ x_0' \end{bmatrix} \equiv \begin{bmatrix} x \\ x' \end{bmatrix}_{s=s_i}$$

Drift
$\mathcal{K}=0$

$$\frac{d^2x}{ds^2} = 0$$

solution:

$$x(s) = x_0 + x_0'(s-s_i)$$

$$x'(s) = x_0'$$

$$\overline{M}(s|s_i) = \begin{bmatrix} 1 & s-s_i \\ 0 & 1 \end{bmatrix}$$

Through drift element:

$$\overline{M}(s_i+\ell, s_i) = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

Focusing Element

$\mathcal{K} > 0$

$$\frac{d^2x}{ds^2} + \mathcal{K}x = 0$$

orbit solution

- Quadrupole

$$\mathcal{K} = \frac{B'}{(B\rho)} > 0 \quad \text{focal}$$

- Dipole Bend (Sector)

$$\mathcal{K} = \frac{1}{\rho^2} > 0 \quad \text{focal}$$

$$x(s) = x_0 \cos\left[\sqrt{\mathcal{K}}(s-s_i)\right] + \frac{x_0'}{\sqrt{\mathcal{K}}} \sin\left[\sqrt{\mathcal{K}}(s-s_i)\right]$$

$$x'(s) = -x_0\sqrt{\mathcal{K}} \sin\left[\sqrt{\mathcal{K}}(s-s_i)\right] + x_0' \cos\left[\sqrt{\mathcal{K}}(s-s_i)\right]$$

$$\bar{M}(s|s_i) = \begin{pmatrix} \cos\left[\sqrt{\mathcal{K}}(s-s_i)\right] & \frac{1}{\sqrt{\mathcal{K}}}\sin\left[\sqrt{\mathcal{K}}(s-s_i)\right] \\ -\sqrt{\mathcal{K}}\sin\left[\sqrt{\mathcal{K}}(s-s_i)\right] & \cos\left[\sqrt{\mathcal{K}}(s-s_i)\right] \end{pmatrix}$$

Through element:

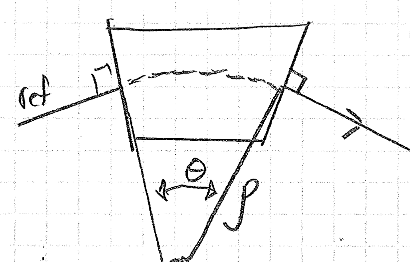
$$\bar{M}(s_i+\ell|s_i) = \begin{pmatrix} \cos(\sqrt{\mathcal{K}}\ell) & \frac{1}{\sqrt{\mathcal{K}}}\sin(\sqrt{\mathcal{K}}\ell) \\ -\sqrt{\mathcal{K}}\sin(\sqrt{\mathcal{K}}\ell) & \cos(\sqrt{\mathcal{K}}\ell) \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{\mathcal{K}}\ell) & \frac{1}{\sqrt{\mathcal{K}}}\sin(\sqrt{\mathcal{K}}\ell) \\ -\sqrt{\mathcal{K}}\sin(\sqrt{\mathcal{K}}\ell) & \cos(\sqrt{\mathcal{K}}\ell) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

For a dipole bend, note that

$$\mathcal{K} = \frac{1}{\rho^2} \qquad \sqrt{\mathcal{K}}\,\ell = \frac{\ell}{\rho} = \Theta_{bend}$$

Θ_{bend} = Bend Angle



<u>Bend</u>

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\ell/\rho) & \rho\sin(\ell/\rho) \\ -\frac{\sin(\ell/\rho)}{\rho} & \cos(\ell/\rho) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Defocusing Element

$\mathcal{K} < 0$

$$\frac{d^2}{ds^2}x - \mathcal{K}x = 0$$

- Quadrupole

$$\mathcal{K} = -\frac{B'}{(B\rho)} < 0 \quad \text{defocus}$$

Orbit solution: $\quad K \equiv -\mathcal{K} > 0$

$$x(s) = x_0 \cosh\left[\sqrt{K}(s-s_i)\right] + \frac{x_0'}{\sqrt{K}} \sinh\left[\sqrt{K}(s-s_i)\right]$$

$$x'(s) = x_0 \sinh\left[\sqrt{K}(s-s_i)\right] + x_0' \cosh\left[\sqrt{K}(s-s_i)\right]$$

$$\overline{M}(s|s_i) = \begin{pmatrix} \cosh\left[\sqrt{K}(s-s_i)\right] & \frac{1}{\sqrt{K}}\sinh\left[\sqrt{K}(s-s_i)\right] \\ \sqrt{K}\sinh\left[\sqrt{K}(s-s_i)\right] & \cosh\left[\sqrt{K}(s-s_i)\right] \end{pmatrix}$$

Through an element:

$$\overline{M}(s_i+\ell|s_i) = \begin{pmatrix} \cosh\left[\sqrt{K}\ell\right] & \frac{1}{\sqrt{K}}\sinh\left[\sqrt{K}\ell\right] \\ \sqrt{K}\sinh\left[\sqrt{K}\ell\right] & \cosh\left[\sqrt{K}\ell\right] \end{pmatrix}, \quad \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\left[\sqrt{K}\ell\right] & \frac{1}{\sqrt{K}}\sinh\left[\sqrt{K}\ell\right] \\ \sqrt{K}\sinh\left[\sqrt{K}\ell\right] & \cosh\left[\sqrt{K}\ell\right] \end{bmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Thin Lens Limit

For focusing / defocusing elements take

$\ell \to 0 \qquad \mathcal{K}\ell$ finite.

Find

Focusing

$$\bar{M} = \begin{bmatrix} \cos(\sqrt{\mathcal{K}}\ell) & \frac{1}{\sqrt{\mathcal{K}}}\sin(\sqrt{\mathcal{K}}\ell) \\ -\sqrt{\mathcal{K}}\sin(\sqrt{\mathcal{K}}\ell) & \cos(\sqrt{\mathcal{K}}\ell) \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -\mathcal{K}\ell & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

$\frac{1}{f} = \mathcal{K}\ell$

$f =$ focal length

Defocusing $\qquad K = -\mathcal{K} > 0$

$$\bar{M} = \begin{bmatrix} \cosh(\sqrt{K}\ell) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}\ell) \\ \sqrt{K}\sinh(\sqrt{K}\ell) & \cosh(\sqrt{K}\ell) \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ |\mathcal{K}|\ell & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}$$

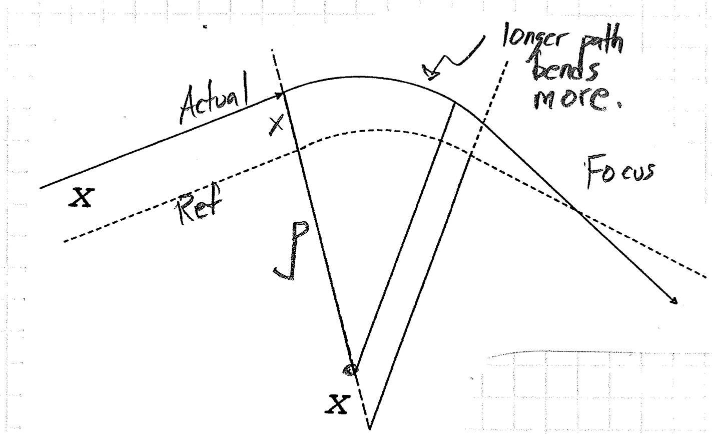
$\frac{1}{f} = |\mathcal{K}|\ell$

$f =$ virtual focal length.

For the focusing case with $\mathcal{K} = \frac{1}{\rho^2}$ for a dipole bend

$$\Rightarrow \frac{1}{f} = \frac{\ell}{\rho^2} = \left(\frac{\ell}{\rho}\right)\frac{1}{\rho} = \frac{\theta_{bend}}{\rho}$$

Consistent with expectation that sector dipole focuses!



More on Bending Dipoles

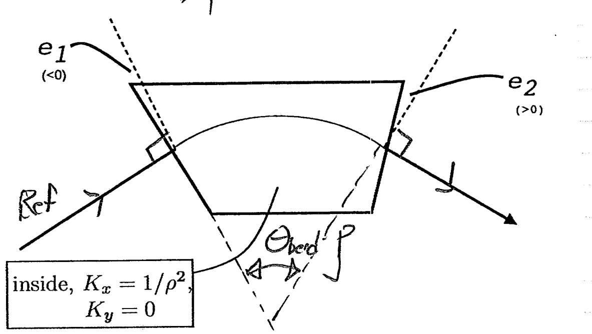
See textbooks for details, but in a bend with arbitrary edge angles:

- x-plane: More/less path length in dipole field leads to focusing/defocusing effect when entering and exiting magnet.
- y-plane: Fringe field structure gives impulse in vertical plane on etering and exiting magnet.

Model these edge effects as thin lens kicks:

$$\overline{M}_{Total} = \underbrace{\overline{M}_{e2}}_{\text{Thin lens exit}} \cdot \underbrace{\overline{M}_{body}}_{\text{Thick lens body (sector)}} \cdot \underbrace{\overline{M}_{e1}}_{\text{Thin lens enter}}$$

Dipole Bend



$$\overline{M}_x = \begin{bmatrix} 1 & 0 \\ \frac{\tan e_2}{\rho} & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\ell/\rho) & \rho\sin(\ell/\rho) \\ -\frac{1}{\rho}\sin(\ell/\rho) & \cos(\ell/\rho) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{\tan e_1}{\rho} & 1 \end{bmatrix}$$

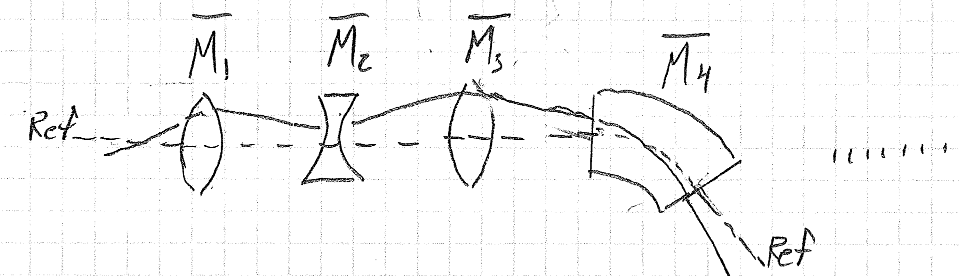
$$\overline{M}_y = \begin{bmatrix} 1 & 0 \\ -\frac{\tan e_2}{\rho} & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}}_{\text{Drift}} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{\tan e_1}{\rho} & 1 \end{bmatrix}$$

Note:
- Sector Dipole: $e_1 = e_2 = 0$
  $\Rightarrow$ No entry/exit corrections
  $\overline{M}_{e1} = \overline{M}_{e2} = \overline{I}$
- Box Dipole: $e_1 = e_2 = \theta_{bend}/2$
  Relevant case: easier to fabricate.

For ρ large corrections are small
But beam may enter/exit many magnets to accumulate.

Summary

We now have the basic parts needed to analyze transverse focusing optics in beamlines:



$$\begin{pmatrix} x \\ x' \end{pmatrix}_N = \bar{M}_N \cdot \bar{M}_{N-1} \cdots\cdots \bar{M}_4 \cdot \bar{M}_3 \cdot \bar{M}_2 \cdot \bar{M}_1 \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Much more to do though:

- Solenoid Focusing
- Energy gain from RF cavities
- Longitudinal physics
- Impact of Nonlinear Fields
- Effect of "off" (not design) momentum
- Statistical properties of beams
- Control / steering
- Space-Charge effects
- Application examples

Also we will first develop more machinery to analyze orbits / Hill's eqn more efficiently: Phase-Amplitude methods + Courant-Snyder invariant.