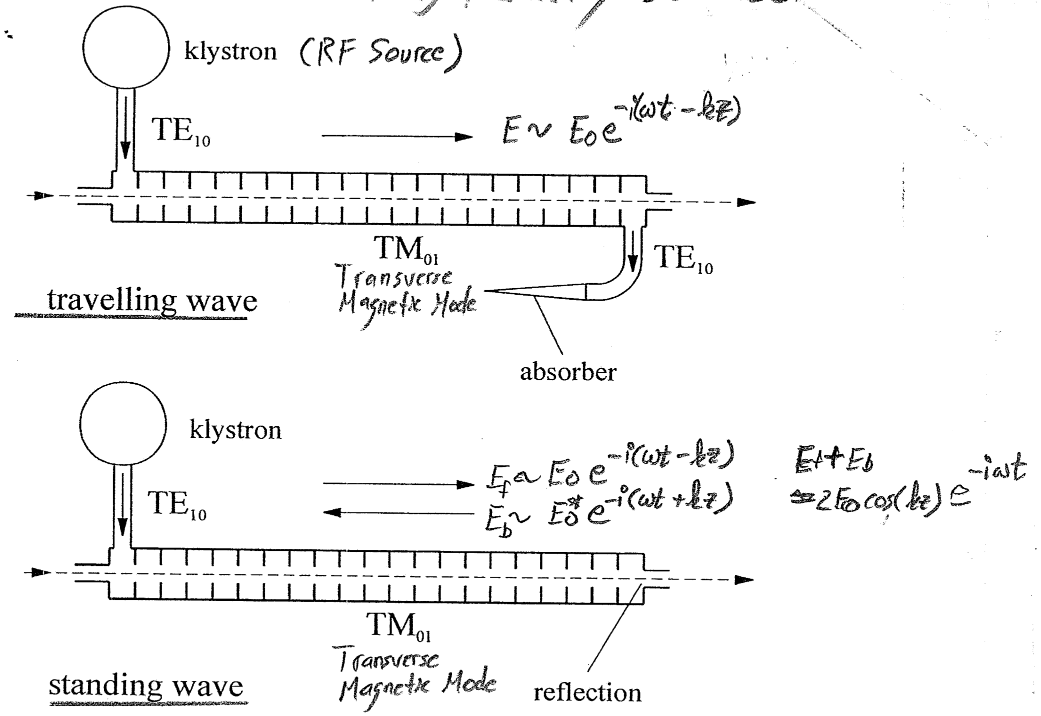
Longitudinal Physics: Beam Acceleration

Different technologies can be employed for beam acceleration

RF: Radio Frequency EM Waves
Tuned to resonate with beam.
that is longitudinally bunched.



$E \sim E_0 e^{-i(\omega t - kz)}$

$E_f \sim E_0 e^{-i(\omega t - kz)}$

$E_b \sim E_0^* e^{-i(\omega t + kz)}$

$E_f + E_b$
$\simeq 2E_0 \cos(kz) e^{-i\omega t}$

Fig. 5.9 The two modes of operation of the linac structure. The upper diagram shows the more commonly used travelling wave mode in which an absorber is installed at the end of the structure to prevent reflections. In the second case the wave is reflected virtually without losses, resulting in a standing wave.

Two basic schemes:
1) Traveling Wave: e^- machines common
2) Standing Wave: most common
Cavities coupled or Individually controlled.

Traveling Wave

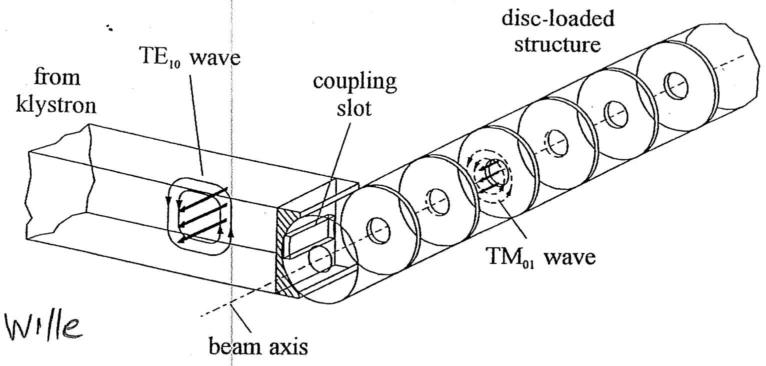


Fig. 5.8 Coupling of the TE_{10} waveguide to the linac structure. The transfer of the wave is achieved without reflections via an appropriately sized coupling slot.

Standing Wave

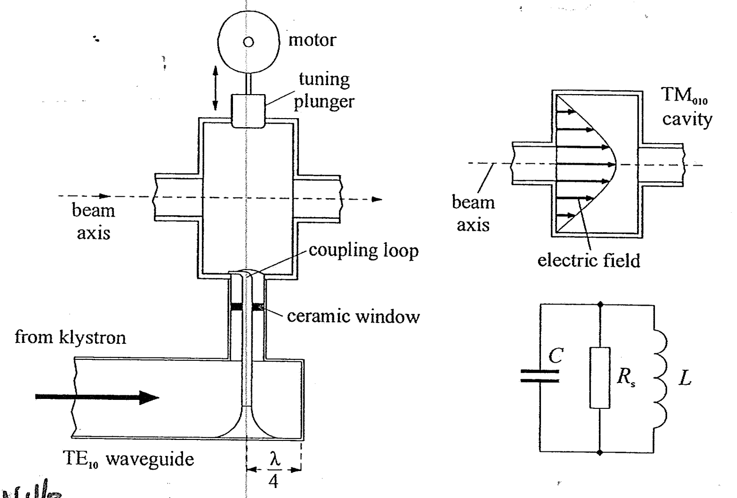


Fig. 5.4 Design of a single-cell accelerating structure using the TM_{010} mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

Induction Acceleration

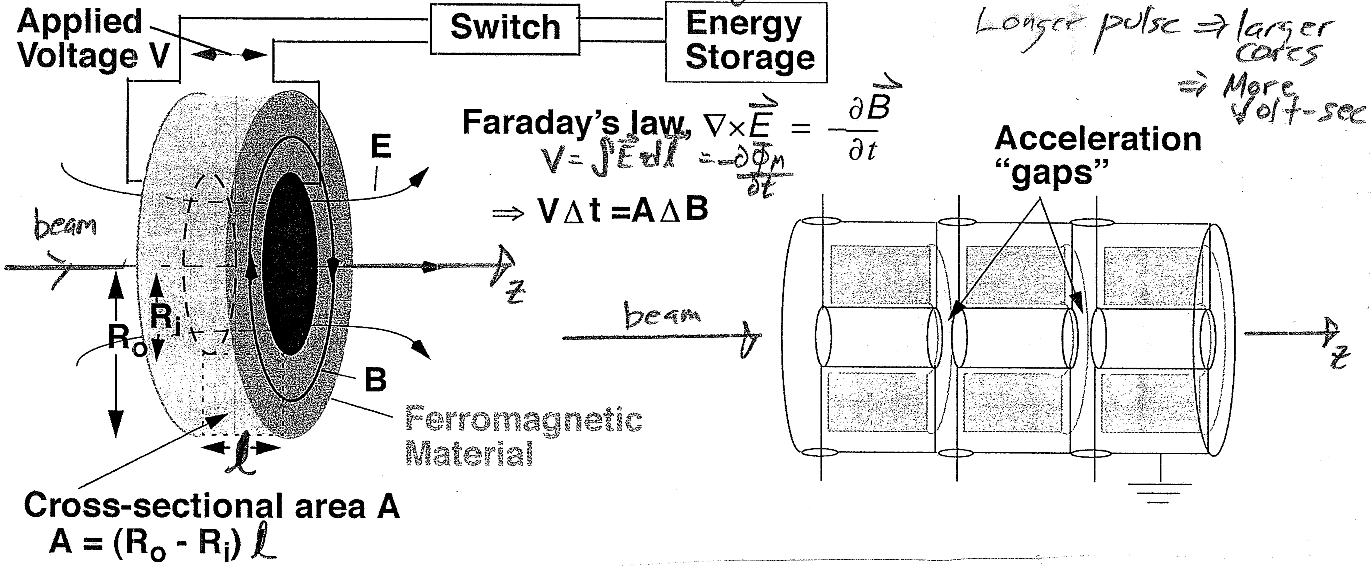
Beam coupled inductively to a pulsed power source.
Operates like a 1:1 transformer
Ferromagnetic core must have sufficient "capacity" (Volt-seconds) to keep voltage from collapsing over pulse duration of beam.

Schematic

Beam pulse can be as long as voltage can be maintained

Longer pulse ⇒ larger cores
⇒ More Volt-sec

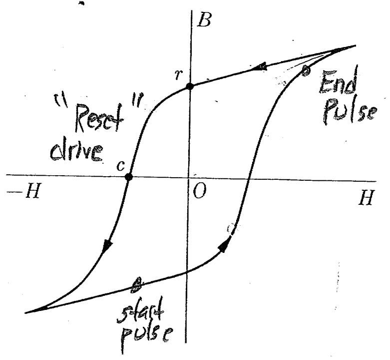
Core "reset" to same point on B(H) curve each pulse.



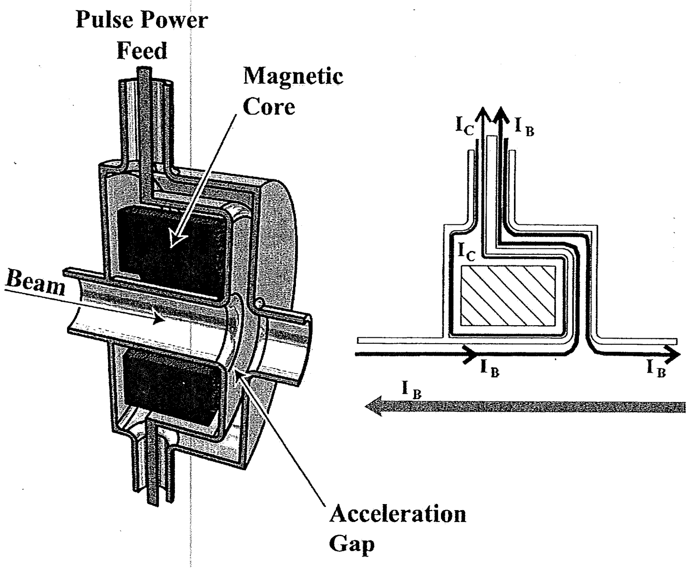
Faraday's law, $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$V = \int \vec{E} \cdot d\vec{l} = -\dfrac{\partial \Phi_m}{\partial t}$

$\Rightarrow V \Delta t = A \Delta B$



More Realistic Geometry



$$\frac{\Delta V \Delta t}{\Delta z} = (R_o - R_i)\,\Delta B \left(\begin{array}{c}\text{Radial}\\ \text{Packing}\\ \text{Frac}\end{array}\right)\left(\begin{array}{c}\text{Axial}\\ \text{Packing}\\ \text{Frac}\end{array}\right)$$

$$\sim 1\text{m} \times 2.5\text{T} \times \sim 0.8 \times \sim 0.8 \sim 1.6 \,\frac{\text{Volt-sec}}{\text{m}}$$

- ★ Losses in material heat core + reset time
  - ⇒ Challenging for rings or CW = Continuous Wave applications
- ★ Easy to shape pulse. Good for low rep rate, high intensity.
- ★ Conceptually simple / appealing and can be efficient, but pulse power control also can be challenge.
  - ~ Wall plug efficiencies ≳ 50% possible

Eletrostatic Acceleration

see Livingston and Blewett, "Particle Accelerators" for more info.

Use DC high voltage electric field to accelerate charged particles falling through a potential well. Beam can be continuous or pulsed.

$* \ \Delta E = q \Delta V$

ΔV = change in E.S. potential

ΔE = " " kinetic energy

Concept

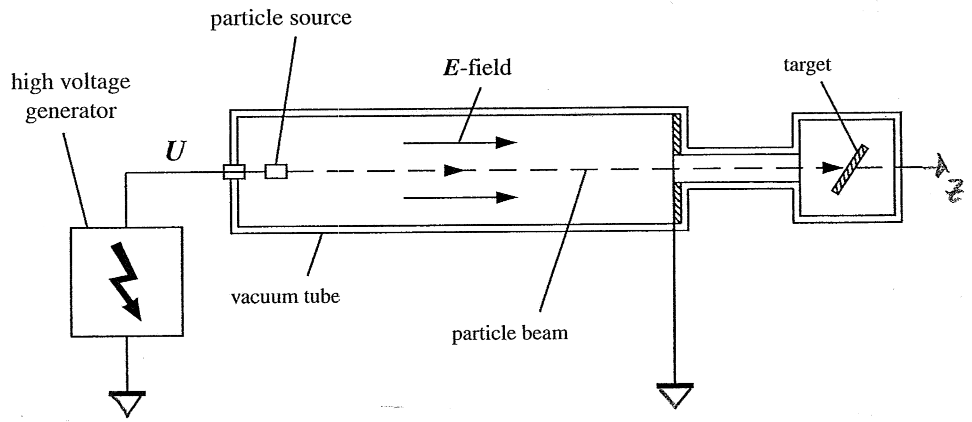
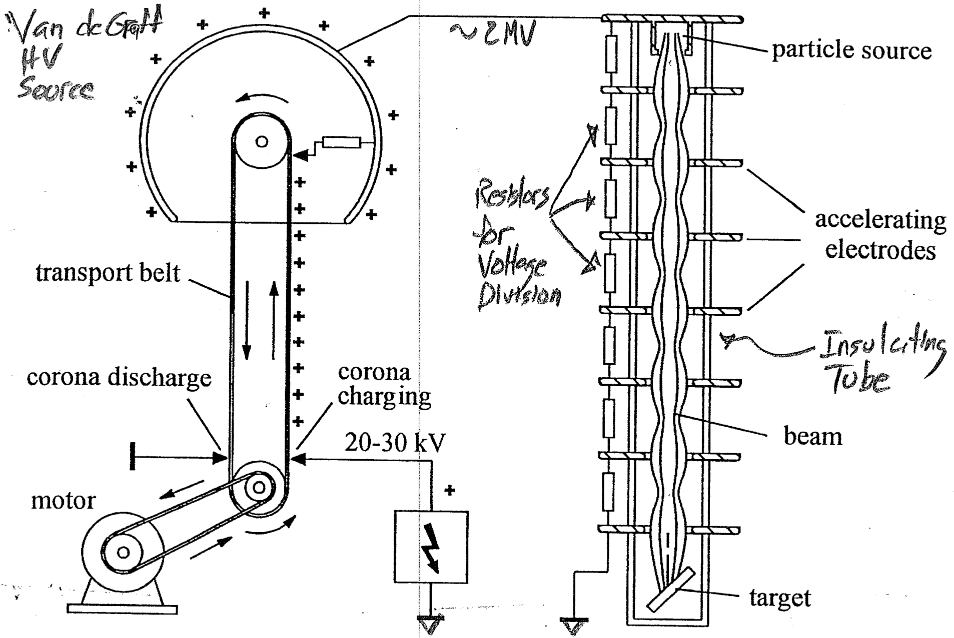


Fig. 1.3 General principle of the electrostatic accelerator.

Wille

Closer to Reality!



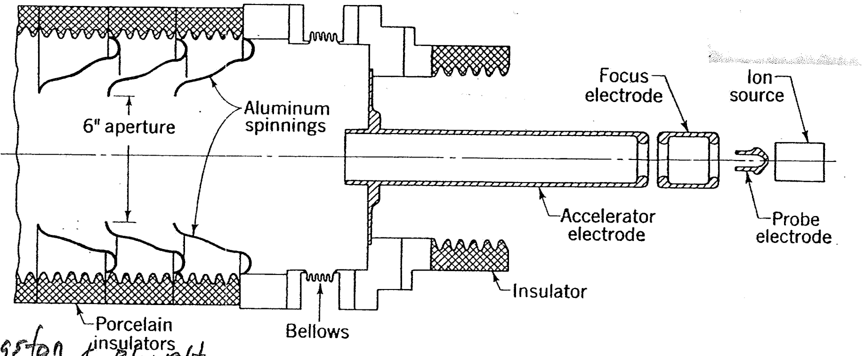
Wille **Fig. 1.7** The Van de Graaff accelerator.

Need DC or long pulse supplies to work!

- Van de Graaff (static electromechanical)
- Cockcroft-Walton (AC to DC voltage mult)
- Marx Generator (long pulse)

⋮

More Realistic Geometry



Livingston & Blewett

Fig. 3-13. Positive-ion source, focusing electrodes, and accelerating-tube structure for the Brookhaven 4-Mv generator.[22]

- ☆ Gratings and voltage division limit local fields to inhibit electrical breakdown.
- ☆ Insulators structured to inhibit avalanche breakdown.
- ☆ Careful attention to details
  - No sharp metal corners near large potential diffs.
  - Metal / Insulator junctions.

Best Efforts result in only few MV max.

Breakdown Scaling

Voltage Holding found to scale as (Handbook Accel. Phys., A. Faltens)

$$V_{max} \lesssim 100\,\mathrm{kV} \begin{cases} \left(\frac{d}{1\,\mathrm{cm}}\right) & d \lesssim 1\,\mathrm{cm} \\ \left(\frac{d}{1\,\mathrm{cm}}\right)^{1/2} & d > 1\,\mathrm{cm} \end{cases}$$

d = characteristic distance

Scaling can be degraded!

- ☆ Under "typical" near injector vacuum conditions $\sim 10^{-7}$ Torr
  - Poor vacuum can degrade.
- ☆ Assumes steps taken to minimize local peak field.
  - Radiused edges
  - Smooth conductors

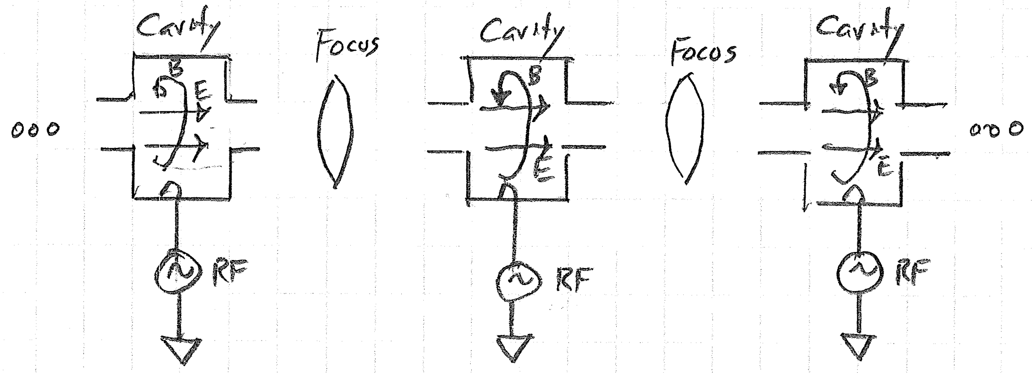
  ⋮
- ☆ Lost particles on conductors or insulators can trigger breakdown.

RF Acceleration

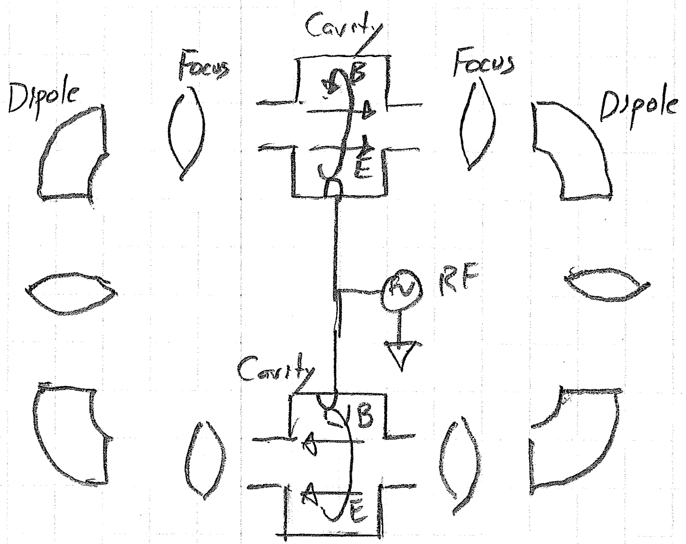
We will concentrate primarily on RF acceleration, first from the perspective of an RF Linac using resonant cavities. But before proceeding to outline how these works, here we frame a range of potential concepts to place in context.

In RF concepts the beam must be longitudinally bunched with bunches maintaining proper synchronism with an oscillating RF wave.

Linear Accelerator
with RF Cavities



Circular Accelerator
with RF Cavities



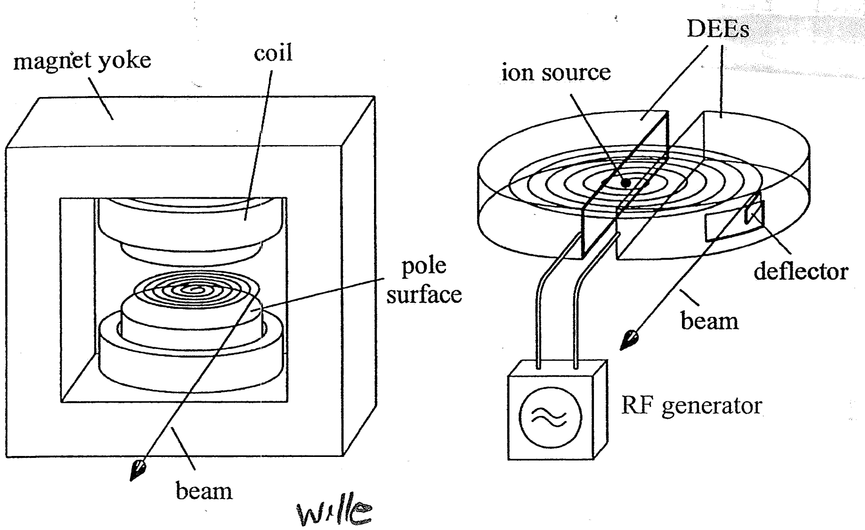
- ★ Cavities placed where particle transit between cavities is phased for energy gain + longitudinal focusing
- ★ Transverse focusing provided by optics between cavities.
- ★ RF sources drive cavities with proper phase control.
  - Heavy ions with low β may require individual phase control
  - Cavities may also be coupled with established phase relationship.

- ★ Cavity phase control (possibly in some high harmonic) setup for energy gain + longitudinal focusing consistent with particle time transit around ring
  - Path length with p + slip factor must be accounted for
- ★ Focus + Bending between cavities
- ★ One or few RF cavities at positions in ring. Cavities have related phase.

Range of RF Concepts : Very Broad ! Only Schematic Outline here.

Cyclotron

See Livingston and Blewett, "Particle Accelerators", chapter 6 for more info.



Wille
Fig. 1.12 The cyclotron.

Non-Relativistic:

$$\omega = \frac{q B_z}{m} = \text{const}$$

$$\frac{1}{\rho} = \frac{B_z}{(B\rho)}$$

ρ increases with energy gain till particle spirals out to deflector.

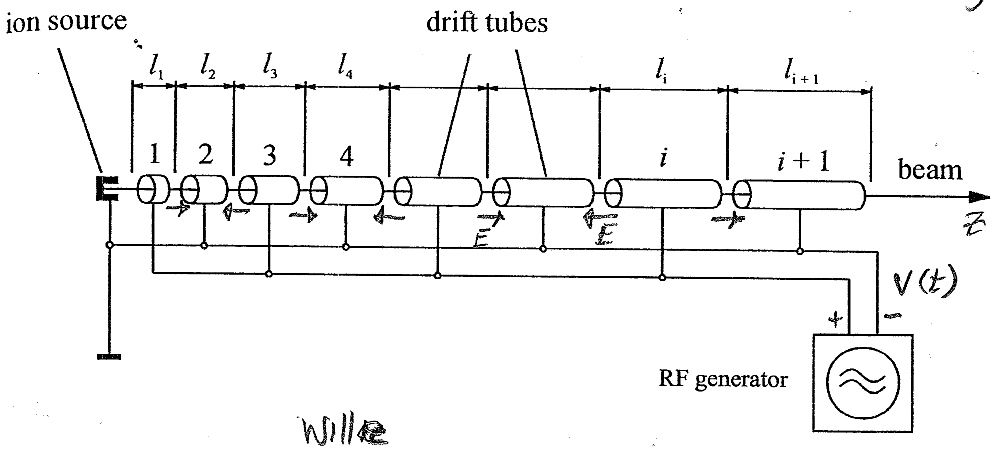
As particle becomes relativistic, synchronism will be broken.

- ★ Not much focusing possible
- ★ Continuous train bunches possible
- ★ Relatively simple.

Already discussed 1st lectures.

Wideröe Linac

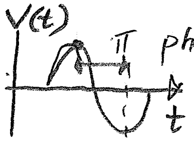
See historical discussions in many accelerator books: Wiedemann, Conte & McKay Wangler, ...



Wille
Fig. 1.9 Wideröe linear accelerator.

Non-Relativistic

$$W_i = \frac{1}{2} m v_i^2 \quad \text{Kinetic Energy}$$



phase advance between gaps

Gap Seperation $\quad l_i = \frac{v_i \tau_{rf}}{2}$

$$= \frac{\beta_i c \tau_{rf}}{2} = \frac{\beta_i \lambda_{rf}}{2} \qquad \lambda_{rf} \equiv c\tau_{rf}$$

for resonance acceleration

Wideröe Linac continued

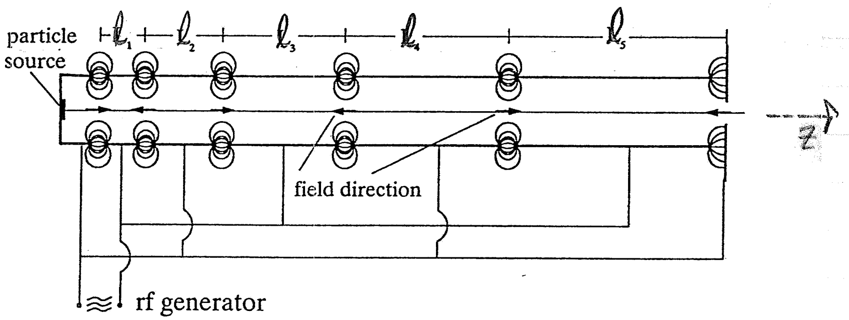


Fig. 2.5. Wideroe linac structure (schematic)
Wiedemann

- ★ Tubes shield particles from deaccelerating (wrong phase) RF till they get to the next gap.
- ★ Resonance condition established by adjusting the tube length.
- ★ Structure is lossy; radiates power.
  $\Rightarrow$ Enclose in tube to make cavity $\Rightarrow$ Alverez structure.

  $\Rightarrow$ Due to losses, Wideröe structure not commonly used today.

Alvarez Linac or Drift Tube Linac (DTL)

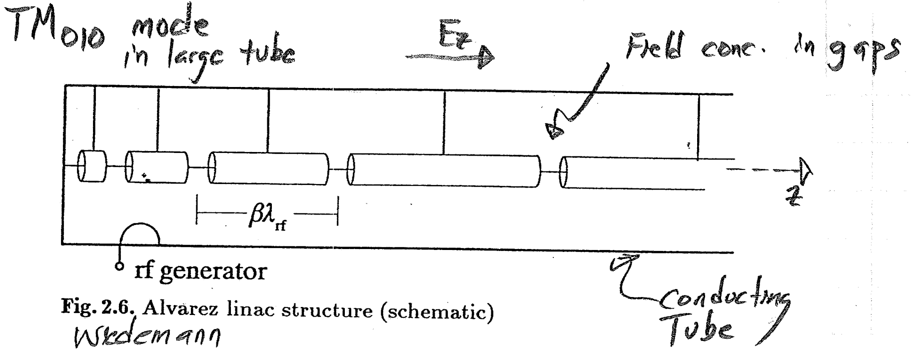


Fig. 2.6. Alvarez linac structure (schematic)
Wiedemann

- ★ Tube to contain radiation boosts efficiency and allows higher freq. RF

  $$\ell_i = \beta \lambda_{rf}$$

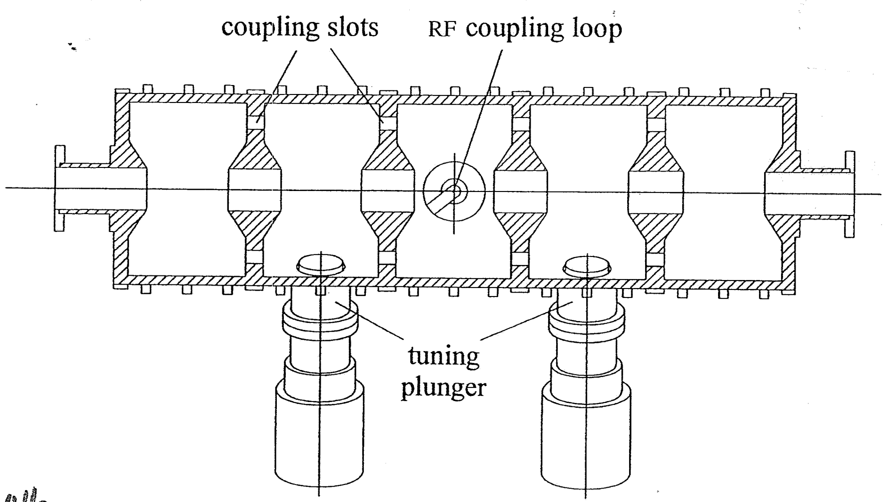
  2π phase advance between gaps

  $\Rightarrow$ Allows use of higher freq.

For more info see:
Wiedemann, Wille, Wangler,
Conte & Mackay, Edwards & Syphers,
''''

- ★ Still common pre-accelerator for protons / ions from injection to a few hundred MeV
  $\beta \simeq 0.04 \sim 0.4$
- ★ Not used for electrons since β is typically too high from injector

Coupled Cavity Linac See Wangler, "RF Linear Accelerators" for more info.

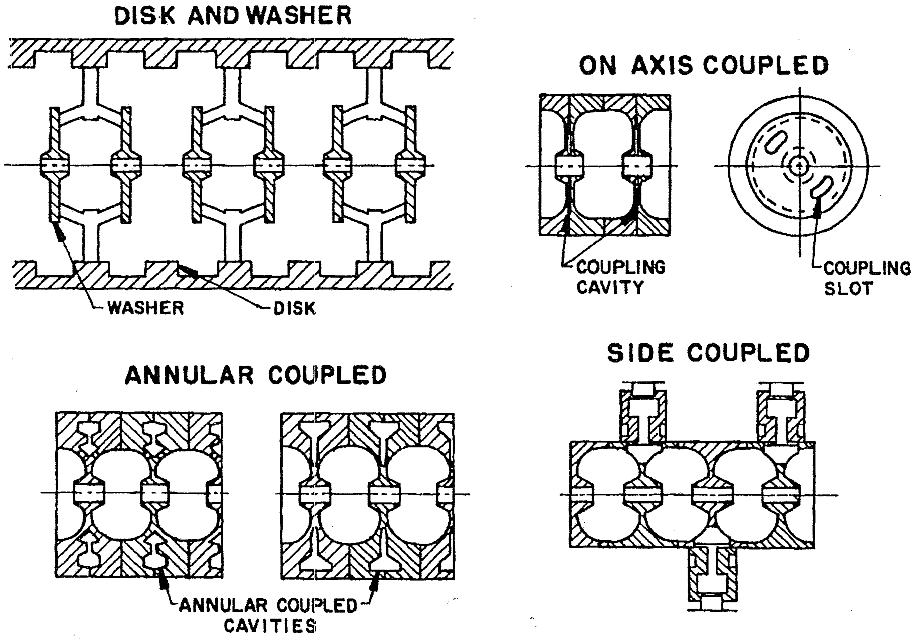


Wille

Fig. 5.5 Layout of a five-cell accelerating structure. The power feed is coupled to the middle cell and two tuning plungers are sufficient for the entire structure.

Banks of RF cavities are coupled together to maintain relative RF phase control needed.

- Very common for high β particles.
- Simplifies RF drive and saves cost.
- Many possible geometries.

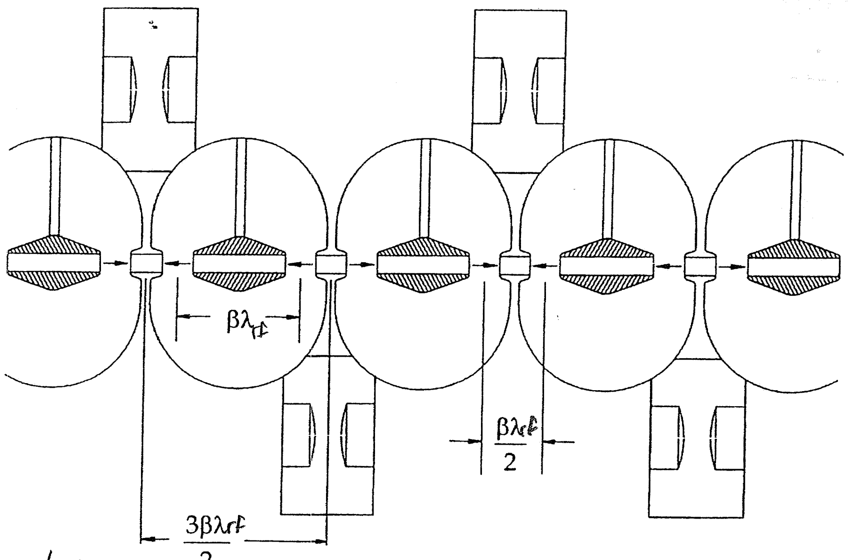


Wangler **Figure 4.17** Four examples of coupled-cavity linacs.

☆ Coupling cavities sometimes in beam line and other times moved off-axis for more efficient packing.

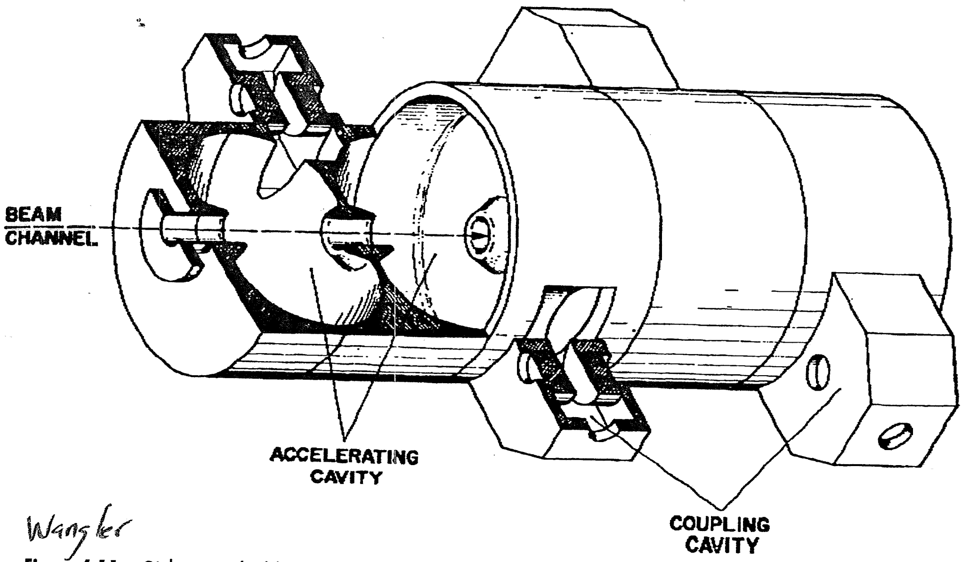
☆ Usually transverse focusing interspersed between banks of coupled RF cavities.

Coupled Cavity Linac   Further examples



Wangler

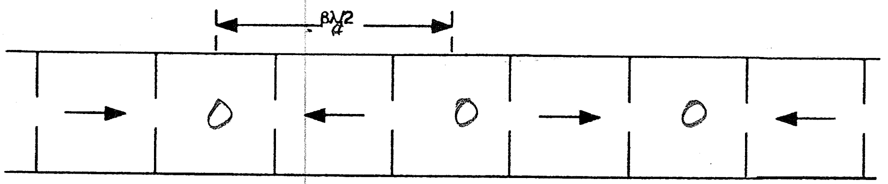
Figure 12.2 Coupled-cavity drift-tube linac (CCDTL) structure with a single drift tube in each accelerating cavity.



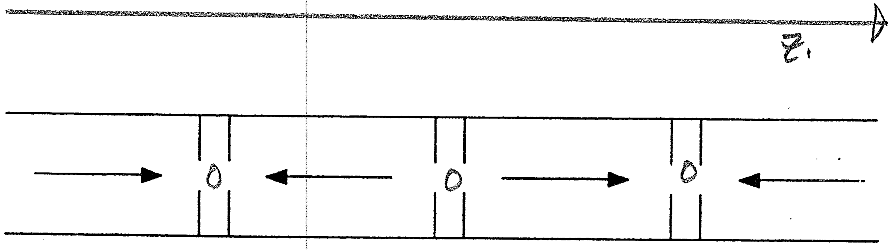
Wangler

Figure 4.11 Side-coupled linac structure as an example of a coupled-cavity linac structure. The cavities on the beam axis are the accelerating cavities. The cavities on the side are nominally unexcited and stabilize the accelerating-cavity fields against perturbations from fabrication errors and beam loading.

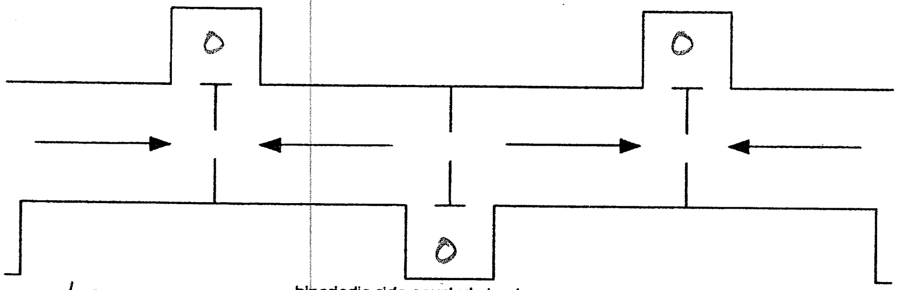
Example phase relations of E field between cavities.



π/2 mode of periodic structure



biperiodic on-axis-coupled structure



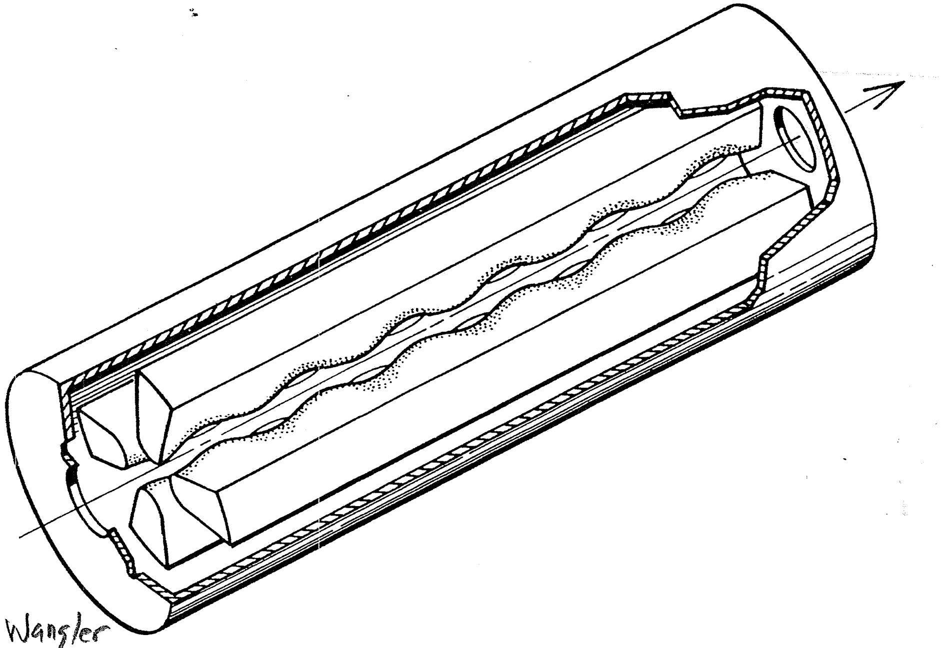
biperiodic side-coupled structure

Wangler

Figure 4.15 π/2-like-mode operation of a cavity resonator chain. From top to bottom are shown a periodic structure in π/2 mode, a biperiodic on-axis coupled-cavity structure in π/2 mode, and a biperiodic side-coupled cavity in π/2 mode.

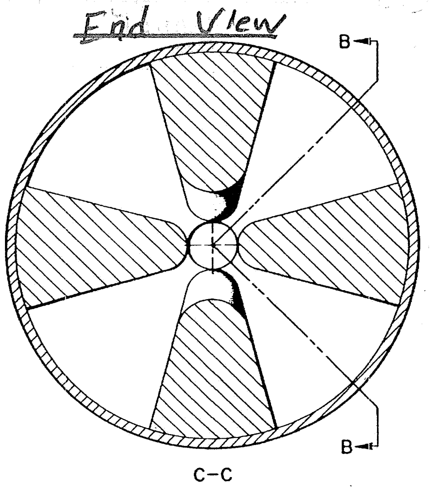
Radio Frequency Quadrupole (RFQ)

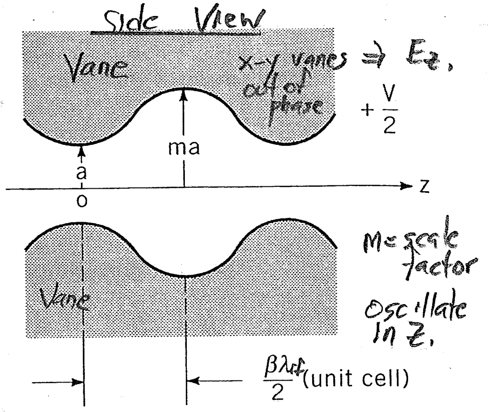
see Wangler, "RF Linear Accelerators" for more info.



Wangler

Figure 1.7 The radio-frequency quadrupole (RFQ), used for acceleration of low-velocity ions, consists of four vanes mounted within a cylindrical cavity. The cavity is excited in a quadrupole mode in which the RF electric field is concentrated near the vane tips to produce an electric transverse restoring force for particles that are off-axis. The modulation of the vane tips produces a longitudinal electric-field component that accelerates the beam along the axis.

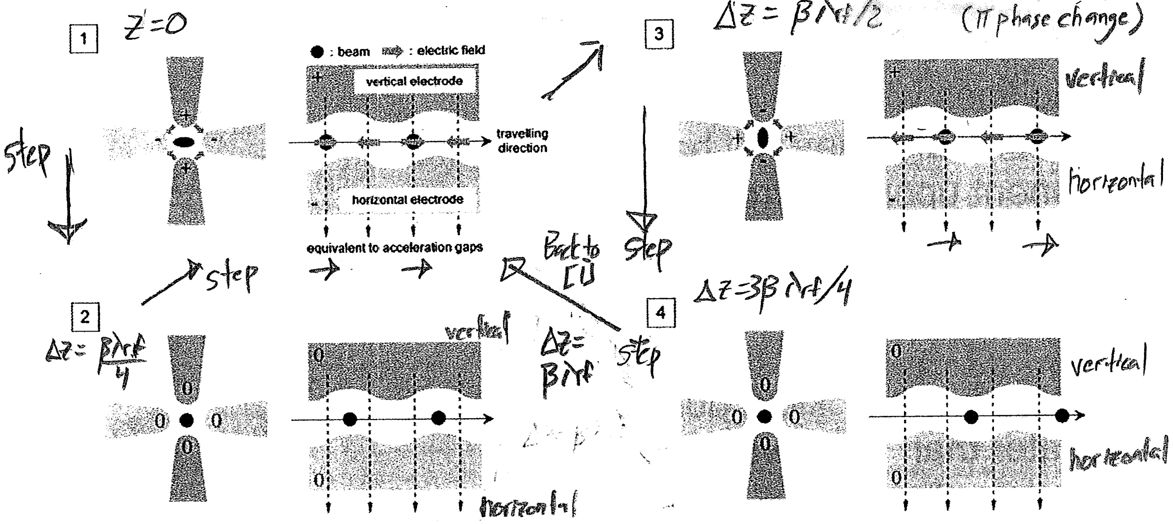




- ★ Electric quadrupole mode excited in cavity with four quadrupole symmetry vane electrodes.
  - Vanes concentrate ⊥ E field to provide strong transverse quadrupole (electric) focusing
  - Longitudinal rippling of vanes provides ∥ E field for longitudinal acceleration
  - Gives simultaneous ⊥ focusing and longitudinal accel & bunching.
- ★ Works for $\beta \sim 0.01 - 0.06$
- ★ Can be setup to bunch and accelerate a DC injected beam from source to match into required bunch structure of RF accelerator.
  - structure can be tapered to enhance bunching or acceleration. Period = $\beta\lambda_{rf}$ varies with energy gain.
- ★ Common choice for front ends.
  - Including FRIB.
- ★ Not used for electrons since low β structure.

Schematic on how an RFQ works! M. Syphers, USPAS Notes.

The Radio Frequency Quadrupole (RFQ)



Many variants: 4-vanes, 4-rods, Al/Cu, large/small

Typical energy range — up to few MeV (protons, ions typically; ~~also electrons~~)

[1]	deFocus x Focus y	Accel. in z	+ phase focus from field variation
[2]	Null	Null	
[3]	Focus x deFocus y	Accel in z	+ phase focus from field variation
[4]	Null	Null	

⇑ AG Focus-DeFocus cycle

Where + Electrode closer
⇒ ϕ on axis +

Where − Electrode closer
⇒ ϕ on axis −

Comment: An RFQ essentially employs AG electric transverse focusing which is strong for low velocity (β) particles.

$$x'' + \frac{(\gamma\beta)'}{(\gamma\beta)}x' + K x = 0$$

$$K = \begin{cases} \dfrac{B'}{(B\rho)} & \text{Magnetic} \\ \dfrac{E'}{(\beta c)(B\rho)} & \text{Electric} \end{cases}$$

extra factor β in denominator

$B' = \left.\frac{\partial B_y}{\partial x}\right|_0 = \left.\frac{\partial B_x}{\partial y}\right|_0 =$ Magnetic Quad Gradient

$E' = \left.\frac{-\partial E_x}{\partial x}\right|_0 = \left.\frac{\partial E_y}{\partial y}\right|_0 =$ Electric Quad Gradient

Traveling Wave Linac

see Wangler, "RF Linear Accelerators" for more info. Chapter 3

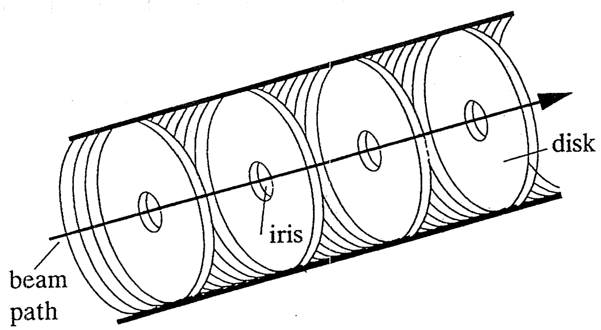
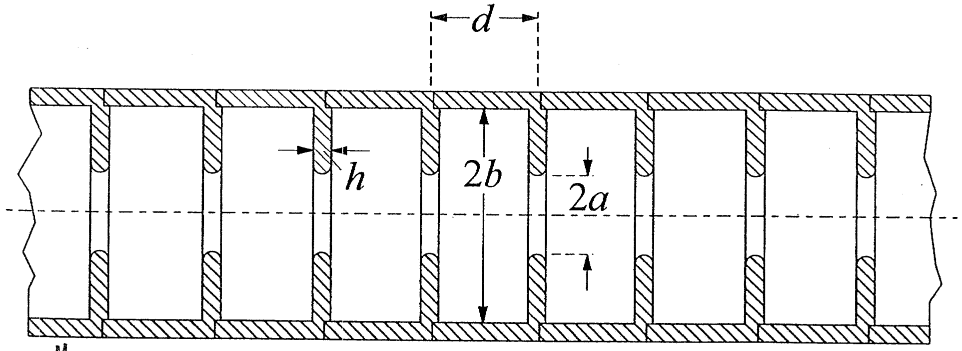


Fig. 2.8. Disk loaded accelerating structure for an electron linear accelerator (schematic)
Wiedemann



Wille
Fig. 5.6 Cross-section through a typical linac structure. The phase velocity of the RF wave is reduced to the particle velocity by the insertion of irises.

Why not use a simple waveguide TM mode to have a longitudinal E_z resonate with beam for acceleration?

Phase velocity of waveguide modes $> c$ so cannot maintain resonance.

But can add periodic structure in waveguide to slow wave and maintain resonance.

- Structure essentially sets up small coupled cavities with part reflections.
- periodic lattice of disks filling cylindrical waveguide commonly used. Example: SLAC electron linac.

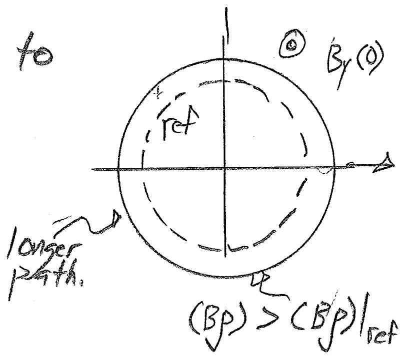
Some aspects will be discussed more later:
- Waveguide modes.
- Traveling wave field to calculate energy gain.

RF Linear (LINAC) Acceleration

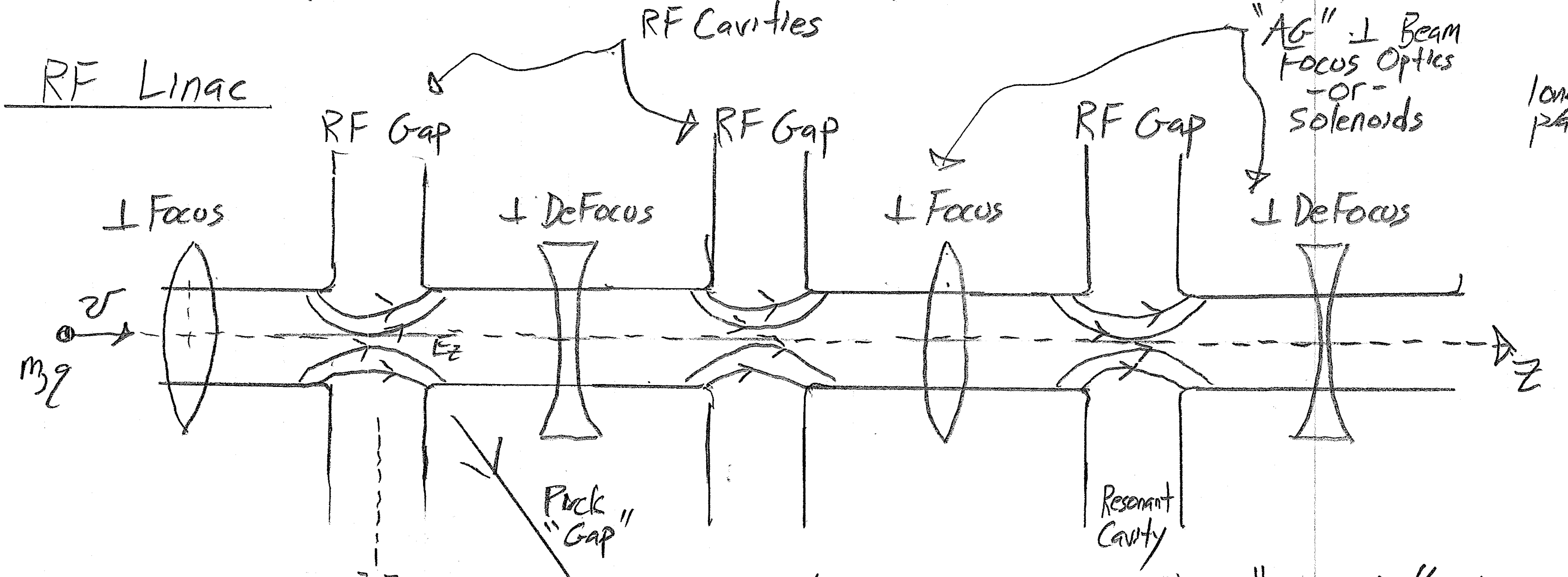
Will follow Wangler, "RF Linear Accelerators" + info from other sources cited. and Barnard & Lund USPAS

We will first cover RF linacs and then modify the formulation to a form appropriate for rings.

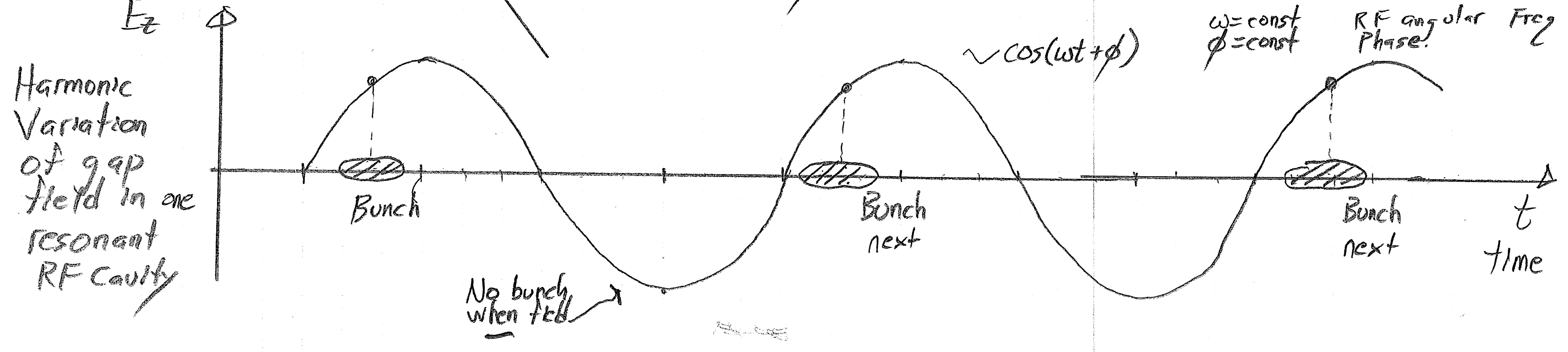
★ Rings require modification of synchronism conditions due to longer path length for larger particle rigidity $(B\rho)$



RF Linac



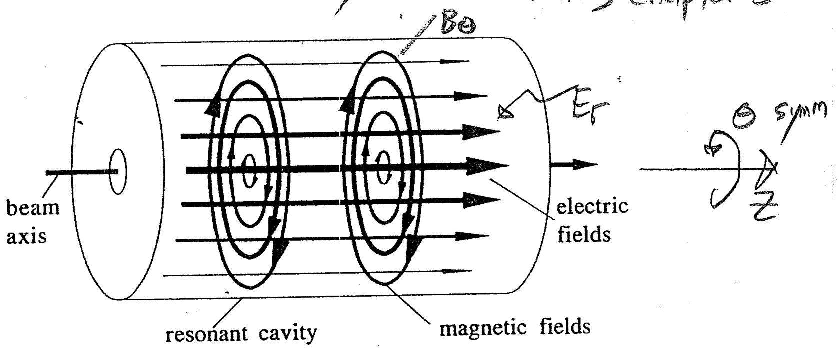
Gap Field in Cavity: all RF "buckets" filled



RF Cavity Fields

"Pillbox" Cavity

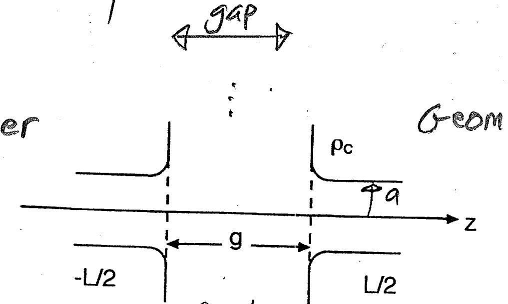
Wangler, §1.2
Conte & Mackay, Chapter 9
Wiedemann, §2.2.2
Wille, Chapter 5



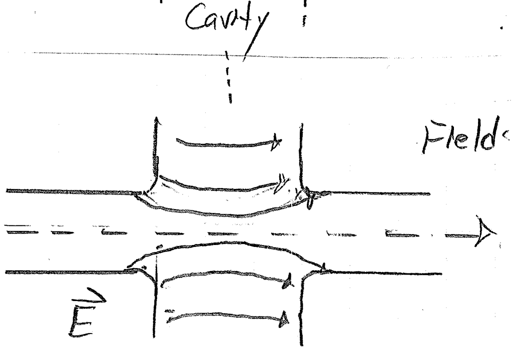
Wiedemann

Structures within the cavity often concentrate the field in an "RF Gap"

Wangler



Geometry



Field

Simplist Case:

Cavity excited harmonically with a lowest order transverse magnetic mode that primarily generates a longitudinal E_z for beam acceleration when particles transit at the right phase.

TM_{010} mode shown

- ☆ Cavities may be coupled (high β) or independently driven (low β) with appropriate phase control.
- ☆ More details on cavities later.

Harmonic TM_{010} Fields

ϕ = const RF phase
ω = const RF angular velocity

$$E_z(r,z,t) = E_z(r,z)\cos(\omega t+\phi)$$

$$B_\theta(r,z,t) = B_\theta(r,z)\sin(\omega t+\phi)$$ out of phase by $\pi/2$.

Allowed Due to finite beam hole + gap structures $\rightarrow$

$$E_r(r,z,t) = E_r(r,z)\cos(\omega t+\phi)$$

- $E_r = 0$ possible if no beam apeture.
- Need apeture for beam enter/exit.

Will discuss cavity fields more later, but for moment motivate form is OK....

Within cavity (vacuum region)

1) $\nabla\cdot\vec{E} = \cancel{\frac{\rho}{\epsilon_0}}^{\,0}$

2) $\nabla\times\vec{B} = \cancel{\mu_0\vec{J}}^{\,0} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

3) $\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$

4) $\nabla\cdot\vec{B} = 0$

$$E_z(r,z,t) = E_z(r,z)\cos(\omega t+\phi)$$

$$B_\theta(r,z,t) = B_\theta(r,z)\sin(\omega t+\phi)$$

$$E_r(r,z,t) = E_r(r,z)\cos(\omega t+\phi)$$

1) $\nabla\cdot\vec{E} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r E_r(r,z)\right) + \frac{\partial E_z(r,z)}{\partial z}\right]\cos(\omega t+\phi) = 0$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r E_r(r,z)\right) + \frac{\partial E_z(r,z)}{\partial z} = 0 \qquad A)$$

2) $\nabla\times\vec{B} = \left[-\frac{\partial B_\theta(r,z)}{\partial z}\hat{r} + 0\,\hat{\theta} + \frac{1}{r}\frac{\partial}{\partial r}\left(r B_\theta(r,z)\right)\hat{z}\right]\sin(\omega t+\phi)$

$$= \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \frac{-\omega}{c^2}\left[E_r(r,z)\,\hat{r} + E_z(r,z)\,\hat{z}\right]\sin(\omega t+\phi)$$

$$\Rightarrow \hat{r}:\ -\frac{\partial B_\theta(r,z)}{\partial z} = \frac{-\omega}{c^2}E_r(r,z) \quad B) \qquad \hat{z}:\ \frac{1}{r}\frac{\partial}{\partial r}\left(r B_\theta(r,z)\right) = \frac{-\omega}{c^2}E_z(r,z) \quad C)$$

3) $\nabla\times\vec{E} = \left[0\,\hat{r} + \left(\frac{\partial E_r(r,z)}{\partial z} - \frac{\partial E_z(r,z)}{\partial r}\right)\hat{\theta} + 0\,\hat{z}\right]\cos(\omega t+\phi)$

$$= -\frac{\partial}{\partial t}\vec{B} = -\omega\, B_\theta(r,z)\cos(\omega t+\phi)$$

$$\Rightarrow \frac{\partial E_r(r,z)}{\partial z} - \frac{\partial E_z(r,z)}{\partial r} = -\omega B_\theta(r,z) \qquad D)$$

4) $\nabla\cdot\vec{B} = \frac{1}{r}\frac{\partial}{\partial\theta}B_\theta(r,z,t) = 0$ ✓ Satisfied

Cavity Field Constraints

Maxwell Eqns reduce to 4 equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r E_r(r,z)\right) + \frac{\partial E_z(r,z)}{\partial z} = 0 \qquad A)$$

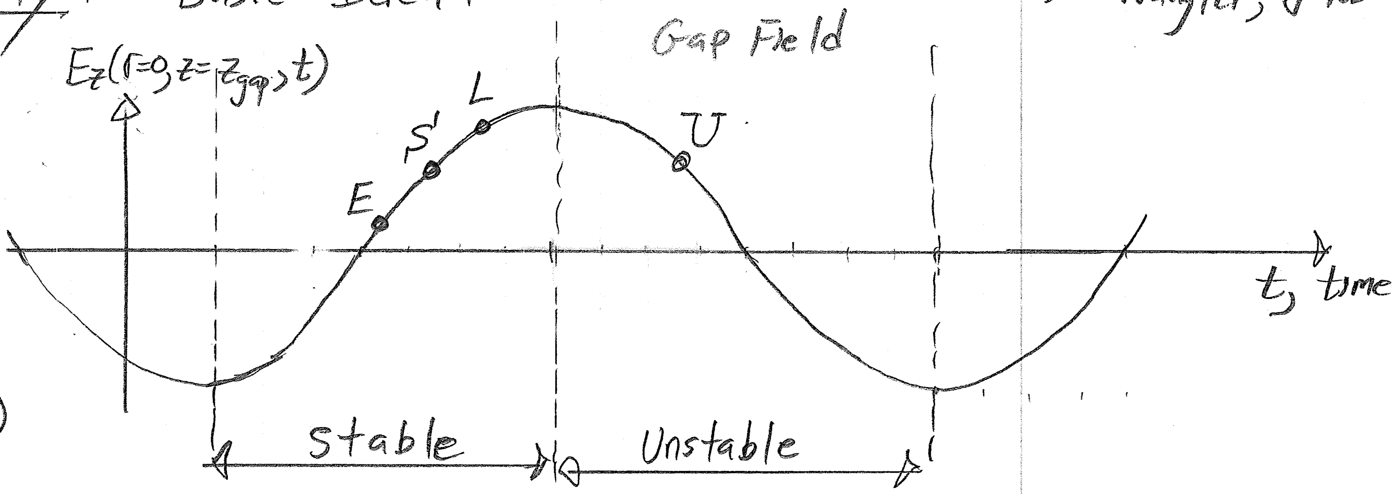
$$\frac{\partial B_\theta(r,z)}{\partial z} = \frac{\omega}{c^2}E_r(r,z) \quad B) \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r B_\theta(r,z)\right) = \frac{-\omega}{c^2}E_z(r,z) \quad C)$$

$$\frac{\partial E_r(r,z)}{\partial z} - \frac{\partial E_z(r,z)}{\partial r} = -\omega B_\theta(r,z) \qquad D)$$

Phase Stability: Basic Idea.

see Wangler, §1.3

Sketch for $q > 0$ (ions)
Easy to modify for $q < 0$
(electrons, neg ions)



Stable on rising E_z-field $\partial E_z/\partial t > 0$; longitudinal focusing

S' : "Synchronous" Particle : Will reach next gap at design time to same position on RF wave.

E : Early : More energetic particle arrives early
E_z lower ⇒ less energy gain ⇒ smaller v increase
⇒ moves toward S' at next gap.

L : Late : Less energetic particle arrives late
E_z higher ⇒ more energy gain ⇒ larger v increase
⇒ moves toward S' at next gap

Unstable on falling E_z-field $\partial E_z/\partial t < 0$, longitudinal defocusing

Cases reversed : early / late will move away from any design particle choice at next gap.

Particle Dynamics in Gap

see Wangler's Chapter 2

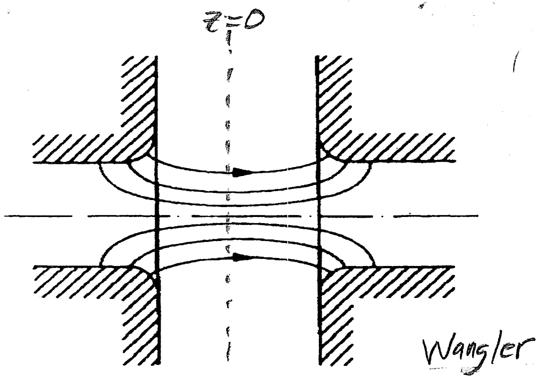


Figure 1.9 Electric-field lines in an accelerating gap.

RF Gap Fields

TM010 - like excitation

$$E_z(r,z,t) = E_z(r,z)\cos(\omega t+\phi)$$

$$B_\theta(r,z,t) = B_\theta(r,z)\sin(\omega t+\phi)$$

$$E_r(r,z,t) = E_r(r,z)\cos(\omega t+\phi)$$

Lorentz Force

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v}\times\vec{B}$$

$$= qE_r\hat{r} + qE_z\hat{z} - qv_zB_\theta\hat{r} + qv_rB_\theta\hat{z}$$

$$\hat{z}: \quad \frac{dp_z}{dt} = qE_z(r,z)\cos(\omega t+\phi) + qv_rB_\theta(r,z)\sin(\omega t+\phi)$$

$$\hat{r}: \quad \frac{dp_r}{dt} = qE_r(r,z)\cos(\omega t+\phi) - qv_zB_\theta(r,z)\sin(\omega t+\phi) \quad + \quad F_r\Big|_{\text{Focusing Optics}}$$

$$\hat{\theta}: \quad \frac{dp_\theta}{dt} = 0$$

From ⊥ focusing elements

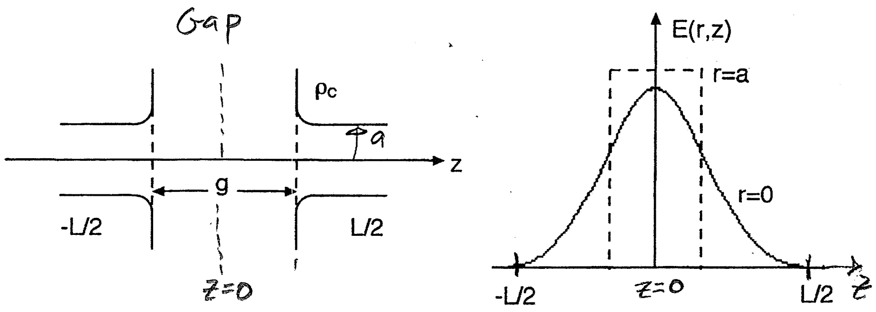
- We will return later to $\hat{r}$ equation: Cavity provides RF focus/defocus
- Examine longitudinal dynamics of $\hat{z}$ equation 1st.

Estimate the kinetic energy gain of a particle in the gap from the on-axis $(r=0)$ fields.

$$E_z(r=0, z, t) \equiv E(0, z)\cos(\omega t(z) + \phi)$$

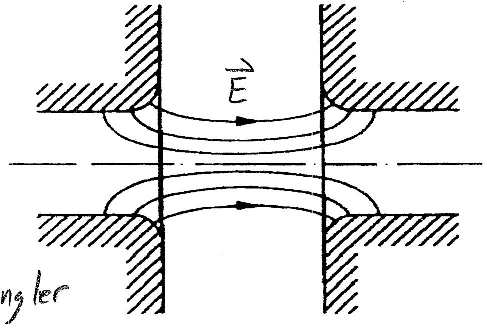
$$B_\theta(r=0, z, t) = 0 \leftarrow$$

Will find later $B_\theta \propto r/r_{cavity}$
$\Rightarrow$ B_θ small on axis of gap of small radial extent in cavity



Wangler

Figure 2.1 Gap geometry and field distribution.



Wangler

Figure 1.9 Electric-field lines in an accelerating gap.

Insert in equations of motion

$$t(z) = \int \frac{dz}{v_z(z)} + \text{const}$$

⋆ Ref: Particle at center of gap $(z=0)$ at time $t=0$.

⋆ $v_z \simeq v = |\vec{v}|$ Paraxial Approx

$$\boxed{t(z) = \int_0^z \frac{d\tilde{z}}{v(\tilde{z})}}$$

Note: At time $t=0$, ϕ is the phase of the E_z field relative to the peak value

$$E_z(r=0, z, t=0) = E(0, z)\cos\phi$$

In one gap examined.
But will vary ϕ
in other gaps to keep this relation true.

Usually use same value ϕ all cavities for reference particle.

Kinetic Energy Gain

See Wanglers, Chapter 2

$$\boxed{W = (\gamma - 1) mc^2} \equiv \text{Particle Kinetic Energy}$$

☆ Use W = kinetic energy in longitudinal dynamics to be consistent with usual notation.

Denote:

ΔW = KE gain through gap.

Denote $E_z(r=0,z) \equiv E(0,z)$

Comment: Use capital W for KE to later distinguish from another variable w.

$$\Delta W = \int_{gap} \vec{E} \cdot d\vec{l} = q\int_{-L/2}^{L/2} E_z(r=0,z,t)\,dz = q\int_{-L/2}^{L/2} E(0,z)\cos[\omega t(z) + \phi]\,dz$$

$$= q\int_{-L/2}^{L/2} E(0,z)\{\cos[\omega t(z)]\cos\phi - \sin[\omega t(z)]\sin\phi\}\,dz$$

L some axial length large enough to contain field

Express the energy gain as:

Notational Definitions:

$$\Delta W \equiv q V_0 T \cos\phi$$

$$V_0 \equiv \int_{-L/2}^{L/2} E(0,z)\,dz = \text{RF Voltage} \qquad [qV_0] = eV$$

$$T \equiv \frac{\int_{-L/2}^{L/2} E(0,z)\cos[\omega t(z)]\,dz}{\int_{-L/2}^{L/2} E(0,z)\,dz} - \tan\phi\,\frac{\int_{-L/2}^{L/2} E(0,z)\sin[\omega t(z)]\,dz}{\int_{-L/2}^{L/2} E(0,z)\,dz}$$

$\equiv$ Transit-Time Factor $\qquad [T] = 1$

Sometimes denote $\boxed{V_0 \equiv E_0 L}$ to define avg field E_0 over gap field extent L

☆ Important: Specify L used here or ambiguous!

↳ Gives: $E_0 = \frac{1}{L}\int_{-L/2}^{L/2} E(0,z)\,dz$

$$\Rightarrow \boxed{\Delta W = q E_0 L T \cos\phi = \text{Panofsky Equation}}$$

Panofsky eqn is deceptively simple appearing: contains much physics via T.

Transit Time

Wangler, §2.2

Much physics contained within the transit time factor T.

★ Time variation of field in gap always reduces energy gain relative to static case: for any RF phase ϕ.

- T provides normalized measure of reduction: $T = 1 \Rightarrow$ static

Wangler

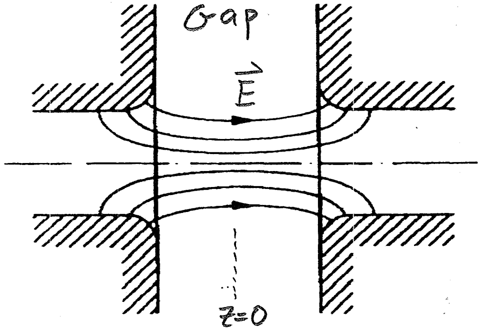


Figure 1.9 Electric-field lines in an accelerating gap.

$E(0,z)$ even function of z in typical gap.

$$\Rightarrow \int_{-L/2}^{L/2} E(0,z) \sin[\omega t(z)]\, dz \simeq 0$$

if change in v within gap negligible: $v =$ const

- Typical small fractional energy gain in cavity
- Weakest field near injector.

If $v \simeq$ const in gap:

$$t(z) = \int_0^z \frac{dz}{v} = \frac{z}{v} \quad \Rightarrow \quad \omega t(z) = \omega \frac{z}{v} = \frac{2\pi z}{v \tau_{rf}} = \frac{2\pi z}{\beta c \tau_{rf}}$$

$$= \frac{2\pi z}{\beta \lambda_{rf}}$$

$\lambda_{rf} \equiv c\tau_{rf} =$ RF Wavelength

Using this

$$T \equiv \frac{\int_{-L/2}^{L/2} E(0,z) \cos[\omega t(z)]\, dz}{\int_{-L/2}^{L/2} E(0,z)\, dz} - \tan\phi \frac{\int_{-L/2}^{L/2} E(0,z) \sin[\omega t(z)]\, dz}{\int_{-L/2}^{L/2} E(0,z)\, dz}$$

$$\boxed{T = \frac{\int_{-L/2}^{L/2} E(0,z) \cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right) dz}{\int_{-L/2}^{L/2} E(0,z)\, dz} - \tan\phi \frac{\int_{-L/2}^{L/2} E(0,z) \sin\left(\frac{2\pi z}{\beta\lambda_{rf}}\right) dz}{\int_{-L/2}^{L/2} E(0,z)\, dz}}$$

Transit time:

$$\boxed{T = \frac{\int_{-L/2}^{L/2} E(0,z)\cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)dz}{\int_{-L/2}^{L/2} E(0,z)dz} - \tan\phi\,\frac{\int_{-L/2}^{L/2} E(0,z)\sin\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)dz}{\int_{-L/2}^{L/2} E(0,z)dz}}$$

For "usual" cases of a symmetric field in the gap:

$$\int_{-L/2}^{L/2} E(0,z)\sin\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)dz \simeq 0$$

and the transit time reduces to

$$\boxed{T = \frac{\int_{-L/2}^{L/2} E(0,z)\cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)dz}{\int_{-L/2}^{L/2} E(0,z)dz}}$$

★ Most "usual" situation and many books define transit-time T using this formula.

- Go back to original definition in cases where it fails
- Corresponds to results in Yue Hao lectures.

Take a simple approximation for the gap field to illustrate T

Constant field in gap, zero outside.

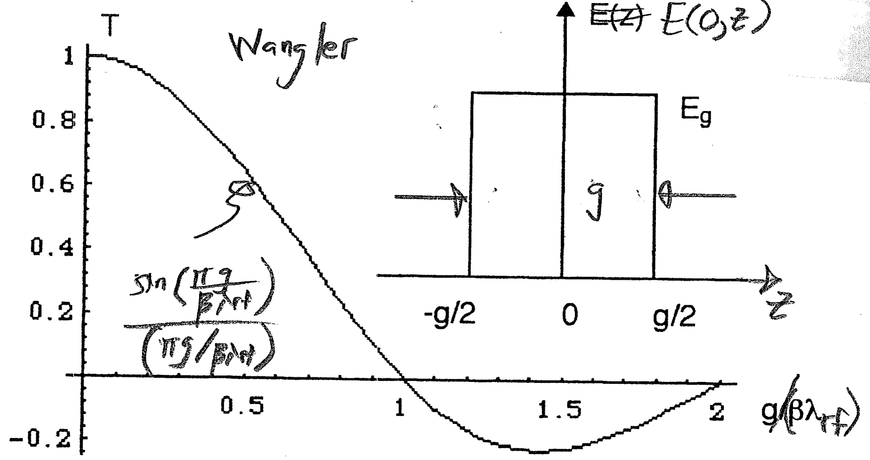


Figure 2.2 Transit-time factor for square-wave electric-field distribution.

☆ $T \to 1$ when $g \ll \beta\lambda_{rf}$

- Want short gap relative to $\beta\lambda_{rf}$ for efficient use of RF cavity accelerating potential.
- For electrons or very energetic protons/ions $\beta \simeq 1$ and want $g \ll \lambda_{rf}$. Approximation $v \simeq$ const in gap very good for $\beta \simeq 1$.

Take: $E(0,z) = E_g = \text{const}$

over $L = g$ and zero otherwise

Then:

$$T = \frac{\int_{-L/2}^{L/2} E(0,z)\cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)dz}{\int_{-L/2}^{L/2} E(0,z)dz}$$

$$= \frac{E_g \int_{-g/2}^{g/2} \cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)dz}{E_g\, g}$$

$$\boxed{T = \frac{\sin\left(\frac{\pi g}{\beta\lambda_{rf}}\right)}{\left(\frac{\pi g}{\beta\lambda_{rf}}\right)} = \text{sinc}\left(\frac{\pi g}{\beta\lambda_{rf}}\right)}$$

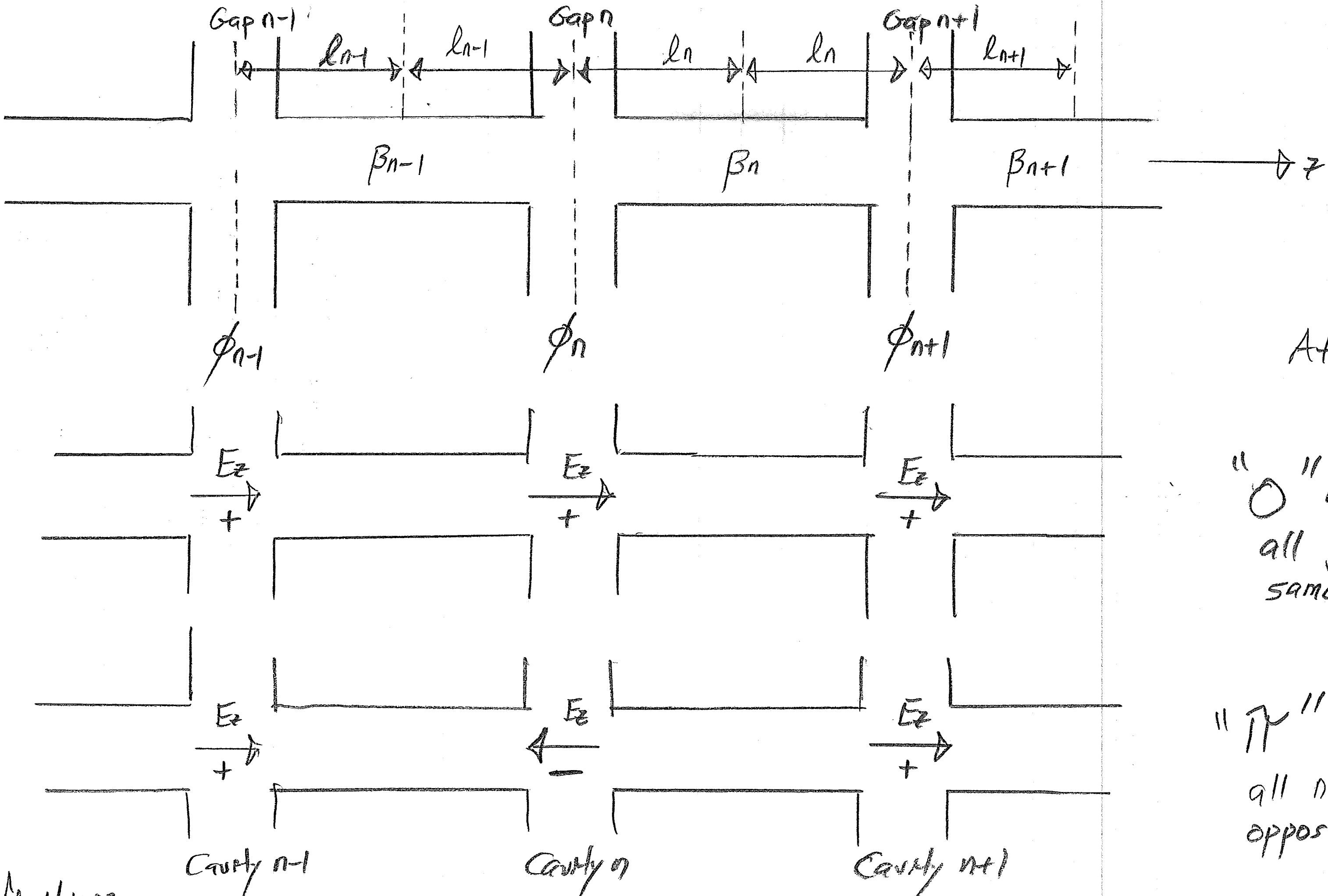
$$\text{sinc}\, x \equiv \frac{\sin x}{x}$$

Numerous expressions for T can be found in literature for a variety of cavities under a range of approximations and idealizations. For examples, see Wangler. Some cavities have 2 or more gaps that may be lumped into T.

Difference Equations for longitudinal motion in a standing-wave linac

Now have parts needed to analyze the longitudinal dynamics.

See Wangler, Chapter 6
Lund and Barnard, USPAS Notes



At time t:

"0" Mode
all gap E_z same direction

"π" Mode
all neighboring gap E_z opposite direction.

Definitions

$2l_n$ = distance from z-center nth gap to n+1'th gap

β_n = β after nth gap (constant between gaps)

W_n = kinetic energy after end nth gap (const between gaps)

ϕ_n = RF phase at z-center each gap.

Design Values

$\beta_{sn}, W_{sn}, \phi_{sn}$ = synchronous (design) particle values

Particle Phase

Transit time between gaps: $\Delta t\big|_{n-1\to n} = \frac{(2l_{n-1})}{\beta_{n-1}\cdot c}$

$\omega\,\Delta t\big|_{n-1\to n}$ = Advance RF phase as particle transits between gaps

For an arbitrary particle, then

$$\boxed{\phi_n = \phi_{n-1} + \frac{\omega(2l_{n-1})}{\beta_{n-1}c} + \begin{cases} 0 & \text{0-Mode} \\ \pi & \pi\text{-Mode} \end{cases}} \quad *)$$

For the synchronous particle:

$$\phi_{s,n} = \phi_{s,n-1} + \underbrace{\frac{\omega(2l_{n-1})}{\beta_{s,n-1}\cdot c} + \begin{cases} 0 & \text{0-Mode} \\ \pi & \pi\text{-Mode} \end{cases}}_{\text{Factor} = 2\pi}$$

For both cases so $\phi_{s,n} = \phi_{s,n-1}$ modulo 2π

$$\Rightarrow \quad (2l_{n-1})\frac{\omega}{\beta_{s,n-1}\cdot c} = \begin{cases} 2\pi & \text{0-Mode} \\ \pi & \pi\text{-Mode} \end{cases}$$

But $\frac{\omega}{c} = \frac{2\pi}{T_{rf}c} = \frac{2\pi}{\lambda_{rf}}$

$$\Rightarrow \quad \boxed{(2l_{n-1}) = \lambda_{rf}\,\beta_{s,n-1}\begin{cases} 1 & \text{0-Mode} \\ 1/2 & \pi\text{-Mode} \end{cases}}$$

Use this to eliminate the inter-gap length $(2l_{n-1})$ in * above:

$$\phi_n = \phi_{n-1} + \left(\frac{\omega\lambda_{rf}}{c}\right)_{\| 2\pi} \frac{\beta_{s,n-1}}{\beta_{n-1}} \cdot \begin{cases} 1 & \text{0-Mode} \\ 1/2 & \pi\text{-Mode} \end{cases} + \begin{cases} 0 & \text{0-Mode} \\ \pi & \pi\text{-Mode} \end{cases}$$

- 01

$$\boxed{\phi_n = \phi_{n-1} + 2\pi \frac{\beta_{s,n-1}}{\beta_{n-1}} \cdot \begin{cases} 1 & \text{0-Mode} \\ 1/2 & \pi\text{-Mode} \end{cases} + \begin{cases} 0 & \text{0-Mode} \\ \pi & \pi\text{-Mode} \end{cases}}$$

For synchronous particle, $\phi_n \to \phi_{s,n}$; $\beta_{n-1} \to \beta_{s,n-1}$ etc.

$$\phi_{s,n} = \phi_{s,n-1} + 2\pi \frac{\beta_{s,n-1}}{\beta_{s,n-1}} \begin{cases} 1 & \text{0-Mode} \\ 1/2 & \pi\text{-Mode} \end{cases} + \begin{cases} 0 & \text{0-Mode} \\ \pi & \pi\text{-Mode} \end{cases}$$

Subtract to measure phase change relative to the synchronous particle going from the n-1'th gap to the n'th gap as:

$$(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) = 2\pi\, \beta_{s,n-1} \left[\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right] \cdot \begin{cases} 1 & \text{0-Mode} \\ 1/2 & \pi\text{-Mode} \end{cases}$$

Denote $\boxed{N \equiv \begin{cases} 1 & \text{0-Mode} \\ 1/2 & \pi\text{-Mode} \end{cases}}$

Giving

$$\boxed{(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) = 2\pi N\, \beta_{s,n-1} \left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right)} \quad *$$

Phase change rel. to sync particle n-1'th gap to n'th gap.

But denoting $\Delta(\text{Measure}) = (\text{Measure}) - (\text{Measure})_s$ $s \sim$ synchronous $\longrightarrow$ $\boxed{\begin{array}{l} \Delta\beta_n = \beta_n - \beta_{s,n} \\ \Delta W_n = W_n - W_{s,n} \end{array}}$

$$\beta_{s,n-1}\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right) = -\left[1 - \frac{\beta_{s,n-1}}{\beta_{n-1}} \right] = -\left[\frac{\beta_{n-1} - \beta_{s,n-1}}{\beta_{n-1}} \right] = -\frac{\Delta\beta_{n-1}}{\beta_{s,n-1} + \Delta\beta_{n-1}} \simeq -\frac{\Delta\beta_{n-1}}{\beta_{s,n-1}}$$

A)

Vary W

$$W = (\gamma - 1) mc^2 \qquad \Delta W = \Delta\gamma\, mc^2 = (1-\beta^2)^{-3/2} \beta_s \Delta\beta\, mc^2 = \gamma_s^3 \beta_s\, mc^2 \Delta\beta$$

$$\gamma = (1-\beta^2)^{-1/2} \qquad \Rightarrow \quad \boxed{\Delta W = \gamma_s^3 \beta_s mc^2 \Delta\beta}$$

Label $\Delta \overline{W}$ at the n-1'th gap consistent using $\Delta \overline{W} \simeq \gamma_s^3 \beta_s mc^2 \Delta\beta$

$$\boxed{\Delta \overline{W}_{n-1} \equiv \overline{W}_{n-1} - \overline{W}_{s,n-1} \simeq \gamma_{s,n-1}^3 \beta_{s,n-1} mc^2 \Delta\beta_{n-1}} \qquad B)$$

Using these, Equation $*$ for the phase becomes:

$$(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) = 2\pi N \boxed{\beta_{s,n-1}\left[\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}}\right]}$$

use A) on pg 25/ $= -\dfrac{\Delta\beta_{n-1}}{\beta_{s,n-1}}$

$$\Delta\phi_n - \Delta\phi_{n-1} = -2\pi N \frac{\Delta\beta_{n-1}}{\beta_{s,n-1}}$$

$$= -\frac{2\pi N}{\gamma^3_{s,n-1}\beta^2_{s,n-1}}\left(\frac{\Delta\overline{W}_{n-1}}{mc^2}\right)$$

$$\Delta\beta_{n-1} = \frac{\Delta \overline{W}_{n-1}}{\gamma^3_{s,n-1}\beta_{s,n-1} mc^2} \qquad B)$$

-or-

$$\boxed{\Delta\phi_n - \Delta\phi_{n-1} = \frac{-2\pi N}{\gamma^3_{s,n-1}\beta^2_{s,n-1}}\left(\frac{\Delta\overline{W}_{n-1}}{mc^2}\right)} \text{---} (1)$$

$\Delta\phi_n = \phi_n - \phi_{s,n}$

$\Delta\overline{W}_n = \overline{W}_n - \overline{W}_{s,n}$

Next apply Panofsky's equation $\Delta W = qE_0 LT\cos\phi$ to the nth gap

$$\boxed{\overline{W}_n - \overline{W}_{n-1} = qE_{0,n}L_n T_n(\beta_n)\cos\phi_n} \qquad C)$$

$T_n = T_n(\beta_n)$ ← func of β_n

For the synchronous particle: $\overline{W}_n \to \overline{W}_{s,n}$ etc giving:

$$\boxed{\overline{W}_{s,n} - \overline{W}_{s,n-1} = qE_{0,n}L_n T_n(\beta_{s,n})\cos\phi_{s,n}} \qquad D)$$

$*$ Taking picture of $\beta \simeq$ const in gap here.

Subtract C) and D) for an energy gain equation

$$(W_n - W_{s,n}) - (W_{n-1} - W_{s,n-1}) = qE_{0,n}L_n\left[T_n(\beta_n)\cos\phi_n - T_n(\beta_{s,n})\cos\phi_{s,n}\right]$$

But, expect that $T_n(\beta_{s,n}) \simeq T_n(\beta_n)$

★ Little variation in T for small changes in β for usual applications.

This Gives:

$$\boxed{\Delta W_n - \Delta W_{n-1} = qE_{0,n}L_n \cdot T_n(\beta_{s,n})\left[\cos\phi_n - \cos\phi_{s,n}\right]} \quad -(2)$$

Summary 1), and 2) + energy gain equation for synchronous particle form a closed system describing the particle evolution in phase-energy phase space.

★ Nonlinearly coupled <u>differance equations</u>

★ Solve numerically for initial values of ϕ_n, ΔW_n

★ Advance synchronous particle also to calculate $\gamma_{s,n}\,\beta_{s,n}$, $T_n(\beta_{s,n})$.

$$\boxed{\begin{aligned} \Delta\phi_n - \Delta\phi_{n-1} &= \frac{-2\pi N}{\gamma_{s,n-1}^3\beta_{s,n-1}^2}\left(\frac{\Delta W_{n-1}}{mc^2}\right) && 1) \\ \Delta W_n - \Delta W_{n-1} &= qE_{0,n}L_n T_n(\beta_{s,n})\left[\cos\phi_n - \cos\phi_{s,n}\right] && 2) \end{aligned}}$$

$$N = \begin{cases} 1 & \text{"0" Mode} \\ \tfrac{1}{2} & \text{"}\pi\text{" Mode} \end{cases}$$

Summary

$$(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) = \frac{-2\pi N}{\gamma^3_{s,n-1}\,\beta^2_{s,n-1}} \frac{\Delta \overline{W}_{n-1}}{mc^2}$$ ~ phase dev.

$\Delta\overline{W}_n = \overline{W}_n - \overline{W}_{s,n}$

$N = \begin{cases} 1 & \text{0-Mode} \\ \frac{1}{2} & \pi\text{-Mode} \end{cases}$

$$\Delta\overline{W}_n - \Delta\overline{W}_{n-1} = qE_{0,n}L_n T_n(\beta_{s,n})\left[\cos\phi_n - \cos\phi_{s,n}\right]$$ ~ energy dev.

$$\overline{W}_{s,n} - \overline{W}_{s,n-1} = qE_{0,n}L_n T(\beta_{s,n})\cos\phi_{s,n}$$ ~ sync. energy gain

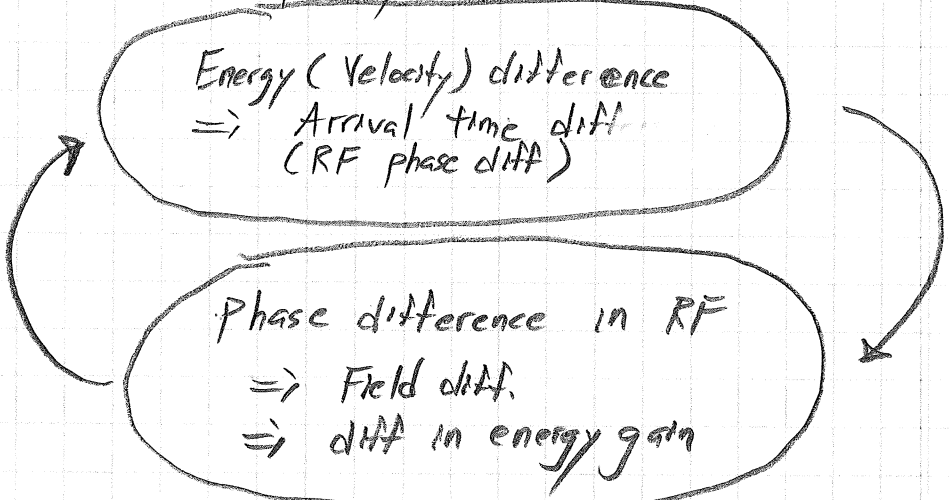
(*)

★ Easy to solve (*) on computer to study phase stability about synchronous particle in terms of evolution of ϕ and $\Delta\overline{W}$.

- Solve for specified initial values ϕ_n, $\Delta\overline{W}_n$

★ Also analyze later in "continuous" approx when cavity changes small.

Graphically



Avg Accel Gradient

On the equation for ΔW_n :

Need to connect $\underset{\text{Panofsky Eq'n}}{L_n}$ to $\underset{\text{Derivation diff eqns}}{l_n}$

$$E_{0n} \cdot L_n = V_{0n} = \int_{-L_n/2}^{L_n/2} E(0,z)\Big|_{n\text{'th gap}} dz$$

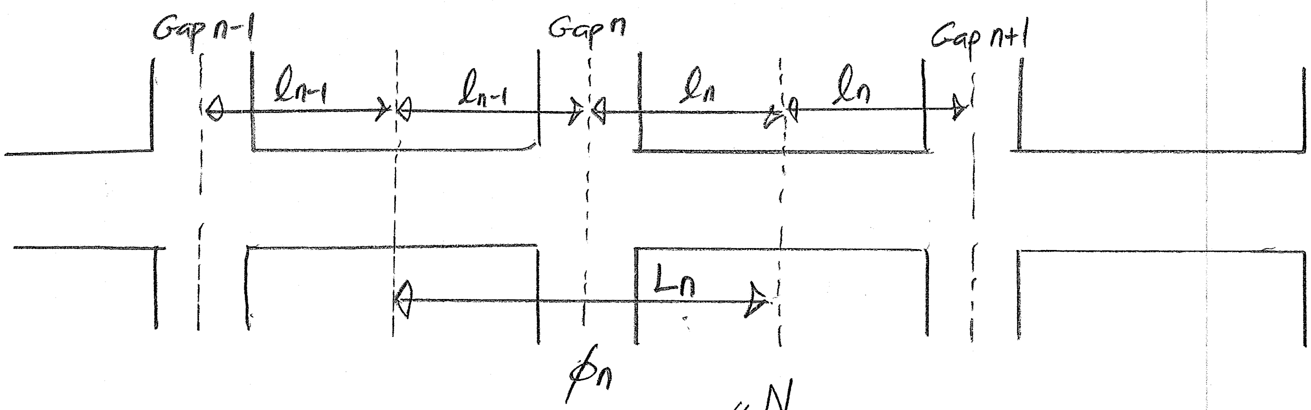
V_{0n} = Accel potential n'th gap

L_n = Length containing full gap fringe field.

E_{0n} = Avg. E-field over gap extent defined by $z_{gap} \pm L_n/2$.

Sometimes, one defines E_{0n} over a length L_n about the n'th gap mid-way between neighboring gaps upstream and downstream.

* Convenient to define avg gradient over "cell" length L_n in a periodic or quasi-periodic lattice.



$$\text{But } (2l_{n-1}) = \beta_{s,n-1}\lambda_{rf} \overset{''\,N}{\begin{cases} 1 & \text{0-Mode} \\ 1/2 & \pi\text{-Mode} \end{cases}} = N\beta_{s,n-1}\lambda_{rf} \Rightarrow \begin{aligned} l_{n-1} &= \frac{N}{2}\beta_{s,n-1}\lambda_{rf} \\ l_n &= \frac{N}{2}\beta_{s,n}\lambda_{rf} \end{aligned}$$

$$\Rightarrow \boxed{L_n = l_{n-1} + l_n = N(\beta_{s,n-1} + \beta_{s,n})\frac{\lambda_{rf}}{2}}$$

This should safely contain the gap fringe extent and define E_{0n} naturally as the average gradient in the cell.

Need care to consistently apply gradient specifications of RF cavities.

Continuous Differential Equations to Model Longitudinal Dynamics

See: Wangler §6.3
Lund and Barnard USPAS notes.

Derived "kick" difference equations to model longitudinal dynamics about the synchronous particle:

$$\boxed{\begin{aligned} &(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) = \frac{-2\pi N}{\gamma^3_{s,n-1}} \frac{\Delta W_{n-1}}{\beta^2_{s,n-1}\, mc^2} \qquad \Delta W_n = W_n - W_{s,n} \\ &\Delta W_n - \Delta W_{n-1} = q E_{0,n} L_n T_n(\beta_{sn}) \left[\cos\phi_n - \cos\phi_{ssn}\right] \end{aligned}}$$

For small gap-to-gap changes, replace discrete kicks by a continuous variation / field.

$$(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) \longrightarrow \frac{d(\phi - \phi_s)}{dn} \qquad \phi_n \to \phi \quad \phi_{s,n} \to \phi_s$$

$$\Delta W_n - \Delta W_{n-1} \longrightarrow \frac{d\,\Delta W}{dn} \qquad \Delta W_n \to \Delta W$$

☆ Treat n as continuous.

Convert from gap index n to axial coordinate s as an independent variable

$$\boxed{\begin{aligned} n &= \frac{(s - s_0)}{N \beta_s \lambda_{rf}} \\ &\equiv \frac{s}{N\beta_s \lambda_{rf}} \end{aligned}}$$

s_0 = axial position of nth gap along reference trajectory

For notational simplicity

$$\Rightarrow \boxed{\frac{d}{dn} = N \beta_s \lambda_{rf} \frac{d}{ds}}$$

☆ Using sync. particle to define coord. s.

Then, the differance eqns

$$\left\{ \begin{aligned} (\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) &= \frac{-2\pi N}{\gamma^3_{s,n-1}} \frac{\Delta W_{n-1}}{\beta^2_{s,n-1}\, mc^2} \\ \Delta W_n - \Delta W_{n-1} &= q E_{0,n} L_n \cdot T_n(\beta_n) \left[\cos\phi_n - \cos\phi_{s,n} \right] \end{aligned} \right.$$

Become:

$$\boxed{\begin{aligned} N \beta_s \lambda_{rf} \frac{d}{ds} (\phi - \phi_s) &= \frac{-2\pi N}{\gamma_s^3 \beta_s^2} \frac{\Delta W}{mc^2} \qquad \Delta W = W - W_s \\ N \beta_s \lambda_{rf} \frac{d}{ds} \Delta W &= q E_0(s)\, T(\beta_s(s))\, L(s) \left[\cos\phi - \cos\phi_s \right] \end{aligned}} \quad 1)$$

★ Take $L_n = L(s)$ with (usually) $L(s) = \text{const.}$

Also, the synchronous particle equation must also be integrated for the gain in energy for the γ_s, β_s factors etc.

$$W_n - W_{n-1} = q E_{0,n} L_n \cdot T_n(\beta_{s,n}) \cos\phi_{s,n}$$

Becomes

$$\boxed{N \beta_s \lambda_{rf} \frac{d}{ds} W_s = q E_0(s)\, T(\beta_s(s))\, L(s) \cos\phi_s} \quad 2)$$

1) and 2) can be analyzed for the longitudinal dynamics of a particle evolving through many small cavity "kicks" smeared out into a continuously acting force.

★ Should work well to understand and in many applications (especially rings).

For simplicity; denote

$$\left.\begin{aligned} E_0(s) &= E_0 = \text{const} \\ T(\beta_s(s)) &= T = \text{const} \\ L(s) &\equiv L = \text{const} \end{aligned}\right\} \text{Constants in a periodic lattice.}$$

Then Eq. 1) becomes:

$$\Rightarrow \quad (\gamma_s \beta_s)^3 \frac{d}{ds}(\phi - \phi_s) = -\frac{2\pi}{\lambda_{rf}} \left(\frac{\Delta W}{mc^2}\right) \qquad \Delta W = W - W_s$$

$$\frac{d}{ds}\Delta W = q E_0 T \left(\frac{L}{N \beta_s \lambda_{rf}}\right)(\cos\phi - \cos\phi_s)$$

$$L = N(\beta_{s,n-1} + \beta_{s,n})\frac{\lambda_{rf}}{2} \Rightarrow \frac{L}{N \beta_s \lambda_{rf}} = 1$$

provided we take L to be the cell spacing: in this context E_0 is the avg. gradient over the cell length.

Giving

$$\boxed{\begin{aligned} (\gamma_s \beta_s)^3 \frac{d}{ds}(\phi - \phi_s) &= \frac{-2\pi}{\lambda_{rf}}\left(\frac{\Delta W}{mc^2}\right) \\ \frac{d}{ds}\Delta W &= q E_0 T (\cos\phi - \cos\phi_s) \end{aligned}} \quad (*)$$

Notes: ★ N has been eliminated. Same formulas for 0-Mode and π-Mode

★ Nonlinear equations

These can be combined to eliminate W-Ws as:

$$\boxed{\frac{d}{ds}\left[(\gamma_s \beta_s)^3 \frac{d}{ds}(\phi - \phi_s)\right] = -\frac{2\pi}{\lambda_{rf}} \frac{q E_0 T}{mc^2}(\cos\phi - \cos\phi_s)}$$

★ 2nd order nonlinear equation for evolution of $\phi(s)$ from initial values $\phi(s_i)$, $\frac{d\phi}{ds}(s_i) = \phi'(s_i)$

Small Amplitude Phase Excursions

see Wangler §6.6, Lund and Barnard, USPAS notes

$$\boxed{\begin{gathered}\frac{d}{ds}\left((\gamma_s\beta_s)^3 \frac{d}{ds}\Delta\phi \right) = \frac{-2\pi}{\lambda_{rf}} \frac{qE_0T}{mc^2}\left[\cos(\phi_s+\Delta\phi) - \cos\phi_s \right] \\ \Delta\phi = \phi - \phi_s \end{gathered}}$$

Nonlinear Phase evolution Equation

Assume:

$\gamma_s\beta_s$ ~ varies slowly ⇒ pull through $\frac{d}{ds}$

$|\Delta\phi| << 1$ ⇒ small phase excursions about synchronous particle.

Then:

$$\cos(\phi_s+\Delta\phi) = \cos\phi_s \underbrace{\cos\Delta\phi}_{\simeq 1 + \mathcal{O}(\Delta\phi^2)} - \sin\phi_s \underbrace{\sin\Delta\phi}_{\simeq \Delta\phi + \mathcal{O}(\Delta\phi^3)}$$

$$\simeq \cos\phi_s - \sin\phi_s\,\Delta\phi + \mathcal{O}(\Delta\phi^2)$$

To obtain:

$$\boxed{\begin{gathered}\frac{d^2\Delta\phi}{ds^2} + k_s^2\,\Delta\phi = 0 \\ k_s \equiv \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{qE_0T \sin(-\phi_s)}{mc^2(\beta_s\gamma_s)^3}} = \text{Synchrotron Wavenumber}\end{gathered}}$$

Linear equation for small phase excursions about synchronous particle.

This implies for:

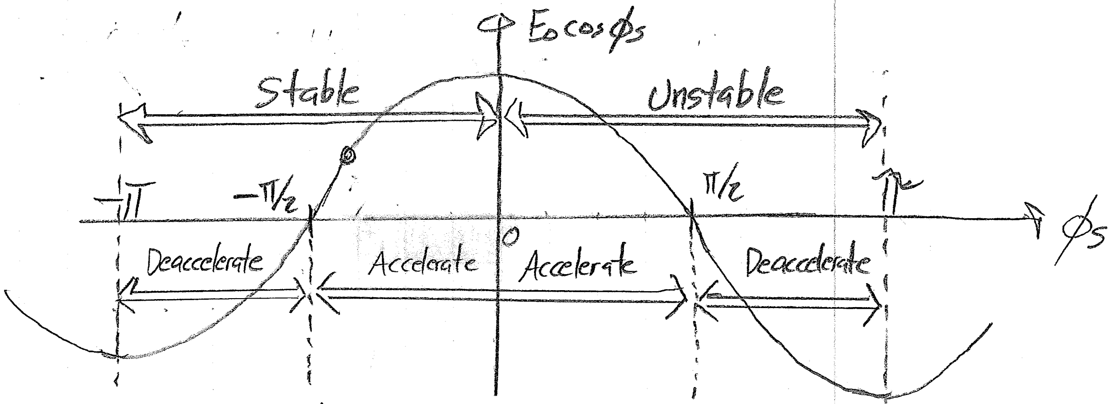
$-\pi < \phi_s < 0$ ⇒ $k_s^2 > 0$ Small amplitude oscillations about synchronous particle <u>Stable</u>

$0 < \phi_s < \pi$ ⇒ $k_s^2 < 0$ Small amplitude oscillations about synchronous particle <u>unstable</u>

Recall the phase is defined relative to the RF wave peak

$$\frac{dW_s}{ds} \sim qE_0 \cos\phi_s$$

$$k_s = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{qE_0T\sin(-\phi_s)}{mc^2(\gamma_s\beta_s)^3}}$$



Stable range ⇒ particle arrives at gap in rising field } consistent with
Unstable range ⇒ particle arrives at gap in falling field. } qualitative expectation

Particle accelerates and is stable for $-\frac{\pi}{2} < \phi_s < 0$

- ★ A commonly taken value of ϕ_s to accelerate with a reasonable phase width for stability (k_s large) is to take:

$$\phi_s \simeq -\frac{\pi}{6} = -30^\circ$$

⇒ Compromise: Accel strength + focusing phase width

- ★ If RF is used for beam bunching rather than acceleration, the strength of k_s is maximized by taking

$$\phi_s = -\frac{\pi}{2}$$

⇒ Bunching: Max focus strength.
But no acceleration.

- ★ If E_0T and ϕ_s remain nearly constant in acceleration:

$$k_s \sim \frac{1}{(\gamma_s\beta_s)^{3/2}}$$

Showing that synchrotron oscillations will <u>slow down</u> (weaker focusing) as the beam accelerates.

- Good intuitive sense: energetic particle more "ridgid"

The corresponding angular frequency to k_s is:

$$\boxed{\omega_s \equiv k_s(\beta c) \qquad \text{Synchrotron angular Freq,}}$$

$$k_s = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{qE_0T\sin(-\phi_s)}{mc^2(\gamma_s\beta_s)^3}}$$

Relative to the RF freq: $\beta \simeq \beta_s$

$$\frac{\omega_s}{\omega} = \frac{f_s}{f_{rf}} = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{\beta_s^2 c^2}{\omega^2} \frac{qE_0T\sin(-\phi_s)}{mc^2(\gamma_s\beta_s)^3}}$$

But $\frac{c}{\omega} = \frac{c\,T_{rf}}{2\pi} = \frac{\lambda_{rf}}{2\pi}$

$\omega = 2\pi f$

$\omega_s = 2\pi f_s$

$\omega = 2\pi f_{rf}$

$$\Rightarrow \boxed{\frac{\omega_s}{\omega} = \frac{f_s}{f_{rf}} = \sqrt{\frac{1}{2\pi(\gamma_s\beta_s)^3}\left(\frac{qE_0T\lambda_{rf}}{mc^2}\right)\sin(-\phi_s)}}$$

From this expect:

* $f_s \ll f_{rf}$ as beam becomes more relativistic,

The linear synchrotron equation of motion can be solved for $k_s = \text{const}$:

$$\frac{d^2\Delta\phi}{ds^2} + k_s^2\,\Delta\phi = 0$$

<u>Initial condition</u>

$$\Delta\phi(s=s_i) = \Delta\phi_i$$

$$\frac{d\Delta\phi}{ds}(s=s_i) = \Delta\phi_i'$$

$$' \equiv \frac{d}{ds}$$

<u>Solution:</u>

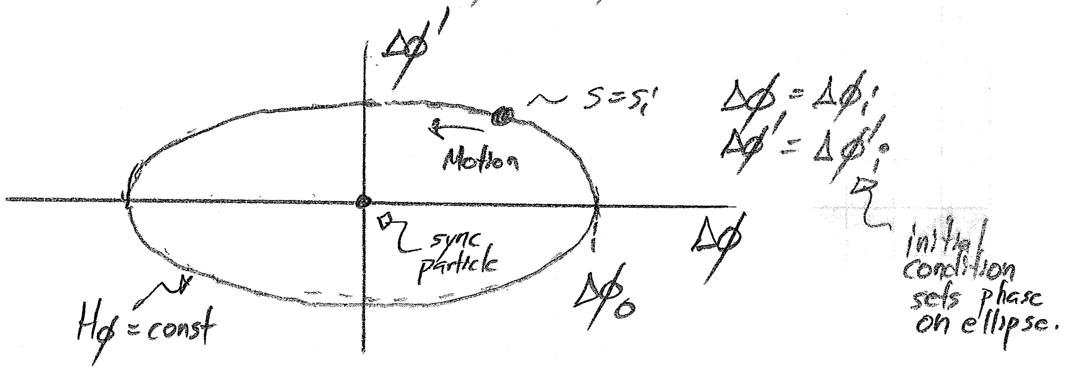
$$\Delta\phi(s) = \Delta\phi_i\cos[k_s(s-s_i)] + \frac{\Delta\phi_i'}{k_s}\sin[k_s(s-s_i)]$$

$$\Delta\phi'(s) = -\Delta\phi_i k_s\sin[k_s(s-s_i)] + \Delta\phi_i'\cos[k_s(s-s_i)]$$

<u>Conservation of Hamiltonian H:</u>

$$H_\phi = \frac{1}{2}(\Delta\phi')^2 + \frac{1}{2}k_s^2(\Delta\phi)^2 = \frac{1}{2}(\Delta\phi_i')^2 + \frac{1}{2}k_s^2(\Delta\phi_i)^2$$
$$= \text{const.}$$

Phase-space in $\Delta\phi - \Delta\phi'$ is an ellipse



When $k_s(s - s_i') = 2\pi$ particle cycles around ellipse.

Denote for convenience:

$\Delta\phi_0 =$ Max Phase excursion $\Rightarrow$ $H_\phi = \frac{1}{2} k_s^2 \Delta\phi_0^2$ since $\Delta\phi' = 0$ at max $\Delta\phi_0$

Then the Hamiltonian conservation is expressed as:

$$\boxed{H_\phi = \frac{1}{2}(\Delta\phi')^2 + \frac{1}{2} k_s^2 (\Delta\phi)^2 = \frac{1}{2} k_s^2 (\Delta\phi_0)^2 = \text{const}}$$

Notation Caution:

small W, capital W

$$w \equiv \frac{\Delta W}{mc^2}$$

Using this and:

$$(\gamma_s\beta_s)^3 \Delta\phi' = \frac{-2\pi}{\lambda_{rf}} \frac{\Delta W}{mc^2} \equiv \frac{-2\pi}{\lambda_{rf}} w$$

The ellipse becomes $\rightarrow$ $\frac{1}{2}(\Delta\phi)^2 = \frac{\pi}{2\lambda_{rf}^2} W^2$

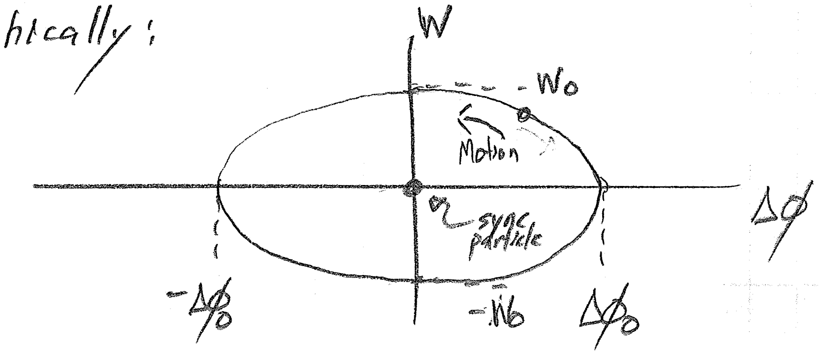
$$\left(\frac{W}{W_0}\right)^2 + \left(\frac{\Delta\phi}{\Delta\phi_0}\right)^2 = 1$$

$$W_0 = \frac{\lambda_{rf}}{2\pi}(\gamma_s\beta_s)^3 k_s \Delta\phi_0$$

$= \frac{1}{2}$ width norm energy deviation

$$= \sqrt{\frac{(\gamma_s\beta_s)^3}{2\pi}\left(\frac{qE_0T\lambda_{rf}}{mc^2}\right)\sin(-\phi_s)\,\Delta\phi_0^2}$$

Graphically:



The phase-space area of the ellipse is:

$$\boxed{\begin{aligned}\text{Area} &= \pi\left(\tfrac{1}{2}\text{-width}_{\text{angle}}\right)\left(\tfrac{1}{2}\,\text{width}_{W}\right)\\ &= \pi\,\Delta\phi_0\,W_0\\ &= \sqrt{\frac{\pi}{2}(\gamma_s\beta_s)^3\left(\frac{qE_0T\lambda_{rf}}{mc^2}\right)\sin(-\phi_s)}\;\Delta\phi_0^2\end{aligned}}$$

Many choices of longitudinal coordinates are employed to study longitudinal dynamics. Some include:

(coord, momentum)

phase-energy: $(\phi, \bar{W})$, $\bar{W} = (\gamma-1)mc^2$ = Kinetic Energy

or $(\Delta\phi, \Delta\bar{W})$ etc.

position-momentum: (z, p_z)

or $(\Delta z, \Delta p_z)$

time-energy $(t, \bar{W})$

or $(\Delta t, -\Delta\bar{W})$

⋮

Proper sets of canonical variables (perhaps rescaled by constants like mc^2) should be employed to measure phase-space areas. Canonical transforms can be applied to connect to other variable choices.

// Aside: Longitudinal Phase-Space Damping with Acceleration

Go back to DE: for $\gamma_s\beta_s \neq \text{const}$

$$\frac{d}{ds}\left((\gamma_s\beta_s)^3 \frac{d}{ds}\Delta\phi \right) = \frac{-2\pi}{\lambda_{rf}}\, q\frac{E_0 T}{mc^2}\left[\cos(\phi_s+\Delta\phi) - \cos\phi_s\right]$$

$$\Rightarrow \boxed{\frac{d^2\Delta\phi}{ds^2} + 3\frac{(\gamma_s\beta_s)'}{(\gamma_s\beta_s)}\frac{d\Delta\phi}{ds} = \frac{-2\pi}{\lambda_{rf}}\, q\frac{E_0 T}{mc^2\gamma_s^3\beta_s^3}\left[\cos(\phi_s+\Delta\phi) - \cos\phi_s\right]}$$

Analogy to Hill's eqn with Accel:

$$\underbrace{x''}_{\text{Inertial}} + \underbrace{\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x'}_{\text{Damping}} + \underbrace{\kappa_x x}_{\text{Focus}} = 0$$

So we expect term $\frac{3(\gamma_s\beta_s)'}{(\gamma_s\beta_s)}$ to induce NL damping, in longitudinal phase-space.

- * Factor 3 changes scale relative to ⊥ physics. – Faster damping.
- * RHS contains both linear ($|\Delta\phi| \ll 1$) and nonlinear restoring forces. when the RHS cannot be approximated by leading order terms.

//

Nonlinear Phase-Space Structure of RF Bucket.

See Wangler, §6.4
Lund and Barnard, USPAS notes.

Cannot use small phase excursion approximation to analyze. Return to nonlinear coupled equations:

$$\boxed{\begin{aligned} (\gamma_s\beta_s)^3 \frac{d\Delta\phi}{ds} &= -\frac{2\pi}{\lambda_{rf}} \frac{\Delta W}{mc^2} \qquad \Delta W = W - W_s \\ & \qquad\qquad\qquad\qquad\;\; \Delta\phi = \phi - \phi_s \\ \frac{d\Delta W}{ds} &= qE_0T[\cos\phi - \cos\phi_s] \end{aligned}}$$

Denote:

$$w \equiv \frac{\Delta W}{mc^2} \qquad A \equiv \frac{2\pi}{\lambda_{rf}(\gamma_s\beta_s)^3} \qquad B \equiv \frac{qE_0T}{mc^2}$$

$$\mathcal{W} = \frac{W}{mc^2}$$

$$\phi = \phi_s + \Delta\phi \quad \Rightarrow \quad \Delta\phi' = \phi'$$

since we take ϕ_s = const (simplicity)

$$' \equiv \frac{d}{ds}$$

Then the nonlinear equations can be expressed as:

$$\boxed{\begin{aligned} \phi' &= -Aw \\ w' &= B[\cos\phi - \cos\phi_s] \end{aligned}}$$

Assume that A and B vary weakly in s
* Likely need for continuous approx to hold

$$\Rightarrow \quad \phi'' = -Aw' = -AB[\cos\phi - \cos\phi_s]$$

$$\boxed{\phi'' = -AB(\cos\phi - \cos\phi_s)}$$

Multiply by ϕ' and integrate: $\phi'' = -AB(\cos\phi - \cos\phi_s)$

$$\phi'\phi'' = -AB(\cos\phi - \cos\phi_s)\phi'$$

$$\int \phi'\phi''\,ds = -AB\int(\cos\phi - \cos\phi_s)\phi'\,ds$$

$$\downarrow \quad \frac{1}{2}\int \frac{d}{ds}\phi'^2\,ds = -AB\int(\cos\phi - \cos\phi_s)\,d\phi \quad \nearrow \quad \boxed{\frac{\phi'^2}{2} + AB(\sin\phi - \phi\cos\phi_s) = \text{const.}}$$

Now use $\phi' = -AW$ and divide by A:

$$\boxed{\frac{AW^2}{2} + B(\sin\phi - \phi\cos\phi_s) = \text{const} \equiv H_\phi}$$

H_ϕ = Synchrotron Hamiltonian.

Analogy: $A\frac{W^2}{2} \Rightarrow$ Interpret as "Kinetic Energy"

$B(\sin\phi - \phi\cos\phi_s) \Rightarrow$ Interpret as "Potential Energy"

To exploit this analogy, denote:

$$V(\phi) \equiv B(\sin\phi - \phi\cos\phi_s) \quad \Rightarrow \quad H_\phi = A\frac{W^2}{2} + V(\phi) = \text{const}$$

$$\frac{\partial V(\phi)}{\partial\phi} = B(\cos\phi - \cos\phi_s) \quad \sim \text{Focus Strength}$$

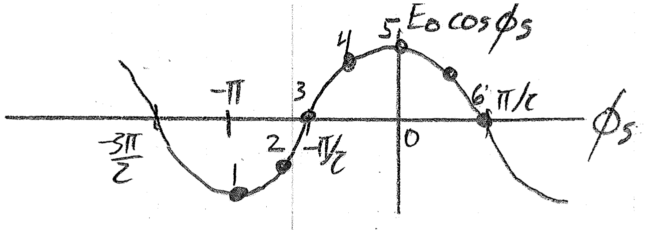
$$\frac{\partial^2 V(\phi)}{\partial\phi^2} = -B\sin\phi \quad \sim \text{Concavity}$$

Same result obtained in small phase excursion limit as should be expected.

Want for stability about synchronous particle:

$$\left.\frac{\partial^2 V(\phi)}{\partial\phi^2}\right|_{\phi=\phi_s} > 0 \quad \Rightarrow \quad -B\sin\phi_s > 0 \quad \Rightarrow \quad \boxed{\begin{array}{c}\sin\phi_s < 0 \\ \pi < \phi_s < 0 \\ \text{for stability}\end{array}}$$

$B > 0$ for $T > 0$



Plots of

$$V(\phi) = B(\sin\phi - \phi\cos\phi_s)$$

$$B = \frac{qE_0T}{mc^2} \geq 0$$

ϕ_s = various values

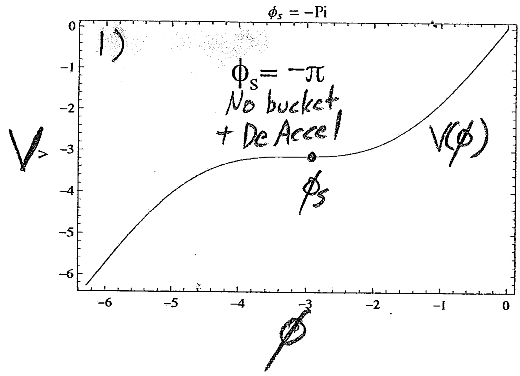
$-\pi < \phi_s < 0$: Stable

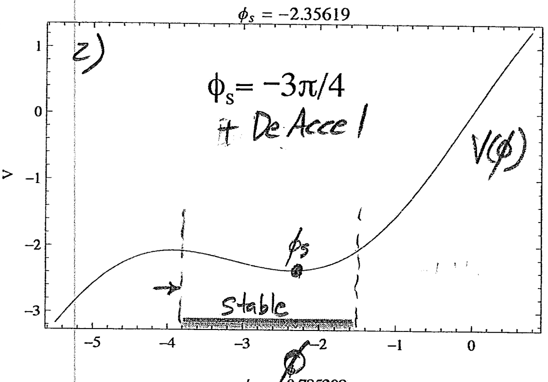
$-\pi/2 < \phi_s < \pi/2$: Accel

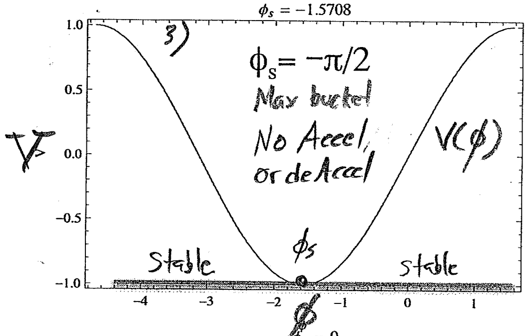
$\pi/2 < \phi_s < 0$ Accel and Focus.

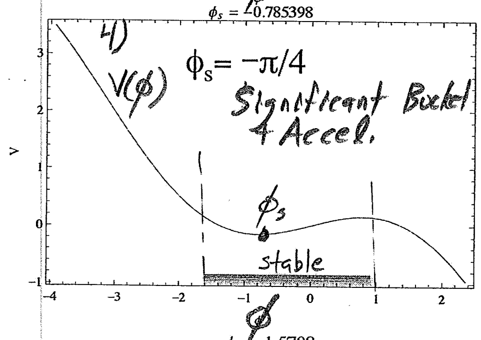
$\phi_s \simeq -30^\circ = -\pi/6$

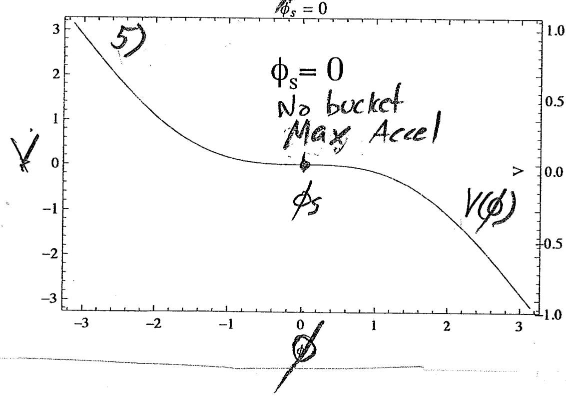
typical value

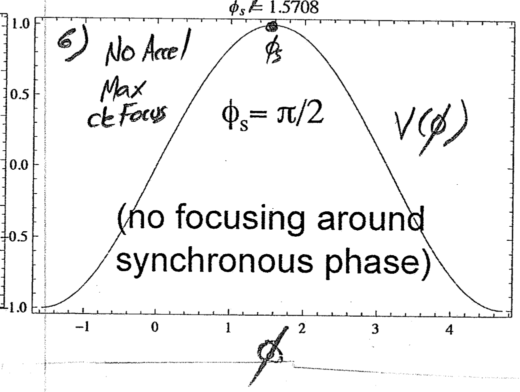












(no focusing around synchronous phase)

$$H_\phi = \frac{Aw^2}{2} + V(\phi)$$

$$V(\phi) = B(\sin\phi - \phi\cos\phi_s)$$

$$A = \frac{2\pi}{\lambda_{rf}} \frac{1}{(\gamma_s\beta_s)^3} > 0$$

$$B = \frac{qE_0T}{mc^2} > 0 \quad (\text{forward accel}) \quad T > 0$$

$H_\phi(w=0, \phi=\phi_s)$ = Stable Fixed Point $\quad \phi_s < 0$

$H_\phi(w=0, \phi=-\phi_s)$ = Unstable Fixed Point

Denote

$$H_\phi(w=0, \phi=\phi_2) = H_\phi(w=0, \phi=-\phi_s) \equiv H_\phi(-\phi_s)$$
$$= B[-\sin\phi_s + \phi_s\cos\phi_s]$$

<u>Seperatrix defining RF "Fish"</u> satisfy:

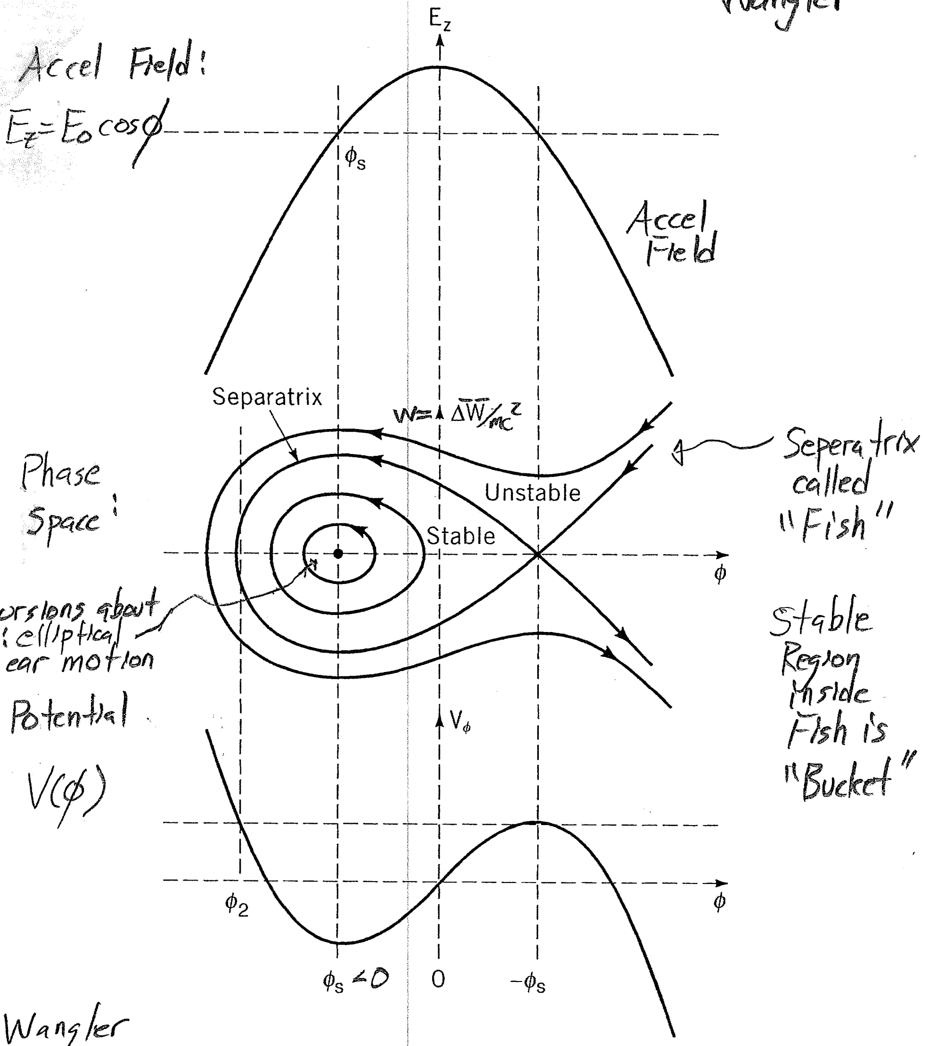
$$H_\phi' = H_\phi(-\phi_s) = H_\phi(w=0, \phi=-\phi_s)$$
$$= -B[\sin\phi_s - \phi_s\cos\phi_s]$$

$$\Rightarrow \boxed{\frac{Aw^2}{2} + B[\sin\phi - \phi\cos\phi_s] = -B[\sin\phi_s - \phi_s\cos\phi_s]}$$

Seperatrix Constraint

Gives stable Bocket

Wangler



For small excursions about sync particle: elliptical P.S. ⟺ linear motion

Wangler

Figure 6.3. At the top, the accelerating field is shown as a cosine function of the phase; the synchronous phase ϕ_s is shown as a negative number, which lies earlier than the crest where the field is rising in time. The middle plot shows some longitudinal phase-space trajectories, including the separatrix, the limiting stable trajectory, which passes through the unstable fixed point at $\Delta W = 0$, and $\phi = -\phi_s$. The stable fixed point lies at $\Delta W = 0$ and $\phi = \phi_s$, where the longitudinal potential well has its minimum, as shown in the bottom plot.

The total phase width of the seperatrix about the synchronous particle is:

$$\boxed{\begin{array}{l} \psi \equiv \text{phase width} = \underset{\substack{\parallel \\ \phi = -\phi_s \\ \text{Right X point}}}{|\phi_s|} + \underset{\substack{\uparrow \\ \text{Left turning point}}}{|\phi_2|} = -\phi_s - \phi_2 \quad \text{for } \begin{array}{l}\phi_s < 0 \\ \phi_2 < 0 \\ \text{also}\end{array} \end{array}}$$

From the seperatrix eqn:

$$H_\phi(\phi = \phi_2, w = 0) = H_\phi(-\phi_s) \qquad \cancel{B}[\sin\phi_2 - \phi_2\cos\phi_s] = -\cancel{B}[\sin\phi_s - \phi_s\cos\phi_s]$$

$$\Rightarrow \boxed{\sin\phi_2 - \phi_2\cos\phi_s = -[\sin\phi_s - \phi_s\cos\phi_s]} \; *$$

* Can be solved numerically for ϕ_2 to calculate the phase width ψ for a given value of ϕ_s.

Analyze phase width approximately:

$$\left.\begin{array}{l} \phi_2 = -\phi_s - \psi \\ \sin\phi_2 = -\sin(\phi_s + \psi) = -(\sin\phi_s\cos\psi + \sin\psi\cos\phi_s) \end{array}\right\} \text{substitute in seperatrix eqn } *$$

$$\sin\phi_s\cos\psi + \sin\psi\cos\phi_s \overset{①}{\cancel{-\phi_s\cos\phi_s}} - \psi\cos\phi_s = \sin\phi_s \overset{①}{\cancel{-\phi_s\cos\phi_s}}$$

$$\Rightarrow (\sin\psi - \psi)\cos\phi_s = (1 - \cos\psi)\sin\phi_s$$

$$\Rightarrow \tan\phi_s = \frac{\sin\psi - \psi}{1 - \cos\psi} = \frac{\psi - \psi^3/6 + \cdots - \psi}{1 - (1 - \psi^2/2 + \cdots)} \underset{\substack{\text{small} \\ \psi}}{\simeq} \frac{\psi - \psi^3/6 - \psi}{\psi^2/2} \simeq -\frac{\psi}{3}$$

$$\boxed{\psi \simeq -3\tan\phi_s}$$

Numerical checks show works well up to $|\phi_s| \approx 1$. : even though approx is "poor"

For case of $\phi_s = -\pi/2$ (Max Focus Case)

seperatrix eqn * $\sin\phi_2 - \phi_2\cos\phi_s = -[\sin\phi_s - \phi_s\cos\phi_s]$

Gives exactly: $\sin\phi_2 - \phi_2\overset{0}{\cos(\pi/2)} = -[-\overset{1}{\sin(\pi/2)} + (\pi/2)\overset{0}{\cos(\pi/2)}]$

$$\sin\phi_2 = 1 \Rightarrow \phi_2 = -3\pi/2 = -270^\circ$$

Exact width $\Rightarrow \psi = \underset{\pi/2}{\underset{\|}{-\phi_s}} \; \underset{3\pi/2}{\underset{\|}{-\phi_2}} = 2\pi = 360^\circ$ Focuses for full RF phase width!

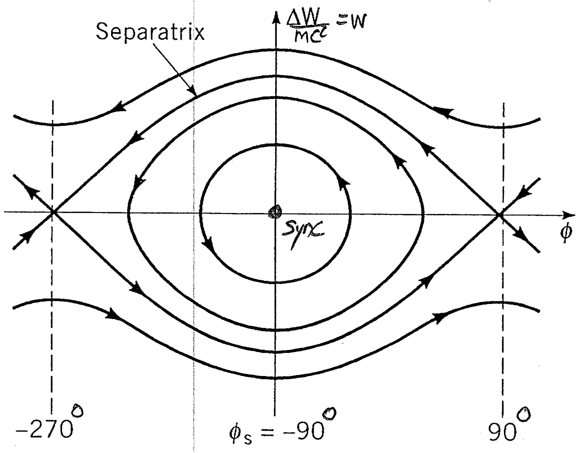


Figure 6.4. Separatrix for $\phi_s = -90^\circ$ (no acceleration).

This choice will give no acceleration but will be most efficient for beam bunching. Note also that the synchrotron wavenumber $k_s = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{qE_0T\sin(-\phi_s)}{(\gamma_s\beta_s)^3 mc^2}}$ is largest for $\phi_s = -\pi/2$.

To estimate the vertical ½-width in W of the seperatrix for arb ϕ_s:

$W = W_{max}$, $\phi = -\phi_s$ in seperatrix eqn:

$$\Rightarrow \quad H_\phi = H_\phi(-\phi_s)$$

$$A\frac{W_{max}^2}{2} + B[\sin\phi_s - \phi_s\cos\phi_s] = -B[\sin\phi_s - \phi_s\cos\phi_s]$$

$$W_{max} = \sqrt{4\frac{B}{A}[\phi_s\cos\phi_s - \sin\phi_s]}$$

$A = \frac{2\pi}{\lambda_{rf}}\frac{1}{(\gamma_s\beta_s)^3}$

$B = \frac{qE_0T}{mc^2}$

$$\boxed{W_{max} = \frac{\Delta W_{max}}{mc^2} = \sqrt{\frac{2qE_0T(\gamma_s\beta_s)^3\lambda_{rf}(\phi_s\cos\phi_s - \sin\phi_s)}{\pi mc^2}}}$$

$W_{min} = -W_{max}$

$W\text{-width} = 2W_{max}$

Approximating crudely, the $\phi - W$ phase-space area of the bucket is:

$$\psi \overset{''}{\simeq} -3\tan\phi_s$$

$$W_{max} = \sqrt{\frac{2qE_0T}{\pi mc^2}(\gamma_s\beta_s)^3\lambda_{rf}(\phi_s\cos\phi_s - \sin\phi_s)}$$

$$\text{Area Bucket} = \int_{\text{Bucket}} d\phi\, dW \simeq \pi (W_{max})(\psi/2)$$

Approx as an ellipse with area $\pi \times (x\text{-radius}) \times (y\text{-radius})$, $\underset{\psi/2}{}$ $\underset{W_{max}}{}$

$$= \left(-\frac{3\pi}{2}\tan\phi_s\right)\cdot W_{max}$$

$$\boxed{\text{Area Bucket} \cong \frac{3\pi}{2}\tan(-\phi_s)\sqrt{\frac{2qE_0T}{\pi mc^2}(\gamma_s\beta_s)^3\lambda_{rf}(\sin(-\phi_s) - \phi_s\cos\phi_s)}}$$

This provides an estimate of the phase-space area that can be accelerated.

Comments:

Relativistic (electrons or very energetic protons/ions)	$\beta_s \simeq 1$	$\Rightarrow$	Field errors ($E_0T \simeq E_0$; $T \simeq 1$) do not change synchronous condition, but shift final energy.
Non-Relativistic (low energy e⁻; protons or ions)	$\beta_s \lesssim 1/2$	$\Rightarrow$	Field errors (E_0T) cause shift to a new synchronous phase.

* Yue Hao lectures give more precise numerical results for the stable bucket area including python code to calculate.

Adiabatic Phase Damping

Ref: Wangler, "RF Linear Accelerators" Secs. 5.12, 6.7.

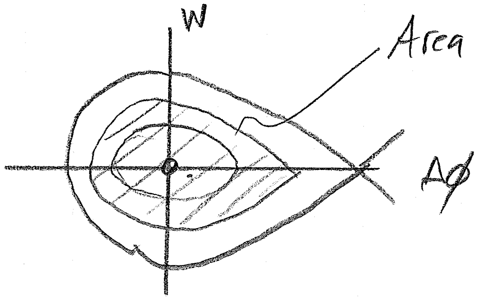
If parameters (focus well) of an oscillator are changed slowly relative to the period of the oscillation, then expect an adiabatic invariant:

$$\boxed{\text{"Action"} = \oint_{\text{cycle}} p\,dq = \text{const.}}$$

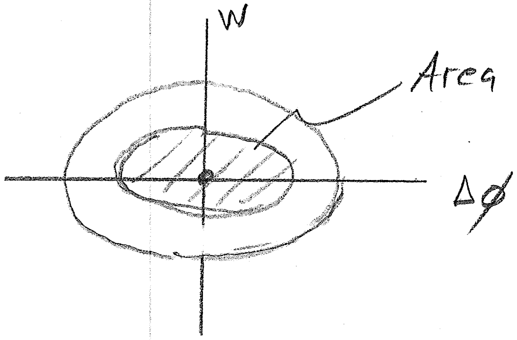
See Landau & Lifshitz "Mechanics", 3rd Edition, pg 154.

- ★ True for any number of parameters varying simultaneously
- ★ For synchrotron motion synchrotron wavenumber $k_s = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{qE_0T\sin(-\phi_s)}{(\gamma_s\beta_s)^3 mc^2}}$ sets the scale to measure slowness for validity.

Nonlinear RF



Linear RF



This result tells us that the longitudinal phase-space area (or emittance) will be conserved as the focusing parameters (say due to acceleration) vary slowly on the synchrotron oscillation period.

Reminder: $k_s = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{qE_0T}{mc^2} \frac{\sin(-\phi_s)}{(\gamma_s\beta_s)^3}}$ = Synchrotron Wavenumber

For the case of linear motion with small phase excursions about the synchronous particle:

$$\boxed{\begin{aligned}\text{"Action"} &= \pi \Delta\phi_0 W_0 = \text{const} \\ &= \pi(\Delta\phi_0)^2 \sqrt{\frac{(\gamma_s\beta_s)^3}{2\pi}\left(\frac{qE_0T\lambda_{rf}}{mc^2}\right)\sin(-\phi_s)}\end{aligned}}$$

$\Delta\phi_0$ = phase ½-width linear orbit.

$W_0 = \frac{\Delta W}{mc^2}$ = $\Delta\phi_0$ corresponding normalized energy deviation of linear orbit.

$$= \sqrt{\frac{(\gamma_s\beta_s)^3}{2\pi}\left(\frac{qE_0T\lambda_{rf}}{mc^2}\right)\sin(-\phi_s)} \times \Delta\phi_0$$

see pg 32, 33

-or-

$$\Delta\phi_0 = \frac{\text{const}}{\left[(\gamma_s\beta_s)^3\left(\frac{qE_0T\lambda_{rf}}{mc^2}\right)\sin(-\phi_s)\right]^{1/4}}$$

(rescaled const)

If we take

$$\phi_s \simeq \text{const}$$
$$E_0T\lambda_{rf} \simeq \text{const}$$

$$\Rightarrow \boxed{\Delta\phi_0 = \frac{\text{const}}{(\gamma_s\beta_s)^{3/4}}}$$

and then for adiabatic invariance

$$\boxed{W_0 = \text{const}\,(\gamma_s\beta_s)^{3/4}}$$

☆ phase width shrinks with adiabatic acceleration. -called "phase damping"

☆ Energy deviation grows with adiabatic accel. for const phase-space area.

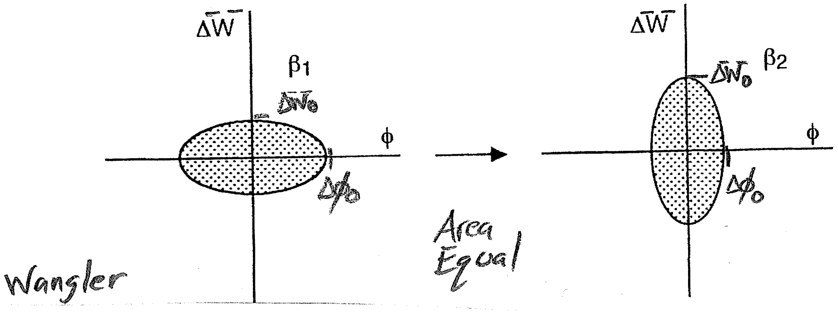
$$W \equiv \frac{\Delta W}{mc^2} \Rightarrow \Delta W_0 = \text{const}\,(\gamma_s\beta_s)^{3/4}$$

recall

or equivalently: $\Delta\phi_0|_i$ = initial value of $\Delta\phi_0$

$$\frac{\Delta\phi_0}{\Delta\phi_0|_i} = \left(\frac{(\gamma_s\beta_s)|_i}{(\gamma_s\beta_s)|_f}\right)^{3/4} \qquad \frac{\Delta\overline{W}_0|_f}{\Delta\overline{W}_0|_i} = \left(\frac{(\gamma_s\beta_s)}{(\gamma_s\beta_s)|_i}\right)^{3/4}$$

Graphically:



Wangler

For RF high energy synchrotrons, $\gamma_s\beta_s$ will vary slowly over many laps and the adiabatic approximation can be well satisfied. For RF linacs, $\gamma_s\beta_s$ may change too rapidly for validity of the adiabatic approximation.

For FRIB linac segment #1 : $k_s \cdot \text{Length} \sim (2\pi)(\sim 10)$ Ref: Q. Zhao

$\Rightarrow$ 10 oscillations

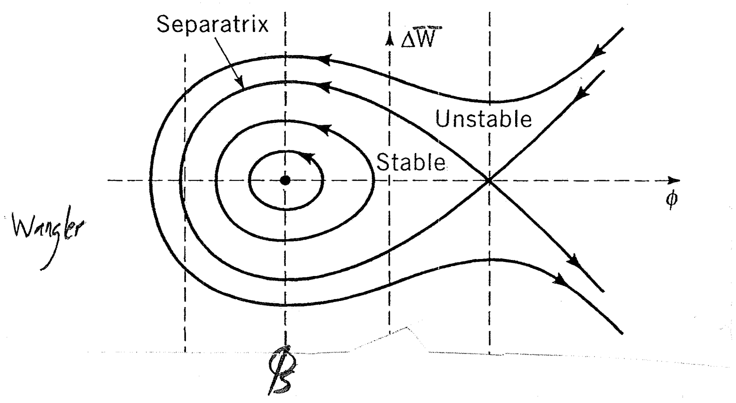
When $\gamma_s\beta_s \neq$ const, $H_\phi \neq$ const and the RF "fish" structure becomes distorted to a more characteristic "golf-club" shape.

✩ Density in phase-space of non-interacting particles governed by Hamiltonian is invariant even if Hamiltonian H is non-constant by Liouville's Theorem.

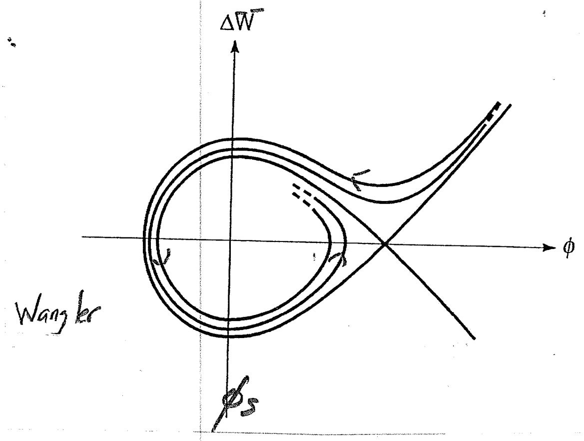
⇒ Phase volume enclosed by surface of fixed density is constant.

⇒ Shape can distort due to acceleration.

$\gamma_s\beta_s$ = const,



<u>$\gamma\beta \neq$ const accelerating</u>



✩ Use H_ϕ = const to analyze

✩ Use difference equations to analyze general case.

✩ Untrapped for H_ϕ = const can move within for bounded orbit.

Yue Hao lectures give interactive programs that can be run with strong accel to see characteristic bucket distortions.

– Ring formulation but physics is analogous. – Rings typically weak accel.

Transverse RF De Focusing

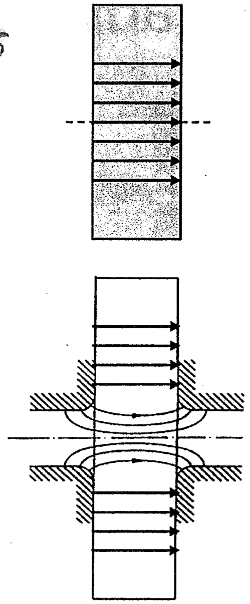
Qualitative:

RF Defocusing

Syphers
USPAS

- When a particle enters a cavity off center, the field lines will have an inward component; and they will have an outward component upon exit from the cavity.
- However, the strength of the field is changing — typically, increasing — during transit.
- Thus, the outward "kick" due to the field will be greater than the inward kick — defocusing effect
- This "RF defocusing" is more important at lower energies

$$\frac{1}{f} = \frac{\Delta x'}{x} \approx \pi \frac{eV_{\text{eff}}}{mc^2} \frac{T \cos \phi_s}{\lambda_{rf}(\beta\gamma)^2}$$



ideal

more realistic

see T. Wangler, *RF Linear Accelerators*

⇐ Ideal pillbox cavity has no radial E-field Er to lead to transverse focusing / defocusing.

⇐ When aperture added to cavity to allow beam to enter / exit this produces an Er. and transverse focusing / defocusing now possible.

Defocusing kick generally larger due to exit field gaining strength due to variation during transit. Part offset due to velocity gain within gap (Einzel lens effect).

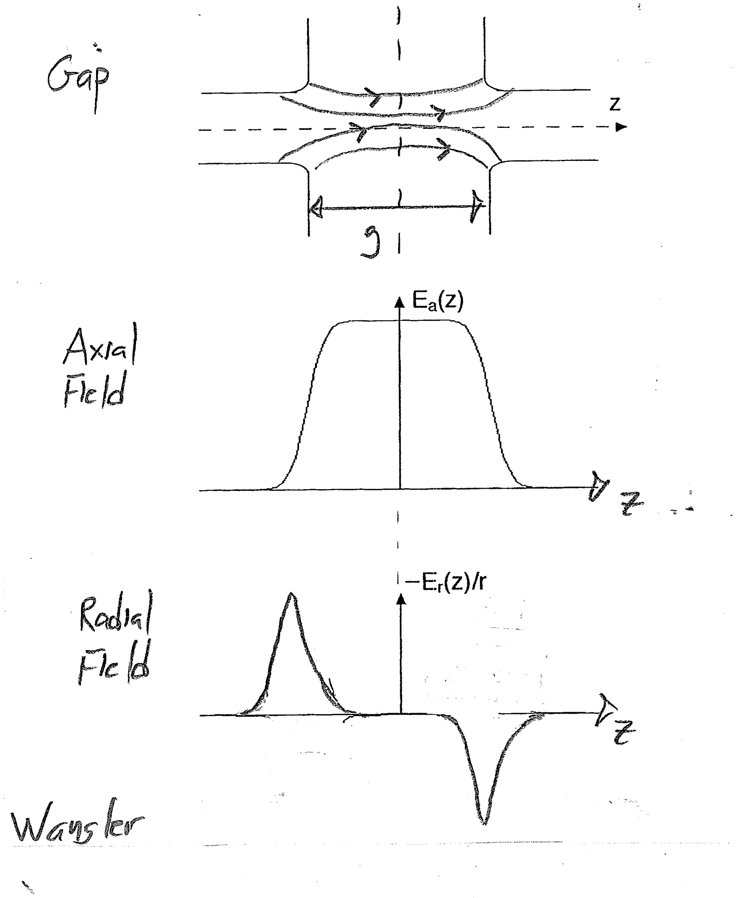
Field rising for stability longitudinally

Transverse RF DeFocusing

Ref: Wangler, "RF Linear Accelerators", § 7.3
Conte and Mackay, "Intro to the Physics of Particle Accelerators", Chapter 9

The field structure of an RF gap can also lead to transverse (radial) beam defocusing. Here we present a simple analysis to calculate the radial impulse a particle experiences when traversing the gap.

Qualitative Picture



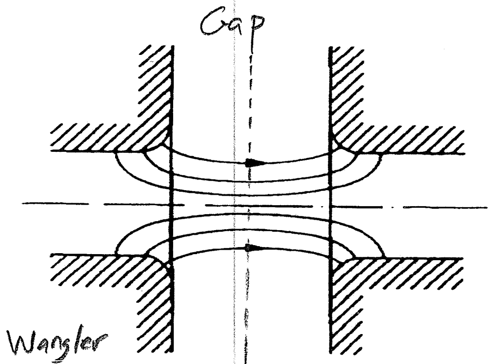
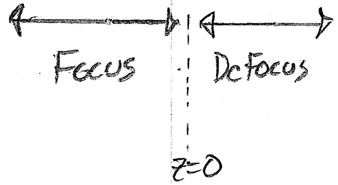


Figure 7.1 Electric field lines in an RF gap.



- ★ Symmetric if field static and $v \simeq$ const (negligible energy gain) ⇒ No optic
- ★ But: RF field rising in time as particle traverses gap ⇒ Larger Defocus expected. Net defocus.
- ★ Counter: v larger to right ⇒ less dwell time in defocus. Can have RF focusing if energy gain large. Like Einzel lens.
- ★ B_θ also present but weaker.

For cavity assume:

$$\boxed{\begin{aligned}\vec{E} &= E_r(r,z,t)\,\hat{r} + E_z(r,z,t)\,\hat{z} \\ \vec{B} &= B_\theta(r,z,t)\,\hat{\theta}\end{aligned}\Bigg\}\ \text{TM type Mode}}$$

Then Lorentz Force Eqn:

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v}\times\vec{B}$$

gives, radial component

$$\frac{dp_r}{dt} = qE_r - qv_z B_\theta$$

$$\boxed{\frac{dp_r}{dt} = qE_r - q\beta c B_\theta}$$

But

$$p_r = m\gamma\frac{dr}{dt} \triangleq m\gamma\beta c\, r'$$

$$r' = \frac{dr}{ds}$$

Giving a radial impulse (change in angle measure)

$$\boxed{\Delta(\gamma\beta r') = \frac{q}{mc}\int_{\text{Gap Transit}}\left[E_r - \beta c B_\theta\right]dt}$$

Maxwell's Equations in Cavity:

$\nabla\cdot\vec{E} = 0$:

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z} = 0 \qquad 1)$$

$\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$:

$$\Rightarrow \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = \frac{-\partial B_\theta}{\partial t} \qquad 2)$$

$\nabla\times\vec{B} = \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t}$:

$$-\frac{\partial B_\theta}{\partial z} = \frac{1}{c^2}\frac{\partial E_r}{\partial t} \qquad 3)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \frac{1}{c^2}\frac{\partial E_z}{\partial t} \qquad 4)$$

$\nabla\cdot\vec{B} = 0$

satisfied by symmetry

Use these equations to approximate the fields near the axis ($r=0$) where we take E_z to be independent of r

Approximate cavity fields near axis where E_z independent of r.

★ Need E_z and B_θ near $r=0$ to calculate impulse.

Using 1) with $\partial E_z/\partial r \simeq 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}(r E_r) + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}(r E_r) = -\frac{\partial E_z}{\partial z} r \quad \xRightarrow{\substack{\text{integrate} \\ E_r(r=0)=0}} \quad \boxed{E_r = -\frac{\partial E_z}{\partial z}\frac{r}{2}} \text{ — ①}$$

Using 3):

$$\frac{\partial B_\theta}{\partial z} = -\frac{1}{c^2}\frac{\partial E_r}{\partial t} \overset{\text{use ①}}{=} \frac{r}{2c^2}\frac{\partial}{\partial z}\frac{\partial E_z}{\partial t} \quad \underset{\substack{\text{integrate} \\ B_\theta(r=0)=0}}{\Rightarrow} \quad \boxed{B_\theta = \frac{r}{2c^2}\frac{\partial E_z}{\partial t}} \text{ — ②}$$

We take for the gap:

$$\boxed{E_z = E_0(z)\cos(\omega t + \phi)} \text{ — ③}$$

harmonic accel. field.
$t=0 \Rightarrow z=0$ $\quad E_z = E_0(z)\cos\phi$

Using ① and ② in the radial impulse formula:

$$= \frac{q}{mc}\int_{\text{Gap Transit}} [E_r - \beta c B_\theta]\, dt$$

$$\Delta(\gamma\beta r') = \frac{q}{2mc}\int_{\text{Gap Transit}} \left[-\frac{\partial E_z}{\partial z}\cdot r - \frac{\beta}{c}\frac{\partial E_z}{\partial t} r\right] dt$$

$$= \frac{-q}{2mc}\int_{-L/2}^{L/2} r\left(\frac{\partial E_z}{\partial z} + \frac{\beta}{c}\frac{\partial E_z}{\partial t}\right)\frac{dz}{\beta c} \qquad dt = \frac{dz}{\beta c}$$

Approximate further in single gap

$r \simeq$ const Impulse approx.
$\beta \simeq$ const Accel weak

} These may break down at very low energies. Then more detailed analysis needed.

Then we can pull r and β through the integral

$$\boxed{\Delta(\gamma\beta r') = \frac{-qr}{2\beta mc^2}\int_{-L/2}^{L/2}\left[-\frac{\partial E_z}{\partial z} + \frac{\beta}{c}\frac{\partial E_z}{\partial t}\right]dz}$$

But

$$\frac{dE_z}{dz} = \frac{\partial E_z}{\partial z} + \frac{dt}{dz}\frac{\partial E_z}{\partial t} = \frac{\partial E_z}{\partial z} + \frac{1}{v_z}\frac{\partial E_z}{\partial t} \simeq \frac{\partial E_z}{\partial z} + \frac{1}{\beta c}\frac{\partial E_z}{\partial t}$$

$$\hookrightarrow \quad \frac{\partial F}{\partial z} \simeq \frac{dE_z}{dz} - \frac{1}{\beta c}\frac{\partial E_z}{\partial t}$$

Using this result to eliminate $\partial E_z/\partial z$:

$$\Delta(\gamma\beta r') = \frac{-qr}{2\beta mc^2}\int_{-L/2}^{L/2}\left[\overset{\nearrow 0}{\frac{dE_z}{dz}} + \frac{1}{c}\left(\beta - \frac{1}{\beta}\right)\frac{\partial E_z}{\partial t}\right]dz$$

/ L contains field so no contribution

$$\left(\beta - \frac{1}{\beta}\right) = \frac{\beta^2 - 1}{\beta} = -\frac{(1-\beta^2)}{\beta} = \frac{-1}{\gamma^2\beta}$$

$$\Rightarrow \boxed{\Delta(\gamma\beta r') = \frac{qr}{2(\gamma\beta)^2 mc^2}\int_{-L/2}^{L/2}\frac{\partial E_z}{\partial t}\,dz}$$

Now use the harmonic accel field

$$E_z = E_0(z)\cos(\omega t + \phi) \quad \Rightarrow \quad \frac{\partial E_z}{\partial t} = -\omega E_0(z)\sin(\omega t + \phi)$$

For gap

$$\omega t = \frac{2\pi}{\beta\lambda_{rf}}\cdot z \quad \Rightarrow \quad \boxed{\frac{\partial E_z}{\partial t} = -\omega E_0(z)\sin\left(\frac{2\pi}{\beta\lambda_{rf}}\cdot z + \phi\right)}$$

Insert this field expression in impulse formula:

$$\Delta(\gamma\beta r') = \frac{-qr\omega}{2(\gamma\beta)^2 mc^3}\int_{-L/2}^{L/2} E_0(z)\sin\left(\frac{2\pi z}{\beta\lambda_{rf}}+\phi\right)dz$$

$$= \frac{-qr\omega}{2(\gamma\beta)^2 mc^3}\int_{-L/2}^{L/2} E_0(z)\left\{\cancelto{0}{\sin\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)\cos\phi} + \cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)\sin\phi\right\}dz$$

if $E_0(z)$ even function: usual for symmetric gap

$$\boxed{\Delta(\gamma\beta r') = \frac{-qr\omega\sin\phi}{2(\gamma\beta)^2 mc^3}\int_{-L/2}^{L/2} E_0(z)\cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)dz}$$

This can be further simplified using our formula for the transit time factor of a symmetric gap:

$$T = \frac{\int_{-L/2}^{L/2} E_0(z)\cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right)dz}{\int_{-L/2}^{L/2} E_0(z)dz}$$

Transit Time

$$E_0 L = \int_{-L/2}^{L/2} E_0(z)dz$$

Avg Field over Cell

(L large enough to contain $E_0(z)$, usually take to be cell length ⇒ E_0 is cell avg field.)

$$\frac{\omega}{c} = \frac{2\pi}{\tau_{rf}\cdot c} = \frac{2\pi}{\lambda_{rf}}$$

Then we have

Radial Impulse from RF Gap

$$\boxed{\Delta(\gamma\beta r') = \frac{\pi}{\lambda_{rf}}\left(\frac{qE_0LT}{mc^2(\gamma\beta)^2}\right)\sin(-\phi) \times r}$$

<u>Comments:</u>

☆ Linear optic: Impulse $\propto r$

☆ For $\phi < 0$ (RF stability) is <u>defocusing</u>

☆ $\sim 1/(\gamma\beta)^3$ ⇒ quickly becomes weak for relativistic particles

⇒ will be stronger for NR heavy ions (FRIB).

☆ More detailed analysis by Gluckstern (see Wangler §7.4) shows that impulse <u>can</u> become focusing or significantly weakened when β varies strongly in gap (low energy ions/protons). In this context the Einzel lens electrostatic focus impulse part compensates or offsets the effect of the rising RF field during transit.

Quasistatic Modeling of RF Gap Field   Wangler, §5.14

In the previous treatment, we took E_z to be independent of r to calculate the approximate cavity defocusing impulse. If one needs a better approx:

1) Import cavity fields from a cavity design code into a particle simulation.

2) Carry out more advanced analysis to better approx. fields and acceleration effects within gap.

Within the context of 2), the so-called quasistatic approx. can be useful to guide improvements

Cavity fields satisfy the wave eqn:

$$\boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0} \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

For harmonically varying fields: $\sim \cos(\omega t + \phi)$

$$\boxed{\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{E} = 0}$$

$\omega = \text{const}$ RF angular freq

$\phi = \text{const}$ RF phase

$\omega = \frac{2\pi}{\tau_{rf}}$

$\tau_{rf} c = \lambda_{rf}$

But

$$\frac{\omega}{c} = \frac{2\pi}{\tau_{rf} c} = \frac{2\pi}{\lambda_{rf}} \Rightarrow \left[\nabla^2 + \left(\frac{2\pi}{\lambda_{rf}}\right)^2\right]\vec{E} = 0$$

If the gap has characteristic length scales $\ell_{gap} << \lambda_{rf}$, expect

$$\nabla^2 \sim \frac{1}{\ell_{gap}^2} >> \left(\frac{2\pi}{\lambda_{rf}}\right)^2 \Rightarrow \boxed{\nabla^2\vec{E} \simeq 0}$$

Vector Laplacian

But $\vec{E}$ satisfies (vector calculus, any field):

$$\boxed{\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}}$$

$\nabla\cdot\vec{E} = 0$ Maxwell Eqn $\nabla\cdot\vec{E} = \rho/\varepsilon_0$; $\rho = 0$ in cavity ; $\nabla\cdot\vec{E} = 0$

$\nabla^2\vec{E} \simeq 0$ Previous page $\nabla^2 >> (2\pi/\lambda_{rf})^2$

Giving

$$\boxed{\nabla\times(\nabla\times\vec{E}) \simeq 0} \Rightarrow \boxed{\nabla\times\vec{E} = 0 \quad \text{solution.}}$$

* Electrostatic form on scales short relative to RF wavelength.

Satisfied if we take

$$\boxed{\vec{E} = -\nabla\phi_e}$$

since $\nabla\times\nabla\phi_e = 0$ for any ϕ_e.

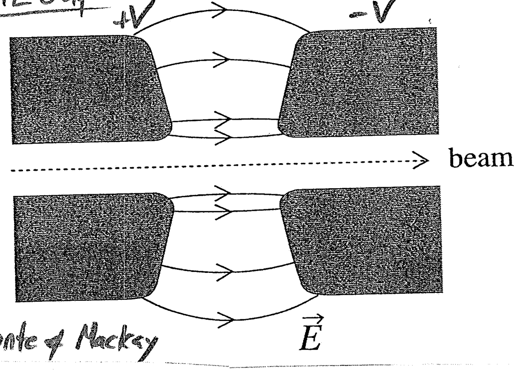
The Potential also must satisfy:

$$\nabla\cdot\vec{E} = -\nabla^2\phi_e = 0 \Rightarrow \boxed{\nabla^2\phi_e = 0}$$

ϕ_e satisfies Electrostatic Laplace equation.

- ☆ Can <u>only</u> apply locally (say near short gap) with $l \ll \lambda_{rf}$
- ☆ Approx decouples electric and magnetic fields since $-\frac{\partial}{\partial t}\vec{B}$ has been neglected in Faraday's Law.

DTL Gap



Conte & Mackey

- ☆ Electrostatic analogy used to guide gap design in R.F cavities.
  - Only apply near gap
  - Use to guide shorter gap design for improved Transit Time T
  - Shape gap to reduce E_r and RF defocusing.
- ☆ Also applied in RFQ analysis (poles ripples small relative to RF wavelength) and analysis of induction accelerator gaps.