RF Cavities

see Conte and Mackay, Chapter 9
Wille, Chapter 5
Wiedemann, § 2.2

Maxwell's equations in vacuum region:

$$\nabla\cdot\vec{E} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \xrightarrow{\nabla\times} \nabla\times(\nabla\times\vec{E}) = -\frac{\partial}{\partial t}\nabla\times\vec{B} \rightarrow \underbrace{\nabla(\overbrace{\nabla\cdot\vec{E}}^{0\ \text{Maxwell Eqn}}) - \nabla^2\vec{E}}_{\text{Vector Identity}} = -\frac{\partial}{\partial t}\overbrace{\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)}^{\text{Maxwell Eqn to eliminate}}$$

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \xrightarrow{\nabla\times} \nabla\times(\nabla\times\vec{B}) = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\vec{E} \rightarrow \nabla(\overbrace{\nabla\cdot\vec{B}}^{0}) - \nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t}\vec{B}\right)$$

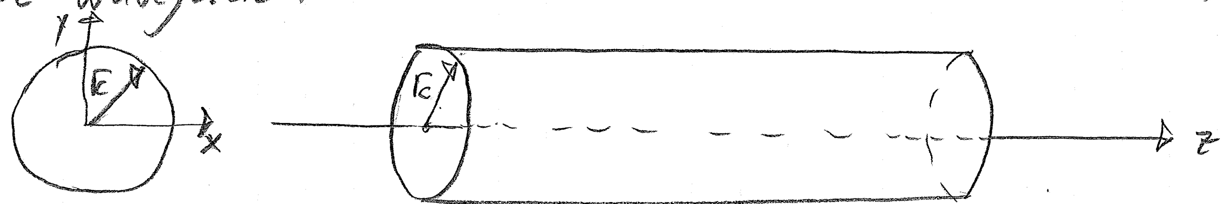
$$\nabla\cdot\vec{B} = 0$$

$$\Rightarrow \boxed{\begin{aligned} \nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} &= 0 \\ \nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} &= 0 \end{aligned}}$$

$\vec{E}, \vec{B}$ satisfy wave equations.

1st step:

We will look for EM wave solutions in a perfectly conducting, cylindrical pipe "waveguide".



End View   Side View

r_c = Radius Cylinder

Maxwell eqns give boundary conds on perfect conductor: $\vec{E}$: Tangential zero
$\vec{B}$: Normal zero

Search for a solution with z-t traveling wave form with harmonic time (t) and z dependance
* $\sim e^{-i\omega t}$ time variation, $i = \sqrt{-1}$, take Re{ } for physical part.

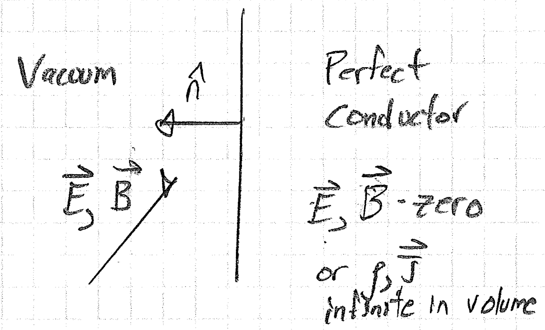
$$\boxed{\left.\begin{aligned} E_z &= E_z(r,\theta)\cdot e^{i(\omega t - kz)} \\ E_r &= E_r(r,\theta)\cdot e^{i(\omega t - kz)} \\ B_\theta &= B_\theta(r,\theta)\cdot e^{i(\omega t - kz)} \end{aligned}\right\} \quad \begin{aligned} &\omega = \text{const Angular Frequency} \\ &k = \text{const Axial Wavenumber} \\ &\text{Transverse Magnetic TM form since want longitudinal } E_z \text{ for acceleration} \end{aligned}}$$

Nonzero field components, in cylindrical-polar coordinates.

Later will restrict $E_z(r = r_c) = 0$ to meet boundary conditions.

Field Boundary Conditions: Conducting walls

Apply Maxwell's eqns at boundary of perfect conductor



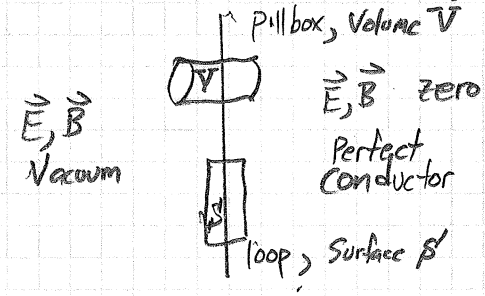
Maxwell Eqns Media

$$\nabla \cdot \vec{D} = \rho$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$

Integrate over $\Longrightarrow$ limiting pill box + loop $V, S \to 0$

Boundary Conds

$$\boxed{\begin{aligned} \hat{n} \cdot \vec{D} &= \Sigma \\ \hat{n} \times \vec{E} &= 0 \\ \hat{n} \times \vec{H} &= \vec{K} \\ \hat{n} \cdot \vec{B} &= 0 \end{aligned}}$$



Σ = Surface Charge Density

$\vec{K}$ = Surface Current Density

In vacuum:

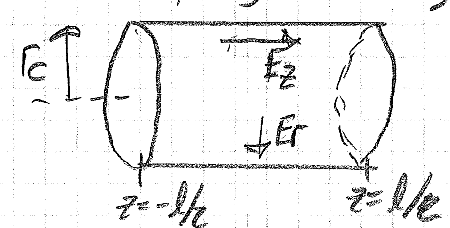
$$\vec{D} = \varepsilon_0 \vec{E}$$
$$\vec{B} = \mu_0 \vec{H}$$

So we have for field boundary conditions in the ideal vacuum / perfect conductor interface:

Exclude $\begin{cases} \vec{E}\,|_{tangential} = 0 \\ \vec{B}\,|_{normal} = 0 \end{cases}$

Allow $\begin{cases} \vec{E}\,|_{normal} \text{ allowed} \\ \vec{B}\,|_{tangental} \text{ allowed} \end{cases}$ $\Rightarrow$ surface charge Σ adjusts to shield conductor; surface current $\vec{K}$ adjusts to shield conductor

Implications in Pipe Segment: E_z, E_r, B_θ allowed



$E_z \to 0$ $\quad r = r_c$: pipe edge

$E_r \to 0$ $\quad z = \pm \frac{l}{2}$: pipe ends

B_θ No restrictions

$$\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2}$$

Examine only E_z in $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$ $\quad \frac{\partial}{\partial t} = i\omega \quad \frac{\partial}{\partial z} = -ik$

$$\nabla^2 E_z - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E_z = 0 \quad \Rightarrow \quad \boxed{\nabla_\perp^2 E_z + \left(\frac{\omega^2}{c^2} - k^2\right) E_z = 0} \qquad \nabla_\perp^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Look for a solution with harmonic azimuthal variation

$E_z \sim \cos(n\theta)$ choose $\theta = 0$ reference to make true.

$$\Rightarrow \boxed{\frac{\partial^2}{\partial r^2} E_z + \frac{1}{r}\frac{\partial}{\partial r} E_z + \left(k_c^2 - \frac{n^2}{r^2}\right) E_z = 0 \qquad k_c^2 \equiv \frac{\omega^2}{c^2} - k^2}$$

Bessel Function Equation.

Recognizing this as Bessel's equation, the general solution is

$$E_z = C_1 J_n(k_c r) + C_2 Y_n(k_c r) \qquad C_1, C_2 \text{ constants}$$

$J_n(x)$ = Ordinary nth order Bessel function of 1st kind
$Y_n(x)$ = Ordinary nth order Bessel function of 2nd kind

$$\lim_{r \to 0} Y_n(k_c r) \to \infty \quad \Rightarrow \quad C_2 = 0 \quad \text{for finite (physical) E-field near } r = 0.$$

Putting back in variation in θ, z, t, we have:

$$\boxed{E_z = E_0 J_n(k_c r) \cos(n\theta)\, e^{i(\omega t - kz)} \qquad E_0 = \text{const. (complex)}}$$

We can now substitute this back in the Maxwells' eqns to find the form of B_θ and E_r consistent. But first, simplify by further restricting to $n=0$ since for accelerating particles we prefer no azimuthal variation.

Then we have:

$$E_z = E_0 J_0(k_c r)\, e^{i(\omega t - kz)}$$
$$E_r = E_r(r)\, e^{i(\omega t - kz)}$$
$$B_\theta = B_\theta(r)\, e^{i(\omega t - kz)}$$

$E_r(r)$, $B_\theta(r)$ must be calculated consistent with E_z form $\Rightarrow$

Maxwell Eqns $\quad \frac{\partial}{\partial z} = -ik, \quad \frac{\partial}{\partial t} = i\omega$

$\nabla\cdot\vec{E} = 0$: $\quad \frac{1}{r}\frac{\partial}{\partial r}(r E_r) - ik E_z = 0 \quad$ 1)

$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ $\quad \hat{r}$: $\quad ik B_\theta = \frac{\omega i}{c^2} E_r \quad$ 2)

$\hat{z}$: $\quad \frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) = \frac{i\omega}{c^2} E_z \quad$ 3)

$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$ $\quad \hat{\theta}$: $\quad \frac{\partial E_z}{\partial r} - ik E_r = -i\omega B_\theta \quad$ 4)

$\nabla\cdot\vec{B} = 0$ $\quad$ satisfied, by symmetry ✓

From 2) $\quad B_\theta = \frac{\omega}{c^2 k} E_r$

From 4) $\quad \frac{\partial E_z}{\partial r} = ik E_r - i\omega \underbrace{B_\theta}_{\text{sub } B_\theta \text{ above}} = \left(ik - \frac{i\omega^2}{c^2 k}\right) E_r$ $\Rightarrow$

$$\boxed{\begin{aligned} E_r &= \frac{-i/k}{1-\omega^2/(c^2k^2)}\frac{\partial E_z}{\partial r} = \frac{ik}{k_c^2}\frac{\partial E_z}{\partial r} \\ B_\theta &= \frac{-i\omega/(c^2k^2)}{1-\omega^2/(c^2k^2)}\frac{\partial E_z}{\partial r} = \frac{i\omega/c^2}{k_c^2}\frac{\partial E_z}{\partial r} \end{aligned}}$$

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

Using $\quad J_0'(x) = -J_1(x)$; $\quad \frac{\partial E_z}{\partial r} = -E_0 k_c J_1(k_c r)\, e^{i(\omega t - kz)}$

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

$$\Rightarrow E_r(r) = -iE_0 \frac{k}{k_c} J_1(k_c r)$$

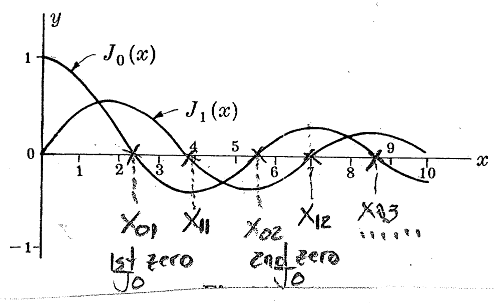
$$\Rightarrow B_\theta(r) = -i\frac{E_0\omega}{c^2 k_c} J_1(k_c r)$$

$$\boxed{\begin{aligned} E_z &= E_0 J_0(k_c r)\, e^{i(\omega t - kz)} \\ E_r &= -iE_0 \frac{k}{k_c} J_1(k_c r)\, e^{i(\omega t - kz)} \\ B_\theta &= -\frac{iE_0\omega}{c^2 k_c} J_1(k_c r)\, e^{i(\omega t - kz)} \end{aligned}}$$

Finally, need $E_z(r = r_c) = 0$ to satisfy tangential $\vec{E} = 0$ on conducting boundary

$\Rightarrow \quad J_0(k_c r_c) = 0 \quad \Rightarrow \quad k_c r_c = X_{0j}$ $\quad j = 1, 2, 3, \ldots$ zero of $J_n(X_{nj}) = 0$

Bessel function:



$X_{01} \simeq 2.405$

1st zero.

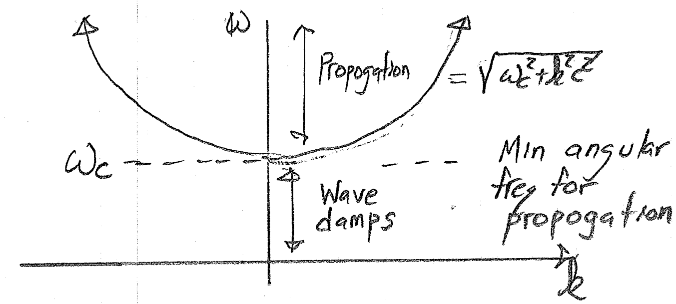
Choose 1st zero to try and get flat field near $r \simeq 0$.

★ Higher zeros increas argument of J_0

$$\Rightarrow \quad k_c r_c = \sqrt{\frac{\omega^2}{c^2} - k^2}\; r_c = X_{01}$$

Dispersion Relation

$$\boxed{\omega^2 = \omega_c^2 + k^2 c^2 \; ; \quad \omega_c \equiv \frac{X_{01}\, c}{r_c} \text{ cutoff freq}}$$



Wave phase velocity

$$\psi = \omega t - kz = \text{const}$$

$$\dot{\psi} = \omega - k\dot{z} = 0 \quad \Rightarrow \quad \dot{z} \equiv v_{phase} = \frac{\omega}{k} \overset{\text{use D.R. above}}{=} \frac{\omega c}{\sqrt{\omega^2 - \omega_c^2}} = \frac{c}{\sqrt{1 - \omega_c^2/\omega^2}} > c \qquad \omega > \omega_c.$$

$$\boxed{v_{phase} = \frac{c}{\sqrt{1-\omega_c^2/\omega^2}} > c \quad \text{Cannot maintain resonance with particle}}$$

Note energy propogation speed at group velocity $\quad \omega = (\omega_c^2 + k^2c^2)^{1/2}$

$$\boxed{v_{group} = \frac{d\omega}{dk} \overset{\text{use D.R.}}{=} \frac{kc^2}{(\omega_c^2 + k^2c^2)^{1/2}} = \frac{c^2}{\omega/k} = \frac{c^2}{v_{phase}} = c\sqrt{1-\omega_c^2/\omega^2} < c}$$

D.R

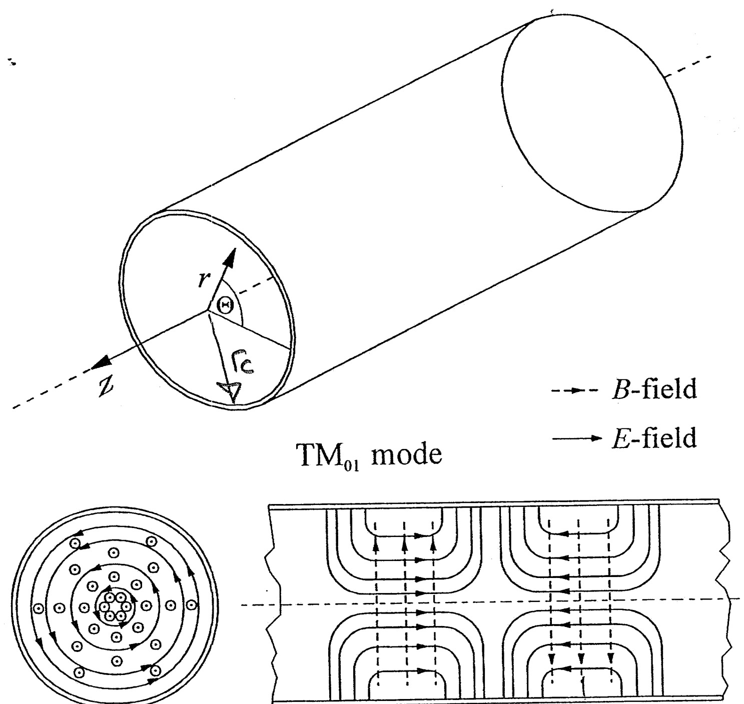
$$\omega = \sqrt{\omega_c^2 + k^2c^2}$$

$$\frac{d\omega}{dk} = \frac{kc^2}{(\omega_c^2 + k^2c^2)^{1/2}}$$

★ $v_{group} < c$ as must be case for physical energy transmission.

Note ! $v_{group} \cdot v_{phase} = c^2 = \text{const.}$

Cylindrical Waveguide TM_{01} Modes



$t =$ const
$\sim e^{i(\omega t - kz)}$
Variation

Wille **Fig. 5.2** Cylindrical waveguide with TM_{01} wave.

Δz $\quad k\Delta z = \pi$

Nonzero Fields:

$$E_z = E_0 \cdot J_0(k_c r)\, e^{i(\omega t - kz)}$$

$$E_r = -i E_0 \frac{k}{k_c} J_1(k_c r)\, e^{i(\omega t - kz)}$$

$$B_\theta = -i \frac{E_0 \omega}{c^2 k_c} J_1(k_c r)\, e^{i(\omega t - kz)}$$

Nomenclature:

TM = Transverse Magnetic (Longitudinal E_z)

$TM_{n_\theta n_r}$

$n_\theta =$ azimuthal θ-harmonic E_z
$= 0 \Rightarrow$ None

$n_r =$ number radial zeros E_z
$= 1 \Rightarrow$ One at $r = r_c$
(min needed for BC with nonzero sol.)

$\Rightarrow$ $\boxed{TM_{01} \text{ Mode}}$

// Side Point: Traveling wave accelerator works by adding disks to waveguide to slow down EM wave phase velocity to maintain particle resonance:

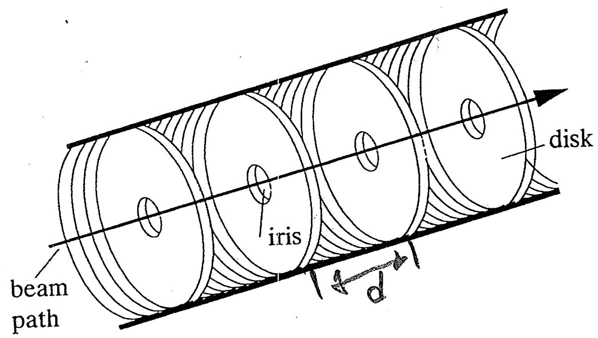
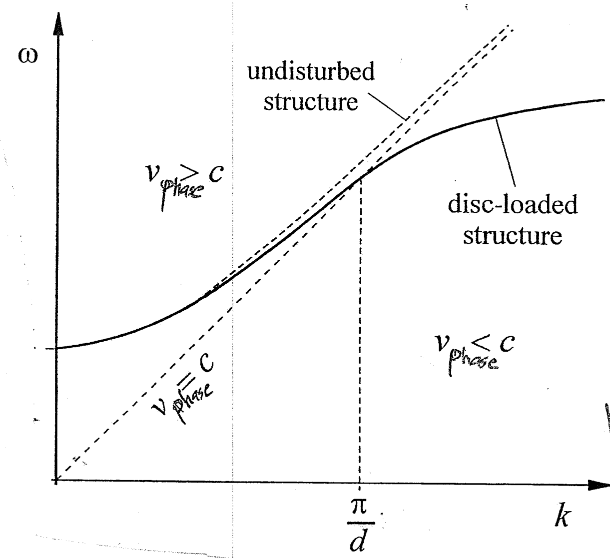


Fig. 2.8. Disk loaded accelerating structure for an electron linear accelerator (schematic)
Wiedemann



Wille

Irises give partial reflections allowing loss free propogation only at RF wavelengths with integer multiples of the iris seperation distance d.

This method is commonly used in e^- accelerators. See Wangler for details.

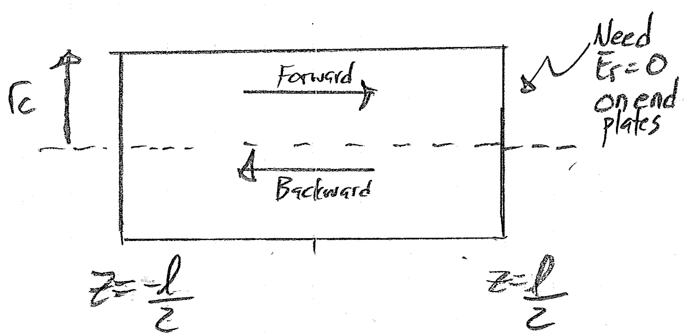
* washer loaded waveguide behaves like (weakly) coupled cavities.

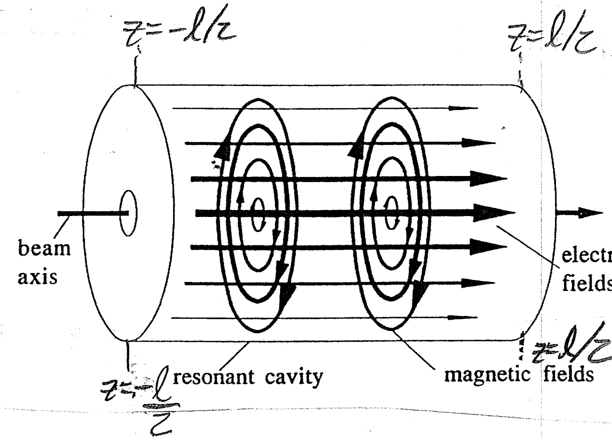
* Treatment analogous to methods used in condensed matter theory to study x-ray scattering in periodic lattices of atoms, Floquet Theory

//

So what do we do in our case? Make resonant cavity.

- ★ Add conducting walls at $z=0$, $z=l$
- ★ Superimpose forward and backward waves in cavity to meet boundary conditions and setup standing wave.
- ★ Time phasing of particles traversing cavity to gain energy and focus
  - Use formulation developed in earlier notes.





Recall: Hole in ends for beam generates E_r field.

Wiedemann

For cavity: Superimpose Waves:

$\underline{E_z}$

$$E_z = \frac{\tilde{E}_0}{2} J_0(k_c r)\, e^{i(\omega t - kz)} + \frac{\tilde{E}_0}{2} J_0(k_c r)\, e^{i(\omega t + kz)}$$

Forward Wave (1/2 Amp) ; Reflected Backward Wave (1/2 Amp) $(k \to -k)$

$$e^{ikz} + e^{-ikz} = 2\cos(kz)$$

$$\Rightarrow \quad E_z = \tilde{E}_0 J_0(k_c r) \cos(kz) e^{i\omega t} \qquad \tilde{E}_0 \text{ Amplitude (Complex)}$$

k will need to be fixed to satisfy end-plate boundary conditions ; see next pg

$$E_r = -\frac{iE_0}{2}\frac{k}{k_c}J_1(k_c r)e^{i(\omega t - kz)} \qquad k = \pm k_{mn} \quad E_0 = \frac{\tilde{E}_0}{2}$$

E_r

$$E_r = -\frac{i\tilde{E}_0}{2}\frac{k}{k_c}J_1(k_c r)\,e^{i(\omega t - kz)} + \frac{i\tilde{E}_0}{2}\frac{k}{k_c}J_1(k_c r)\,e^{i(\omega t + kz)}$$

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

Forward Wave (1/2 Amp)

Reflected Backward Wave ($k \to -k$) (1/2 Amp)

$$i\left[e^{ikz} - e^{-ikz}\right] = 2i\,i\sin(kz) = -2\sin(kz)$$

$$E_r = -\tilde{E}_0\frac{k}{k_c}J_1(k_c r)\sin(kz)\,e^{i\omega t}$$

To meet end-plate boundary conditions $E_r\big|_{z=\pm l/2} = 0$

$$\sin(kz)\Big|_{z=\pm l/2} = 0 \Rightarrow \boxed{\frac{kl}{2} = n_z\pi \qquad n_z = 0, 1, 2, \ldots}$$

$$B_\theta = -\frac{iE_0\,\omega}{c^2 k_c}J_1(k_c r)e^{i(\omega t - kz)}$$

B_θ

$$B_\theta = -\frac{i\tilde{E}_0\omega}{2c^2k_c}J_1(k_c r)\,e^{i(\omega t - kz)} - \frac{i\tilde{E}_0\omega}{2c^2k_c}J_1(k_c r)\,e^{i(\omega t + kz)}$$

Forward Wave (1/2 Amp)

Reflected Backward Wave ($k \to -k$) (1/2 Amp)

$$e^{ikz} + e^{-ikz} = 2\cos(kz)$$

No issues meeting boundary conditions at end-plates

$$B_\theta = -\frac{i\tilde{E}_0\omega}{c^2k_c}J_1(k_c r)\cos(kz)\,e^{i\omega t}$$

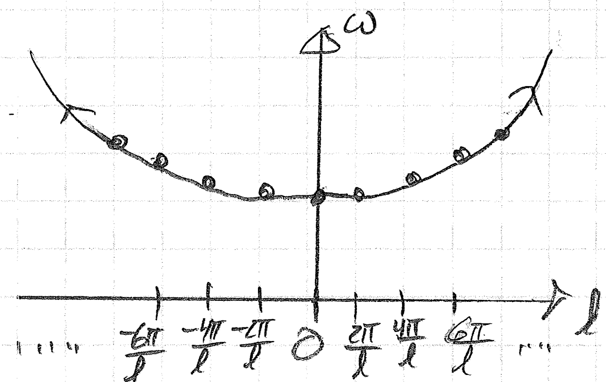
For the Pill-box cavity, due to E_r boundary condition

$$\boxed{k = \frac{2 n_z \pi}{l} \qquad n_z = 0, 1, 2, 3}$$

Inserting in the previous dispersion relation

$$\boxed{\omega^2 = \omega_c^2 + k^2 c^2 = \omega_c^2 + \left(\frac{2 n_z \pi c}{l}\right)^2}$$

$$\omega_c \equiv \frac{x_{01} c}{r_c}$$



Only discrete values k now allowed, for standing wave.

Choose the simplest possible solution

$$n_z = 0 \quad \Rightarrow \quad k = 0$$

Also gives no z-variation in E_z, which is desirable for simple gap dynamics.

Label (Nomenclature) $\boxed{TM_{n_\theta n_r n_z} \Rightarrow TM_{010} \text{ mode}}$

$$\Rightarrow \quad \boxed{\begin{aligned} E_z &= \tilde{E}_0 J_0(k_c r) e^{i\omega t} \\ E_r &= 0 \\ B_\theta &= -i \frac{\tilde{E}_0 \omega}{c^2 k_c} J_1(k_c r) e^{i\omega t} \end{aligned}}$$

$$\omega = \omega_c = \frac{x_{01} c}{r_c} \; ; \; k_c \overset{\text{or}}{=} \frac{\omega_c}{c} = \frac{\omega}{c} = \frac{x_{01}}{r_c}$$

$$\frac{\omega}{c k_c} = 1$$

$$\boxed{\begin{aligned} E_z &= \tilde{E}_0 J_0\left(\frac{x_{01}}{r_c} r\right) e^{i\omega t} \\ E_r &= 0 \\ B_\theta &= -\frac{i\tilde{E}_0}{c} J_1\left(\frac{x_{01} r}{r_c}\right) e^{i\omega t} \end{aligned}}$$

$$\boxed{\tilde{E}_0 \equiv E_0 e^{i\phi} \qquad \begin{aligned} E_0 &= \text{Amp. (Real)} \\ \phi &= \text{Phase (Real)} \end{aligned}}$$

and take the fields to be given by the Real part of the complex expression

$$\mathrm{Re}\left[\tilde{E}_0 e^{i\omega t}\right] = \mathrm{Re}\left[E_0 e^{i(\omega t+\phi)}\right] = E_0 \cos(\omega t+\phi)$$

$$\mathrm{Re}\left[i\tilde{E}_0 e^{i\omega t}\right] = \mathrm{Re}\left[iE_0 e^{i(\omega t+\phi)}\right] = -E_0 \sin(\omega t+\phi)$$

Giving

TM010 cavity fields

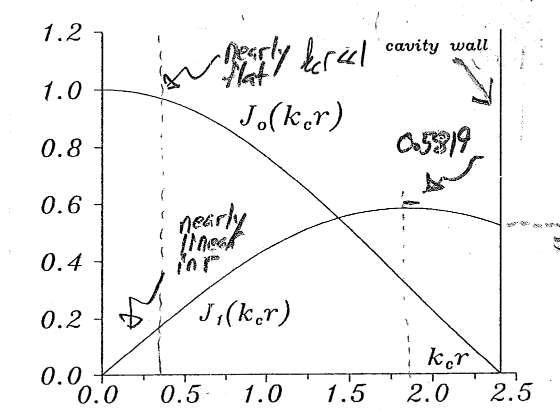
$$\boxed{\begin{aligned} E_z &= E_0 J_0\left(\frac{x_{01} r}{r_c}\right)\cos(\omega t+\phi) \\ E_r &= 0 \\ B_\theta &= -\frac{E_0}{c} J_1\left(\frac{x_{01} r}{r_c}\right)\sin(\omega t+\phi) \end{aligned}}$$

Used phase choices for previous convention
(t=0 at z=0 center of cavity)

Used $k_c = \frac{\omega}{c}$ for $k=0$
in B_θ + reflection terms with consistent phase choices

Comments:

* All other field terms zero. $E_r = 0$ due to $k=0$
* Finite beam apeture at ends will allow $E_r \neq 0$ for this mode.



Wiedemann

$$k_c = \frac{x_{01}}{r_c} \simeq \frac{2.405}{r_c}$$

$J_1(x_{01}) \simeq 0.519$

* Beam will only fill a small fraction of r_c $\Rightarrow k_c r/r_c \ll 1$

$J_0(k_c r) \simeq 1$ Nearly uniform E_z

$J_1(k_c r) \simeq \frac{k_c r}{2}$; $B_\theta \propto r$ $\Rightarrow$ Linear focus optic.

(Usually limited impact)

Reminder: In RF defocus analysis: approximated $\left.\begin{aligned} E_z(r,z) &\simeq \text{const} \\ B_\theta(r,z) &\propto r \end{aligned}\right\}$ Near $r=0$

This verifies!

Note:

Max B_θ at $k_c r = 1.891$ where $J_1(k_c r) = J_1(1.891) \simeq 0.5819$ @ End-Plates

Max $E_z = E_0$ at $r = 0$ where $J_0(0) = 1$

Therefore: $$\frac{cB_{Max}}{E_{Max}} = \frac{J_1(1.891)}{J_0(0)} = \frac{0.5819}{1} = 0.5819$$

This number can have implications for the cavity field stress / breakdown.

$E_{Max} = E_0$ as large as possible for strong acceleration.

However, larger E_{Max} can trigger breakdown issues and larger $E_{Max} \Rightarrow$ larger B_{Max} (on cavity ends) which can also induce a quench for superconducting cavities. Realistic cavities shaped to try to limit these issues. $\Rightarrow$ Elliptical Cavities for Superconducting RF (SRF) applications.

Pillbox cavity resonant frequency:

$$\omega = 2\pi f = \omega_c = \frac{X_{01} c}{r_c} \qquad X_{01} \simeq 2.405$$

$$\boxed{f = \frac{2.405\, c}{2\pi r_c}}$$ Cavity Frequency

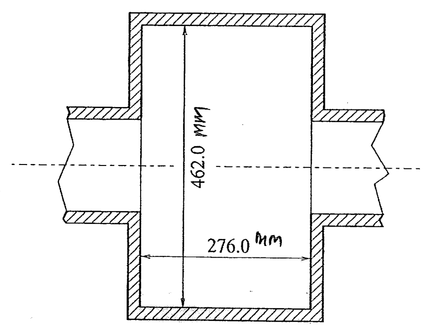
Some numbers:

$$2r_c = \frac{2.405 c}{\pi f}$$

Cavity freq f	Cavity Diameter $2r_c$
1 MHz	240 m
10 MHz	24 m
50 MHz	5 m
100 MHz	2.5 m
500 MHz	45.9 cm
1 GHz	25 cm
3 GHz	8 cm

Higher frequencies desired to limit size of cavities and control cost.

DORIS Storage Ring Cavity
German Electron Synchrotron Lab DESY



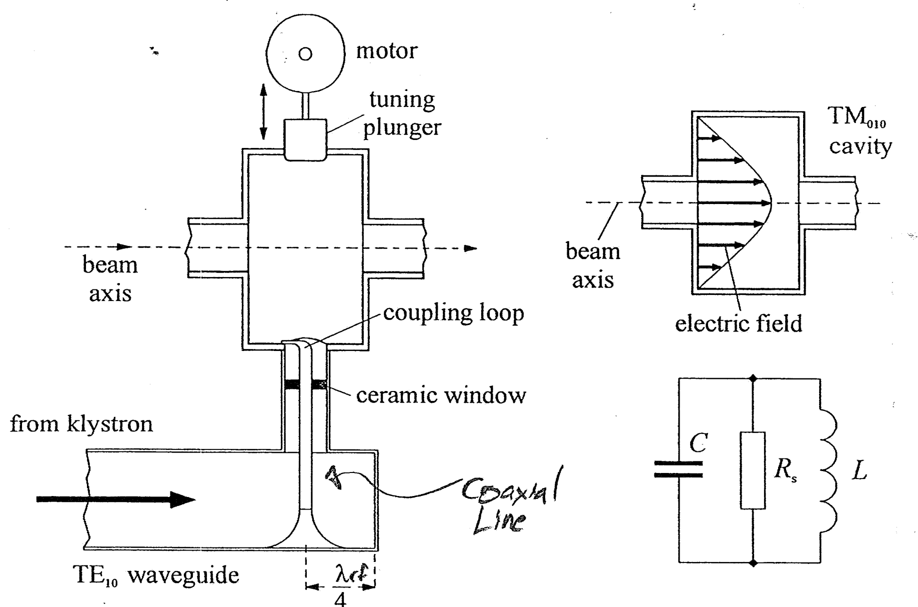
Wille

Fig. 5.3 Example of a single-cell cavity. It is chosen to have the dimensions $D = 462$ mm and $l = 276$ mm used in the accelerating structure developed for the storage ring DORIS, designed for a resonant frequency of 500 MHz.

Cavities must be connected to an RF source such as a klystron. Typical connection sketched below.

- Waveguide carries TE_{10} mode from klystron.
- Waveguide terminated near RF cavity
- Coaxial cable pickup $\sim \lambda_{rf}/4$ from waveguide termination ($\sim E$ max location)
- Connections shaped to inhibit reflections / losses.
- Ceramic window separates waveguide / coaxial cable (normal pressure) from cavity (high vacuum) without impeding RF wave.
  + Cavity window technology demanding for high power/voltages!
- RF wave coupled to TM_{010} symmetry of cavity by a loop.
  + Loop gives magnetic coupling where B_θ is near max on the outer radial wall of cavity.

Many details to do optimally; Just a brief outline here.: MSU RF Power Engineering course + USPAS.



Wille

Fig. 5.4 Design of a single-cell accelerating structure using the TM_{010} mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

A stable standing wave will exist in cavity only if the resonance condition of the TM_{010} mode is precisely satisfied.

Following an identification of cavity equivalent circuit parameters, will show that

$$Q = \frac{\omega_{res}}{\Delta\omega} = \frac{R_s}{|Z|} \gg 1$$

$\Rightarrow \Delta\omega$ small

ω_{res} = resonant cavity ω

$\Delta\omega$ = Frequency bandwidth for ½ power.

Cavity Stored Energy: Pillbox Cavity TM_{010} mode

At any given instant in time t the energy stored in an RF cavity is:

$$\boxed{U = \frac{\varepsilon_0}{2}\int_{cavity} \vec{E}^2 d^3x + \frac{1}{2\mu_0}\int_{cavity} \vec{B}^2 d^3x = \text{Stored EM Energy}}$$

Field Energy Densities

$$\rho_E = \frac{\varepsilon_0}{2}\vec{E}^2$$

$$\rho_M = \frac{1}{2\mu_0}\vec{B}^2$$

Use Pillbox cavity fields and take $\omega t + \phi = 0$; $U = \text{const}$ so can take any time.

☆ This choice ⇒ all energy in E-field.

$$E_z = E_0 J_0(k_c r)\cos(\omega t + \phi) = E_0 J_0(k_c r)$$

$$B_\theta = \frac{-E_0}{c} J_1(k_c r)\sin(\omega t + \phi) = 0$$

$$k_c = \frac{X_{01}}{r_c}$$

and:

$$U = \frac{\varepsilon_0}{2}\int_{cavity} E_z^2 d^3x = \frac{\varepsilon_0}{2}(2\pi) E_0^2 \ell \int_0^{r_c} [J_0(k_c r)]^2 r\,dr$$

$$\int_{cavity} d^3x \ldots = \int_{-\pi}^{\pi} d\theta \int_0^{r_c} dr\, r \int_{-\ell/2}^{\ell/2} dz \ldots$$

$$\int_{cavity} dz = \ell = \text{length cavity}$$

$$\int_{cavity} d\theta = 2\pi \quad \text{Angular Range.}$$

Using integral tables:

$$\int_0^1 t J_n(x_{nj}t) J_n(x_{nk}t)\,dt = \frac{1}{2}[J_n'(x_{nj})]^2 \delta_{jk}$$

$$\int_0^1 t J_0(X_{01}t)^2 dt = \frac{1}{2}[J_0'(X_{01})]^2 = \frac{1}{2}[J_1(X_{01})]^2$$

$$J_0'(t) = -J_1(t)$$

We have

$$\int_0^{r_c} \left[J_0\left(\frac{X_{01}}{r_c} r\right)\right]^2 r\,dr = r_c^2 \int_0^1 [J_0(X_{01}t)]^2 t\,dt = \frac{r_c^2}{2}[J_1(X_{01})]^2$$

$$k_c r = \frac{X_{01}}{r_c} r$$

$$t = r/r_c$$

$$\Rightarrow \boxed{U = \frac{\varepsilon_0}{2} E_0^2 \pi r_c^2 \ell \cdot [J_1(X_{01})]^2}$$

Numerically:

$$J_1(X_{01}) \simeq J_1(2.405) \simeq 0.51911$$

$$\frac{\pi}{2}[J_1(X_{01})]^2 \simeq 0.423$$

$$U \simeq 0.423\,\varepsilon_0 E_0^2 r_c^2 \ell$$

Cavity Dissipation: Pillbox Cavity ; Ref: Pick favorite EM Book.

No perfect conductors exist, but conductivity can be high:

Copper $\frac{1}{\sigma} \simeq 1.7 \times 10^{-8}\ \Omega \cdot m$

For a good but imperfect conductor, the fields penetrate the conductor in a thin surface layer where they fall off rapidly beyond a "skin depth" δ for fields varying at harmonic frequency ω:

$$\boxed{\text{Skin Depth} \quad \delta = \sqrt{\frac{2}{\sigma \mu_0 \omega}}}$$

Copper @ 100 MHz
$\Rightarrow \delta \simeq 10^{-6}\, m = 1\ \mu m$

Because of skin depth AC and DC resistances are not equal.

$$\boxed{\text{RF Surface Resistance} \quad R_{surf} = \frac{1}{\sigma \delta} = \sqrt{\frac{\mu_0 \omega}{2\sigma}}} \propto \omega^{1/2} \sim [\text{ RF frequency }]^{1/2}$$

AC and DC resistance varies.

Copper @ 100 MHz
$R_{surf} \sim$ milli-Ohm.

Electromagnetic theory texts show that the time averaged power loss to the walls over the RF cycle for a Harmonic varying fields:

$$\langle P_{loss} \rangle_{rf} = \frac{1}{\tau_{rf}} \int_0^{\tau_{rf}} P_{loss}\, dt = \frac{R_{surf}}{2} \int_{\substack{\text{Cavity}\\\text{Surface}}} |\vec{H}_t|^2\, ds$$

P_{loss} = Instantaneous lost Power

$\vec{H}_t = \hat{n} \times \vec{H}$ = Tangential component $\vec{H}$ at conductor. $\hat{n}$ = normal conductor

$\sim e^{-i\omega t}$ vary

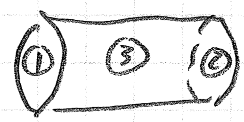
Interpretation: $H_t \rightarrow$ surface current.

Integrate loss over cavity surface.

Apply this loss formula to the RF pillbox cavity

$$\langle P_{loss} \rangle_{rf} = \frac{R_{surf}}{2} \int |\vec{H}_t|^2 \, dS \qquad B_\theta = \mu_0 H_\theta = \frac{-E_0}{c} J_1(k_c r) e^{\overset{\circ}{j}\omega t} \qquad k_c = \frac{x_{01}}{r_c}$$

Will have surface contributions



①,② ends $\quad |H_t| = \frac{E_0}{\mu_0 c} J_1\left(\frac{x_{01} r}{r_c}\right)$; Amplitudes

③ Outer Pipe $\quad |H_t| = \frac{E_0}{\mu_0 c} J_1(x_{01})$

$$\langle P_{loss} \rangle_{rf} = \frac{R_{surf}}{2} \left\{ \underbrace{2}_{\text{2 ends}} \cdot \times \underbrace{2\pi}_{\theta \text{ Integral}} \left(\frac{E_0}{\mu_0 c}\right)^2 \int_0^{r_c} J_1^2\left(\frac{x_{01} r}{r_c}\right) r \, dr + \underbrace{(2\pi r_c)}_{\text{cylinder circumfrance}} \underbrace{(l)}_{\text{cylinder length}} \underbrace{\left(\frac{E_0}{\mu_0 c}\right)^2 [J_1(x_{01})]^2}_{(\text{Field const})^2 \times \text{Area}} \right\}$$

But from integral tables and properties of Bessel functions

$$\int_0^{r_c} J_1^2\left(\frac{x_{01} r}{r_c}\right) r \, dr = r_c^2 \int_0^1 J_1^2(x_{01} t) \, t \, dt = r_c^2 \int_0^1 J_0^2(x_{01} t) \, t \, dt$$

Apply prior result used for stored energy U

$$= \frac{r_c^2}{2} [J_1(x_{01})]^2$$

$$\Rightarrow \boxed{\langle P_{loss} \rangle_{rf} = \pi r_c (r_c + l) R_{surf} \cdot \left(\frac{E_0}{\mu_0 c}\right)^2 [J_1(x_{01})]^2} \simeq 0.847 \, r_c (r_c + l) R_{surf} \left(\frac{E_0}{\mu_0 c}\right)^2$$

Numerically

$$\pi [J_1(x_{01})]^2 \simeq 0.847$$

Typical Cavity Result

* Loss depends on surface resistance (R_{surf}), peak field (E_0), and geometric parameters (Cavity geom specific)

* Need Low R_{surf} for low losses.

Scaling of Rsurf:

Normal Conducting

$$R_{surf} = \sqrt{\frac{\mu_0 \omega}{2\sigma}} \propto f_{rf}^{1/2}$$

Room Temp Copper at $f_{rf} \sim 100$ MHz

$R_{surf} \sim$ milli - Ohm

Superconducting Niobium   Ref. Wangler

$$R_{surf} = \underbrace{9\times 10^{-5} \frac{f_{rf}^2(\text{GHz})}{T(^\circ K)} \exp\left(-\eta \frac{T_c}{T}\right) \Omega}_{\text{BCS Theory}} + \underbrace{R_{residual}}_{\text{Material Imperfections}}$$

$\eta = 1.92$

$T_c = 9.2^\circ K$ Critical Temp. (Niobium)

$R_{residual}$ = Residual Resistance $\sim 10^{-9}-10^{-8}\,\Omega$ typical

* Supercond perfect at DC but has AC resistance due to moving Cooper Pairs

$R_{surf} \propto f_{rf}^2$ for high freqs

$$\underbrace{R_{surf} \sim 10^{-5} \times (R_{surf}\ \text{Copper})}_{\text{Typical}}$$

Dramatic reduction, but SRF materials expensive and fragile + cryogenic cooling is costly.

Quality Factor :

Define in full generality (any cavity):

$$\boxed{\text{Quality Factor} = Q = 2\pi \frac{U}{\langle P_{loss}\rangle_{rf} \cdot \tilde{T}_{rf}} = 2\pi \times \frac{\text{Energy Stored}}{\text{Energy Dissipated in RF Cycle}}}$$

$$Q = \frac{2\pi}{\tilde{T}_{rf}} \frac{U}{\langle P_{loss}\rangle_{rf}} = \frac{\omega U}{\langle P_{loss}\rangle_{rf}} \Rightarrow \boxed{Q = \frac{\omega U}{\langle P_{loss}\rangle_{rf}}}$$

Pillbox Cavity Q

Using previous results for pillbox cavity

$$Q = \frac{\omega \overbrace{\left[\frac{\varepsilon_0}{2} E_0^2 \pi r_c^2 l \left[J_1(x_{01})\right]^2\right]}^{\text{"} U}}{\underbrace{\left[\pi r_c (r_c + l) R_{surf} \left(\frac{E_0}{\mu_0 c}\right)^2 \left[J_1(x_{01})\right]^2\right]}_{\text{"} \langle P_{loss}\rangle_{rf}}}$$

$$= \frac{\omega \overbrace{(\varepsilon_0 \mu_0 c^2)}^{\text{"} 1} \mu_0}{2 R_{surf}} \frac{r_c^2 l}{r_c (r_c + l)}$$

$c^2 = \frac{1}{\mu_0 \varepsilon_0}$

But at resonant frequency $\frac{\omega}{c} = \frac{x_{01}}{r_c}$ $x_{01} \simeq 2.405$

$$= \frac{\omega}{c} \frac{c \mu_0}{2 R_{surf}} \frac{r_c l}{r_c + l}$$

Pillbox Cavity

$$\boxed{Q = \frac{x_{01} \sqrt{\mu_0/\varepsilon_0}}{2 R_{surf}} \frac{1}{1 + r_c/l}} \underset{\text{Numerically}}{\simeq} 1.203 \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{R_{surf}} \frac{1}{1 + r_c/l}$$

Want very high Q for cavity

$\Rightarrow$ R_s low : good conductor or super conductor

NC Example: DESY DORIS pill box Cu cavity $Q \simeq 38{,}000 \sim 10^5$ @ 500 MHz

SC Example: FRIB Quarter Wave SRF Cavity $Q \sim 10^9 - 10^{10}$ range.

High Q corresponds to:

* ★ Low heat generation
* ★ High efficiency
* ★ High stability : to variations in RF drive and beam loading

To understand the stability point, suppose an isolated cavity has stored energy U_0 in oscillatory mode with angular frequency ω: If the drive is removed the energy U will chang as:

$$\frac{dU}{dt} = -\langle P_{loss}\rangle_{rf} = \frac{-\omega U}{Q} \qquad \text{since} \quad Q \equiv \frac{\omega U}{\langle P_{loss}\rangle_{rf}}$$

This has solution:

$$U(t) = U_0 \cdot e^{-\omega t/Q}$$

$\Rightarrow$ slow decay for Q large, giving good stability

A commonly used figure of merit of an RF acceleration system is the so-called <u>shunt impedance</u>   See Wangler, Sec. 2.5

$$V_0 = E_0 L = \text{Effective cavity voltage}$$

$$\boxed{\text{Shunt Impedance:} \quad R_s \equiv \frac{V_0^2}{\langle P_{loss}\rangle_{rf}}}$$

Note Ohms Law: $V = IR$, $P = VI = \frac{V^2}{R}$ $\Rightarrow$ $R = \frac{V^2}{P}$

Caution: (Beware factor of 2!) Sometimes defined as $R_s = \frac{V_0^2}{2\langle P_{loss}\rangle_{rf}}$ due to interpretation of harmonic averaging factors.

Large shunt impedance $\Rightarrow$ Large accelerating potential / relative to cavity dissipation. for economical acceleration.

But due to transit time factor, the accel potential V_0 is not fully imparted to particles. Therefore, define an "effective shunt impedance" to take this into account using synchronous phase $\phi_s = 0$ (Max accel.)

$$\Delta W = q(E_0 L) T \cos\phi_s \qquad \text{Panofsky Equation}$$

$$\Rightarrow \Delta W_{Max} = q V_0 T \qquad T = \text{Transit Time Factor} \Rightarrow \begin{array}{l} E_0 L \Rightarrow E_0 L T \\ V_0 \Rightarrow V_0 T \end{array}$$

in previous formulas for "effective" measures.

$$\boxed{\text{Effective Shunt Impedance} \quad R_{s,eff} = \frac{(V_0 T)^2}{\langle P_{loss}\rangle_{rf}} = \left(\frac{V_0^2}{\langle P_{loss}\rangle_{rf}}\right) T^2 = R_s T^2}$$

Sometimes these are analyzed per axial length L for long systems:

$$\boxed{\frac{R_{s,eff}}{L} = \frac{E_0^2 L^2 T^2}{L \langle P_{loss}\rangle_{rf}} = \frac{(E_0 T)^2}{\langle P_{loss}\rangle_{rf}/L}}$$

Typically given in MΩ/meter

Another figure of merit is "R over Q":

$$\boxed{\text{R over Q}: \quad \frac{R}{Q} \equiv \frac{R_{s,eff}}{Q} = \frac{(V_0 T)^2}{\langle P_{loss}\rangle_{rf}} \cdot \frac{\cancelto{1}{\langle P_{loss}\rangle_{rf}}}{\omega U} = \frac{(V_0 T)^2}{\omega U}}$$

☆ Measures efficiency acceleration per unit stored energy at specific frequency RF

☆ Function only of cavity geometry. – Independent of surface properties of power loss.

Energy imparted to beam particles must also come from RF cavity fields.

Instantaneous Power Delivered by Beam

$$\boxed{P_B = (\#\ \text{Particles}) \cdot \Delta W = \frac{I_{beam}(t)}{q} \Delta W}$$

I_{beam} = beam electrical current. (instantaneous)

The total average power delivered will be

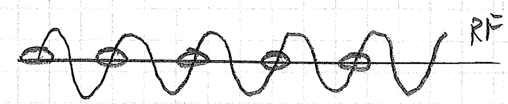
$$\langle P_{Total} \rangle_{rf} = \langle P_{loss} \rangle_{rf} + \langle P_B \rangle_{rf}$$

Take $\langle P_B \rangle_{rf} = \eta_{fill} \dfrac{\langle I_{beam} \rangle_{rf}}{q} \Delta W$ $\quad$ η_{fill} = Bucket Fill-Factor

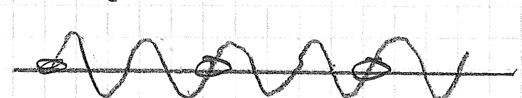
$$\langle I_{beam} \rangle_{rf} = \frac{Q_{bunch}}{T_{rf}}$$

$Q_{bunch} / q = N_{bunch}$ = # particles in bunch

η_{fill} = Bucket fill fraction in machine pulse



$\eta_{fill} = 1$ $\quad$ All buckets filled (every rf period)



$\eta_{fill} = \frac{1}{2}$ $\quad$ Half buckets filled (every other rf period)

★ $\eta_{fill} < 1$ occurs when transitioning to higher freq rf structures.

$$\boxed{\langle P_{Total} \rangle_{rf} = \langle P_{loss} \rangle_{rf} + \eta_{fill} \frac{N_{bunch} \cdot \Delta W}{T_{rf}}}$$

The efficiency of the accelerating structure can be

$$\boxed{\mathcal{E} \equiv \frac{\langle P_B \rangle_{rf}}{\langle P_{Total} \rangle_{rf}}}$$ Efficiency

For "wall-plug" efficiency must account for other losses:

- ☆ RF Generation
- ☆ Focusing + Bending magnet dissipation
- ☆ Front end
- ☆ Cryo-Plant efficiencies for any superconducting systems

More efficient accelerators opens the door for more applications:

- ☆ Material processing
- ☆ Energy Production : Subcritical reactors, Actinide Burning, Fusion drivers
- ⋮

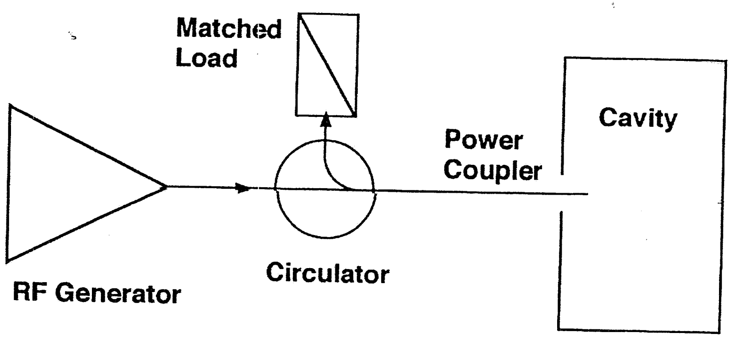
Generally want more beam current for high efficiency and this can make accelerator physics much more difficult due to beam space-charge effects, cavity loading, etc.

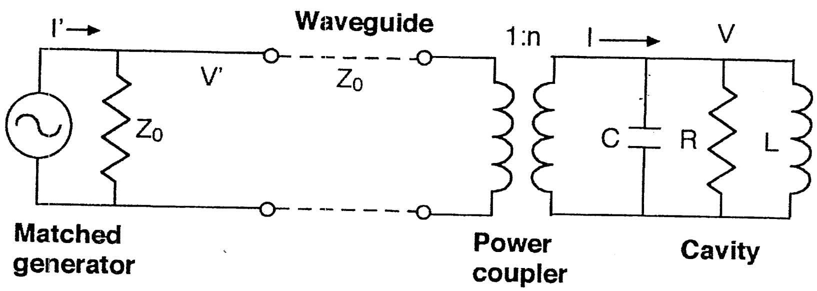
- ☆ Much room for future improvements to enable more applications.

Equivalent Circuit for RF Cavity

Motivated by the qualitative correspondance to circuit parameters for the RF cavity the response of the system is idealized in terms of an equivalent circuit:

Equivalent Circuit

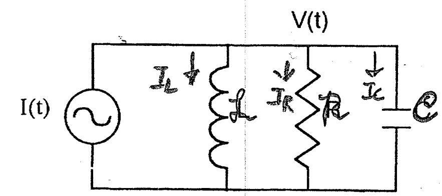




Wangler

Figure 5.3 (*a*) Block diagram of RF system components and (*b*) the equivalent circuit.

Cavity Component (idealized)



Wangler

$V(t) \Rightarrow$ Cavity Voltage $\sim E_0 L$

$\mathcal{L}$ = Cavity Inductance
$\mathcal{R}$ = Cavity Resistance
$\mathcal{C}$ = Cavity Capacitance

Current Conservation / Kirchoff's Law

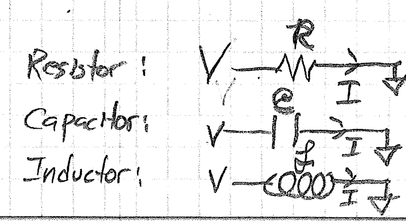
$$I(t) = I_L + I_R + I_c$$

$$= \frac{\int V dt}{\mathcal{L}} + \frac{V}{\mathcal{R}} + \mathcal{C}\frac{dV}{dt}$$

$$\dot{I} = \frac{V}{\mathcal{L}} + \frac{\dot{V}}{\mathcal{R}} + \mathcal{C}\ddot{V}$$

$V(t)$ = RF cavity Voltage $\sim E_0 \cdot L$

Recall:



$V = I\mathcal{R}$
$I = \mathcal{C}\, dV/dt$
$V = \mathcal{L}\, dI/dt$

Driving current $I(t)$ produces voltage $V(t)$

$\boxed{\begin{array}{l} V(t) \sim V_0 e^{j\omega t} \iff \text{Axial accelerating voltage } V = E_0 L \text{ of cavity with harmonic variation.} \\ \qquad \star\text{ No Transit time factor ... cavity only.} \\ \frac{1}{2} C V_0^2 = U \iff \text{Energy } U \text{ stored in the cavity. Sets capacitance } C \\ \langle P_{loss} \rangle_{rf} = \frac{1}{2}\frac{V_0^2}{R} \iff \text{Power lost in cavity. Sets restance } R \end{array}}$

Express equation $\dot{I} = \frac{V}{L} + \frac{\dot{V}}{R} + C\ddot{V}$ as:

$$\ddot{V} + \underbrace{\frac{1}{RC}\dot{V}}_{\text{Damping}} + \underbrace{\frac{1}{LC}V}_{\text{Restore}} = \underbrace{\frac{\dot{I}}{C}}_{\text{Drive}}$$

Express as:

$$\boxed{\ddot{V} + \frac{\omega_{res}}{Q}\dot{V} + \omega_{res}^2 V = \frac{\dot{I}}{C}}$$

$\boxed{\omega_{res} = \frac{1}{\sqrt{LC}} = \text{Resonant Freq} \iff \text{Set } L \text{ to get correct angular freq.}}$ 1)

Motivated from prev damping analysis: pg 78/

$$\frac{1}{RC} = \frac{\omega_{res}}{Q} = \frac{1}{\sqrt{LC}\,Q} \Rightarrow Q = R\sqrt{\frac{C}{L}} = \omega_{res}\frac{U}{\langle P_{loss}\rangle_{rf}} = \omega_{res} RC$$

$\boxed{Q = \omega_{res}\frac{U}{\langle P_{loss}\rangle_{rf}} = \omega_{res} RC \iff \text{Set } R \text{ to get correct damping}}$ 2)

Reminder:

$\boxed{U = \frac{1}{2} C V_0^2 \iff \text{set } C \text{ to get correct stored energy}}$ 3)

1), 2), 3) to fix circuit params

Search for a harmonic steady-state solution ($t \to \infty$) of circuit.

$$I(t) = I_0 e^{i\omega t}$$

$\omega = \text{const}$ angular freq. (need not satisfy $\omega = \omega_{res}$)
$I_0 = \text{const}$

Analysis shows that (electrical engineering texts)

$$\boxed{V(t) = \frac{R I_0 e^{i(\omega t + \phi)}}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}\right)}} \qquad \phi = -\tan^{-1}\left[Q\left(\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}\right)\right]}$$

Denote

$$\boxed{\Delta\omega = \omega - \omega_{res}}$$

Then

$$Q\left[\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}\right] = Q\left[1 + \frac{\Delta\omega}{\omega_{res}} - \frac{1}{1 + \frac{\Delta\omega}{\omega_{res}}}\right] \simeq 2Q\frac{\Delta\omega}{\omega_{res}}$$

The frequency shift $\Delta\omega$ to reduce the voltage amplitude to $1/\sqrt{2}$ the value (i.e., the 1/2 power value) relative to on resonance is:

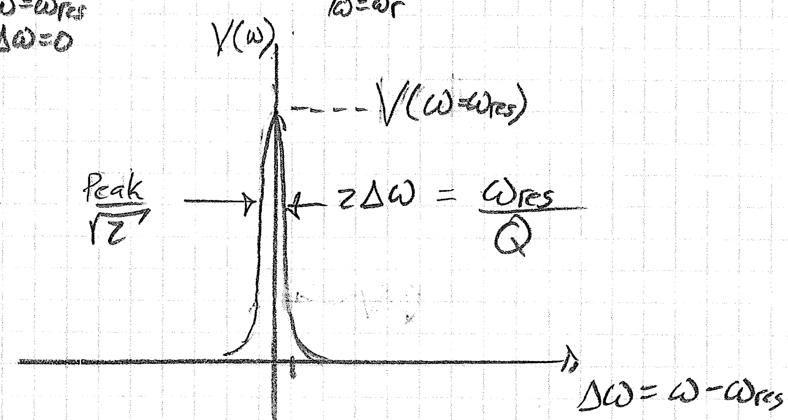
$$V_{res}(t) = V(t)\Big|_{\substack{\omega=\omega_{res}\\ \Delta\omega=0}} = R I_0 e^{i\omega_r t}\Big|_{\substack{\phi=0\\ \omega=\omega_r}}$$

$$V(t)\Big|_{\omega=\omega_{res}+\Delta\omega} = \frac{\left(R I_0 e^{i\omega_r t}\right) e^{i(\Delta\omega t + \phi)}}{\sqrt{1 + 4Q^2\frac{\Delta\omega^2}{\omega_{res}^2}}} \equiv V(\omega) e^{i\omega t}$$

$$= \frac{\left(V(t)\big|_{\Delta\omega=0}\right)}{\sqrt{2}} \cdot \text{phase}$$

for $4Q^2\frac{\Delta\omega^2}{\omega_{res}^2} = 1 \Rightarrow \Delta\omega = \frac{\omega_{res}}{2Q}$

$\sim \frac{1}{2}$ Power $\rightsquigarrow$



$\boxed{\therefore \text{ High } Q \text{ means very sharply tuned resonant frequency.}}$

Frequency Scaling in RF Cavity Figures of Merit   Wangler 2.7

One of the most important parameters to choose in design is the cavity frequency f_{rf}

$$\boxed{\omega = \frac{2\pi}{T_{rf}} = 2\pi f_{rf}}$$

Take:

$$\left.\begin{aligned} E_0 &= \text{const} \\ \Delta W &= \text{const} \end{aligned}\right\}$$ fixed independent of f_{rf} and fix length L

Scale all other cavity dimensions with RF wavelength $\lambda_{rf} = c T_{rf} = \frac{c}{f_{rf}}$

Then

Transit Time T independent of f_{rf} (regard energy gain fixed so $E_0 T/L = \text{const}$)

Cavity surface Area S $\sim r_c \sim \frac{1}{f_{rf}}$

Cavity Volume V $\sim r_c^2 \sim \frac{1}{f_{rf}^2}$ $\Rightarrow$ Cavity Stored Energy U $\sim \frac{1}{f_{rf}^2}$

Surface Resistance $R_{surf} \sim \begin{cases} f_{rf}^{1/2} & \text{Normal Cond (NC)} \\ f_{rf}^2 & \text{Superconducting (SC)} \end{cases}$ ← Skin depth scaling ← Neglect residual resistance (good approx large f_{rf})

Avg. Power Loss $\langle P_{loss} \rangle_{rf} \sim R_{surf} \frac{|B|^2}{\mu_0} S \sim S R_{surf} \sim \frac{1}{f_{rf}} \begin{cases} f_{rf}^{1/2} & NC \\ f_{rf}^2 & SC \end{cases} \sim \begin{cases} f_{rf}^{-1/2} & NC \\ f_{rf} & SC \end{cases}$

Quality Factor $Q = \frac{(\omega)\overset{\sim V}{(U)}}{\langle P_{loss} \rangle_{rf}} \sim (f_{rf})(1/f_{rf}^2) \begin{cases} f_{rf}^{1/2} & NC \\ f_{rf}^{-1} & SC \end{cases} \sim \begin{cases} f_{rf}^{-1/2} & NC \\ f_{rf}^{-2} & SC \end{cases}$

Effective Shunt "Impedance"

$$R_{s,\text{eff}} = \frac{(E_0 T)^2}{\langle P_{loss} \rangle_{rf}} \sim \frac{1}{\langle P_{loss} \rangle_{rf}} \sim \begin{cases} f_{rf}^{1/2} & NC \\ f_{rf}^{-1} & SC \end{cases}$$

★ Effective shunt impedance per unit axial length scales the same as $R_{s,\text{eff}}$.

R over Q

$$\frac{R}{Q} \equiv \frac{R_{s,\text{eff}}}{Q} \sim \frac{(V_0 T)^2}{\cancel{\langle P_{loss} \rangle_{rf}}} \frac{\cancel{\langle P_{loss} \rangle_{rf}}}{\omega U} \sim \frac{1}{\omega U} \sim f_{rf} \sim \begin{cases} f_{rf} & NC \\ f_{rf} & SC \end{cases}$$

★ R over Q scales same for NC and SC since it should be independent of surface properties.

Phase-space Area Bucket that can be accelerated

$$\simeq \frac{3\pi}{2} \tan(-\phi_s) \sqrt{\frac{Z_q E_0 T (\gamma_s \beta_s)^3 \lambda_{rf} (\sin(-\phi_s) - \phi_s \cos\phi_s)}{\pi m c^2}} \sim f_{rf}^{-1/2}$$

$f_{rf} \lambda_{rf} = c$

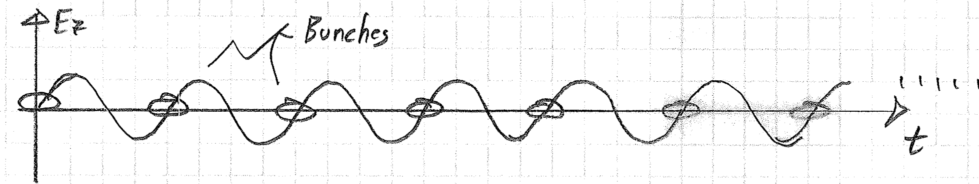
★ Higher frequency will lead to <u>lower longitudinal "acceptance"</u> for phase space area that can be accelerated by bucket.

– "Matching" important too if frequency transitions.

Comment: If linac has frequency transitions only harmonics and sub-harmonics are possible for a wave train of RF buckets. In certain cases only a limited fraction of buckets will be filled.

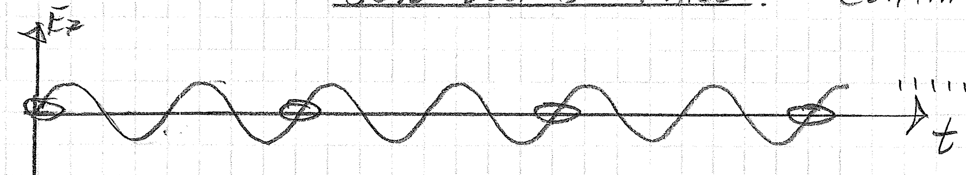
RF Bunch Structures

All Buckets Filled Continuous Wave



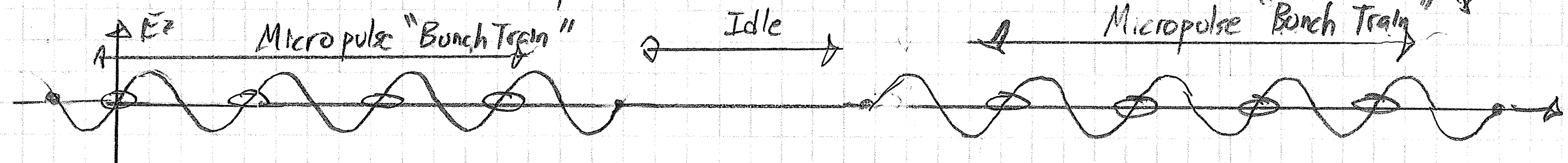
* Highest intensity on target
* Max use of RF
* Ideal for cyclotrons, high power RF linacs, etc.

50% Buckets Filled. Continuous Wave



* Skip any # buckets to reduce intensity on target.

2 Micropulses With all Buckets Filled



"Bunch Train" also called "Batch" in Synchrotrons

* Trains of bunches for consistency with sources etc.

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\+ Many Variants.

Many reasons for various micro-pulse structures.

* RF Structure limits in Power (more idle time)
* Source limitations of particles
* Frequency changes: transitions to higher frequencies for more compact structures.
* Target limitations
* Machine fill cycles of circular machines

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More on Cavities

RF Cavities very diverse topic. Can teach whole courses on just aspects of technology.

Beam tube on pillbox cavity adds complication:

* Want field concentrated on gap for larger transit-time factor.
* Opening large enough to get beam in and out of cavity $\Rightarrow$ E_r generated.
* Peak E may no longer be on-axis.

$$E_{acc} = \text{Accelerating E-field}$$

$$E_{peak} \sim 2\text{-}3 \times E_{acc}$$

$$\text{Figure of Merit} = \frac{E_{peak}}{E_{acc}}$$

SC Cavities

High E_{peak} $\Rightarrow$ Field emission e^-'s decreased efficiency + damage possible

NC Cavities

High E_{peak} $\Rightarrow$ Electric Breakdown Cavity damage + loss of E_{acc}

* Resonant cavity angular freq ω more sensitive to cavity dimensions.

* Large B_θ on outer walls of cavity can quench if superconducting critical magnetic field exceed. The critical field depends on temperature.

$$B_{critical} \sim 0.2 \text{ Tesla} \quad \text{for } 2\text{-}4.2\,^\circ K \quad \text{Niobium}$$

Impurities reduce:

$$B_{max} \sim 0.1 \text{ Tesla typical for operation.}$$

For pill box cavity $\frac{cB_{max}}{E_{max}} = \frac{cB_{max}}{E_{acc}} = 0.5819$

but this value can increase on drift-tubes, nose cones, etc.

Elliptical cavities shaped to reduce B_θ at outer wall.

Electron Field Emission   Limits SC Cavity E_{Max} ; Wangler 5.10

e^- emitted from surface in strong $\vec{E}$ field. $\Rightarrow$ strike cavity after gaining energy and generate heat + x-rays when stopping.

Lowers Q

Fowler-Nordheim Law:

$$\boxed{\text{Current Density } J \propto \frac{E_{peak}}{\Phi} \exp\left(-\frac{a\,\Phi^{3/2}}{E_{peak}}\right)}$$

Φ = Work function $\approx 4.3\,eV$ for Niobium

E_{peak} = peak electric field on surface.

a = const.

$$E_{peak} \sim 250 \times \begin{pmatrix} E_{Max} \text{ of Cavity} \\ \text{on Surface} \end{pmatrix}$$

Due to surface roughness.

Very important for superconducting surfaces to be clean and smooth!

RF Electric Breakdown   Limits NC Cavity E_{Max} ; Wangler 5.11

It is found empherically by Kilpatrick (Rev. Sci Inst., 28, 824 (1957).

* for a given freq f_{rf} the peak E field on the surface before breakdown given by

$$\boxed{f_{rf}(MHz) = 1.64\, E_{Max}^2\, e^{-8.5/E_{Max}} \quad E_{Max} \text{ in } MV/m} \quad (*)$$

$\Rightarrow$ Plot

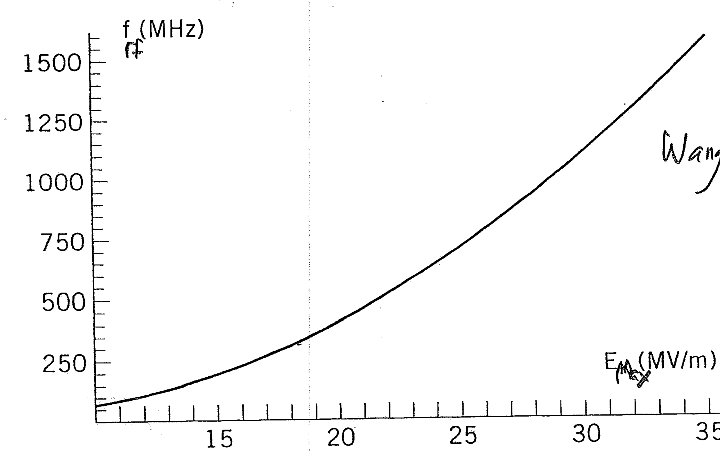


Figure 5.14 Kilpatrick formula from Eq. ~~(5.80)~~. *

* Somewhat conservative, often take

E_{Max} = B (E_{Max} from Kilpatrick)

B = "bravery factor" 1-2 typical

Idealized Pillbox Cavity is distorted to better optimize.

Translate circuit model to a cavity model: Directly driven, re-entrant RF cavity

Outer region: Large, single turn Inductor

$$\mathcal{L} = \frac{\mu_o \pi a^2}{2\pi(R+a)}$$

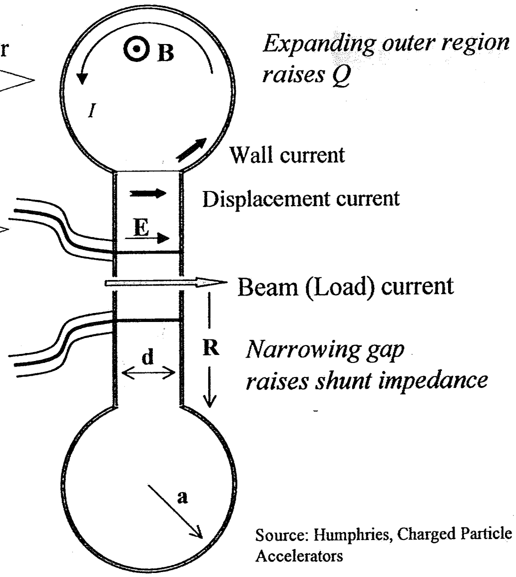
Central region: Large plate Capacitor

$$\mathcal{C} = \varepsilon_o \frac{\pi R^2}{d}$$

$$\omega_o = 1/\sqrt{\mathcal{LC}} = c\left[\frac{2((R+a)d}{\pi R^2 a^2}\right]^{1/2}$$

Q – set by resistance in outer region

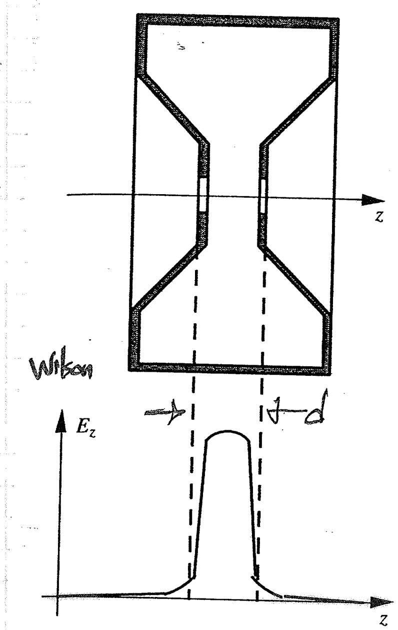
$$Q = \frac{\sqrt{L/C}}{R} \quad ?$$



Source: Humphries, Charged Particle Accelerators

Barletta

US PARTICLE ACCELERATOR SCHOOL



$E_{acc} \sim V_0$

$\mathcal{U} = \frac{1}{2}\mathcal{C} V_0^2$

$\langle P_{loss}\rangle_{rf} = \frac{1}{2} V_0/R$

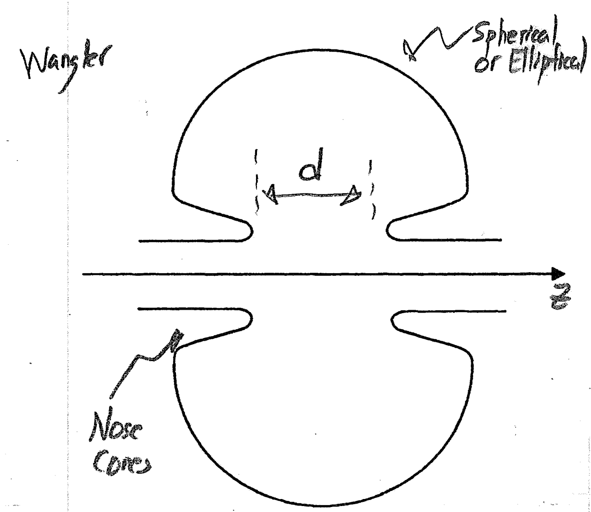
$\omega_{res} = \frac{1}{\sqrt{\mathcal{LC}}}$

$Q = \omega_{res}\frac{\mathcal{U}}{\langle P_{loss}\rangle_{rf}}$

Elliptical Cavity

want:

- Small gap d for efficient acceleration
  - Transit time Factor T large
  - Raise effective shunt impedance $R_{s,eff}$
- Expand outer region, raises Q
  - Distribute B_θ and reduce intensity for given V_0, d.

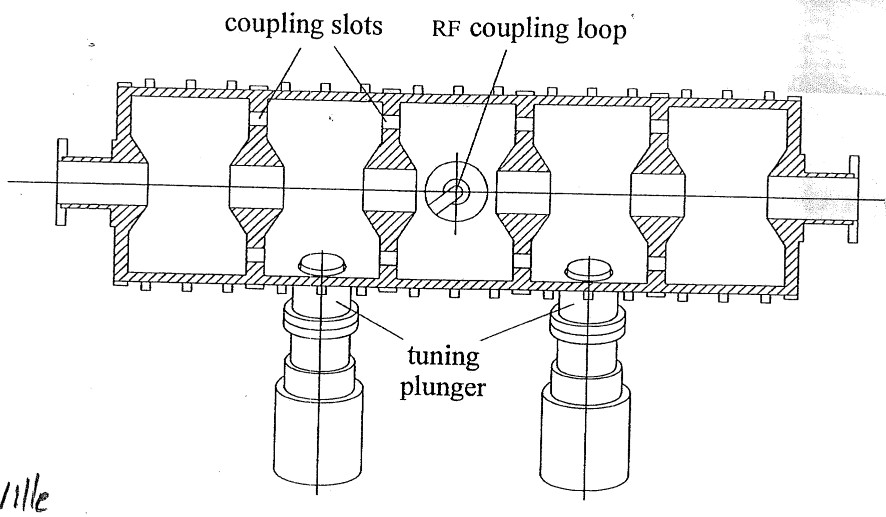


Coupled Cavities Groups of adjacent RF cavities are coupled together to maintain relative RF Phase control

* Common for high β particle acceleration
  - Simplifies RF drive
  - Saves cost
  - Many possible geometries

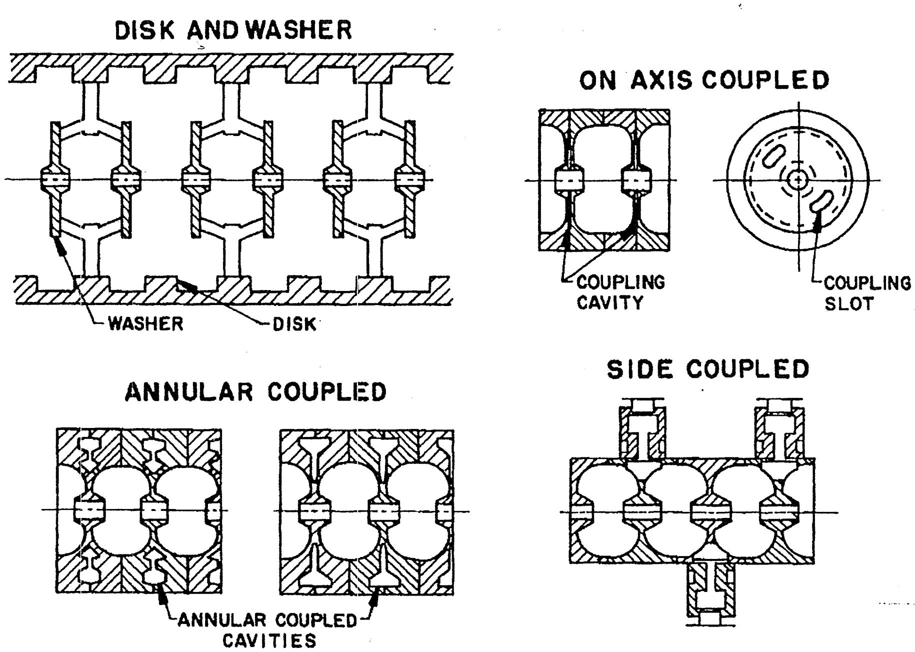
* Coupling can be through beam apertures or slots, or sometimes special coupling cavities
  - Coupling cavities sometimes off axis or minimal length to save space.

* Usually transverse focusing placed between banks of cavities.



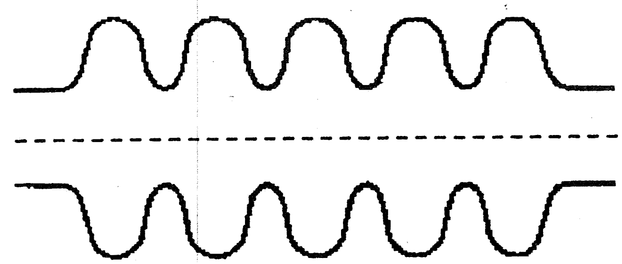
Wille

Fig. 5.5 Layout of a five-cell accelerating structure. The power feed is coupled to the middle cell and two tuning plungers are sufficient for the entire structure.



Wangler **Figure 4.17** Four examples of coupled-cavity linacs.

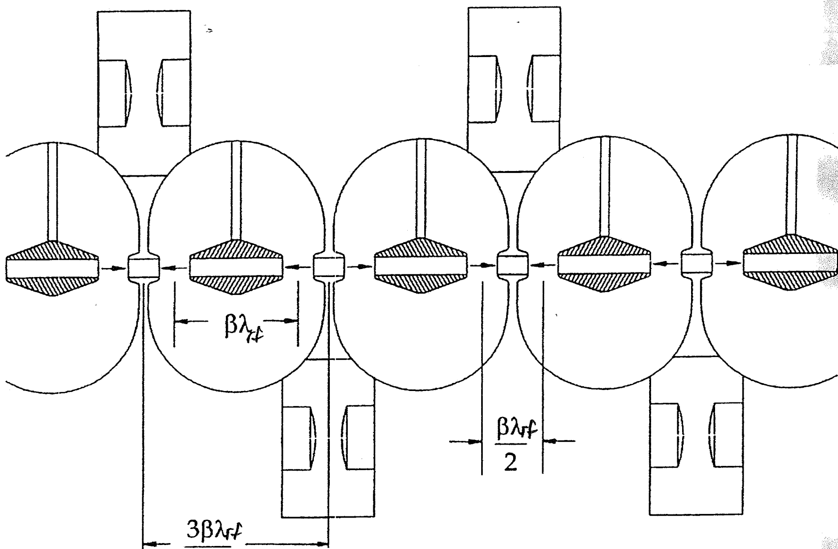
Elliptical : 5-Cell Bank



Wangler

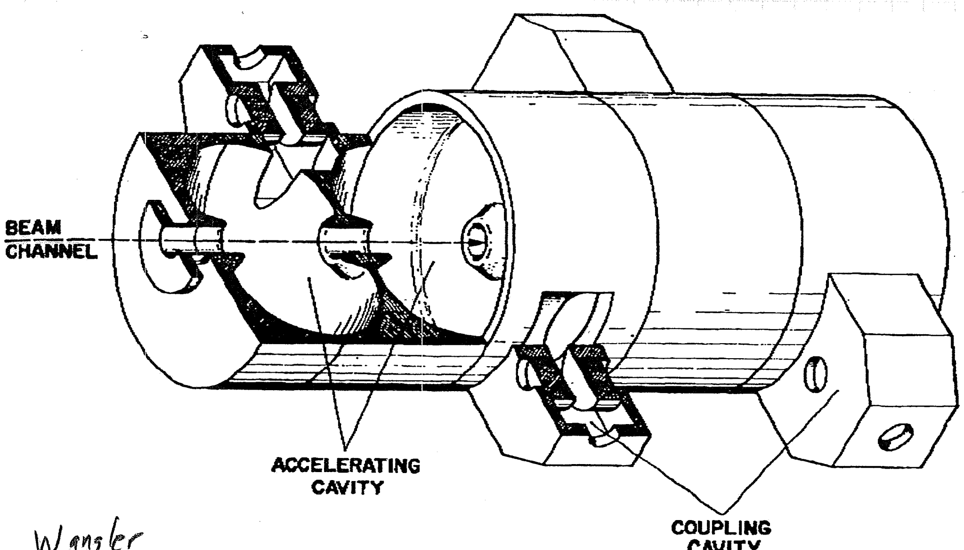
Figure 2.5 Cross section of a $\beta = 0.82$ elliptical cavity designed for a superconducting proton linac. The cross section for each cell consists of an outer circular arc, an ellipse at the iris, and a connecting straight line.

Phase relations between E-fields in cavities can vary.



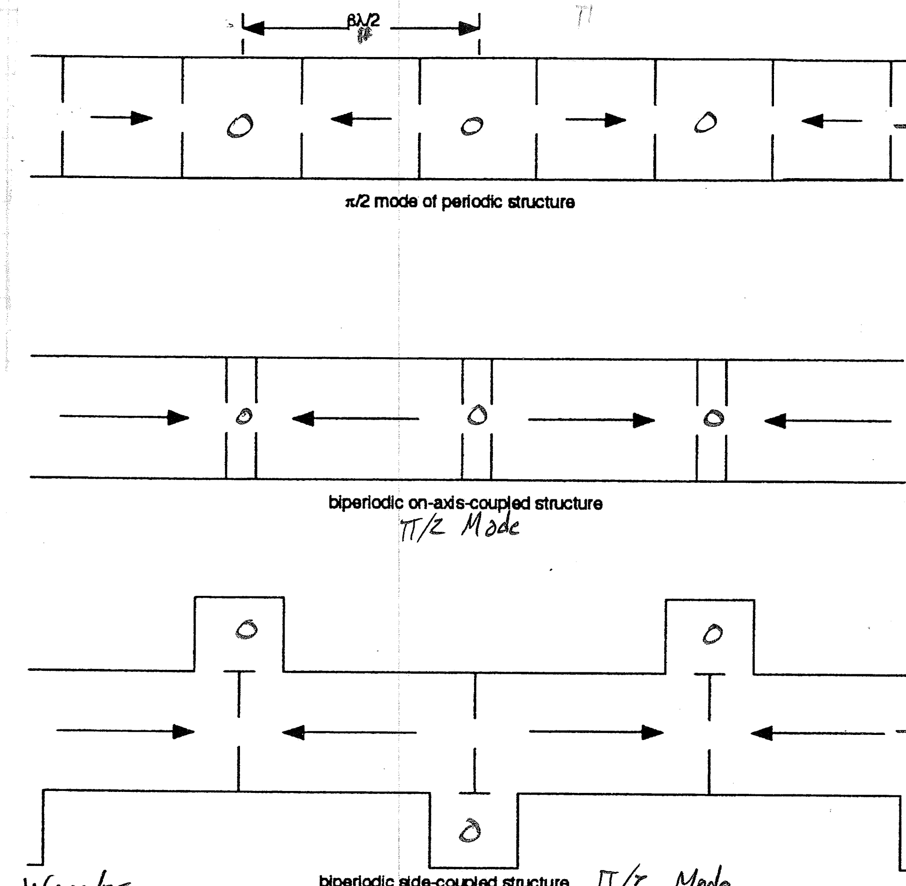
Wangler

Figure 12.2 Coupled-cavity drift-tube linac (CCDTL) structure with a single drift tube in each accelerating cavity.



Wangler

Figure 4.11 Side-coupled linac structure as an example of a coupled-cavity linac structure. The cavities on the beam axis are the accelerating cavities. The cavities on the side are nominally unexcited and stabilize the accelerating-cavity fields against perturbations from fabrication errors and beam loading.



Wangler

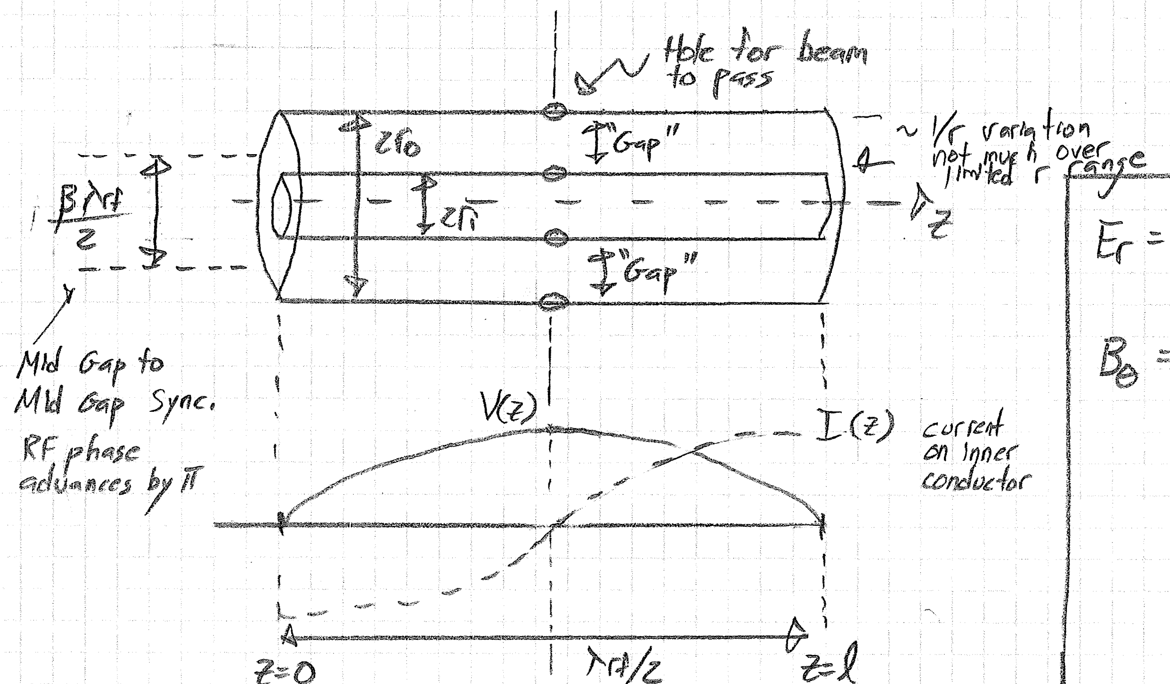
Figure 4.15 $\pi/2$-like-mode operation of a cavity resonator chain. From top to bottom are shown a periodic structure in $\pi/2$ mode, a biperiodic on-axis coupled-cavity structure in $\pi/2$ mode, and a biperiodic side-coupled cavity in $\pi/2$ mode.

Low Frequency Half and Quarter Wave RF Structures

For low freq. ion acceleration with cavities operating with $f_{rf} \lesssim 100$ MHz, cavities based on coaxial resonators are employed.

* Used in FRIB. 1/4 and 1/2 wave SRF cavities.

Basic Idea: Half Wave Structure



Will show on a homework problem that an EM standing wave solution exists with

$$E_r = -2\sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{I_0}{2\pi r} \sin\left(\frac{p\pi z}{\ell}\right) \sin(\omega t + \phi)$$

$$B_\theta = \frac{\mu_0 I}{\pi r} \cos\left(\frac{p\pi z}{\ell}\right) \cos(\omega t + \phi)$$

$$p = 1, 2, 3, \ldots$$

$$p = 1 \Rightarrow \text{Half-Wave}$$

$$\omega = \frac{p\pi c}{\ell}$$

I_0 = Amplitude of traveling wave current components on inner conductor.

$$V = \int_{r_i}^{r_o} E_r \, dr = \text{Accel. Voltage.}$$

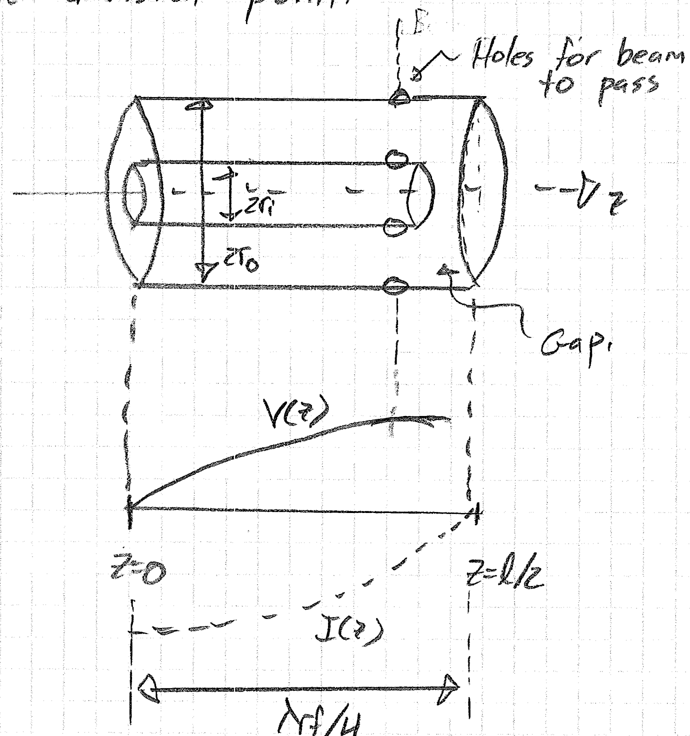
* Beam holes at $z = \ell/2$ where voltage is maximum
* Beam moves on radial path sees no field when inside inner conductor (like drift tube).
* RF phase advances by π when traversing the inner conductor so that the particle can be accelerated on both entrance and exit sides.
  - Conductor radii chosen for max energy gain on each side
* Effectively forms 2 gap cavity. 2 Gap Transit Time factor of HW problems applies.

Will also show in homework problems for the ½-wave coaxial resonator:

$$\boxed{\begin{aligned} U &= \frac{\mu_0 \ell I_0^2}{2\pi} \ln(r_0/r_i) && \text{RF Energy Stored} \\ Q &= \frac{p\pi}{R_{surf}} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\ln(r_0/r_i)}{\ell[\,1/r_i + 1/r_0\,] + 4\ln(r_0/r_i)} && \text{Quality Factor} \end{aligned}}$$

Quarter Wave Structure

Essentially split the half-wave structure divided in two with a capacitive termination at the division point.



★ Has a lesser degree of symmetry and fields will be distorted more than in the half-wave resonator.

Design formulas including the contribution to the fields from the capacitive gap termination can be found in

Moreno, Microwave Transmission Design Data, Dover, NY 1948, pp. 227-230.

Both Quarter and Half-Wave structures produce more compact low freq. cavities:
★ Save RF power
★ Cheaper superconducting (less material, less losses to cool, ...)

Coupling to RF Cavities

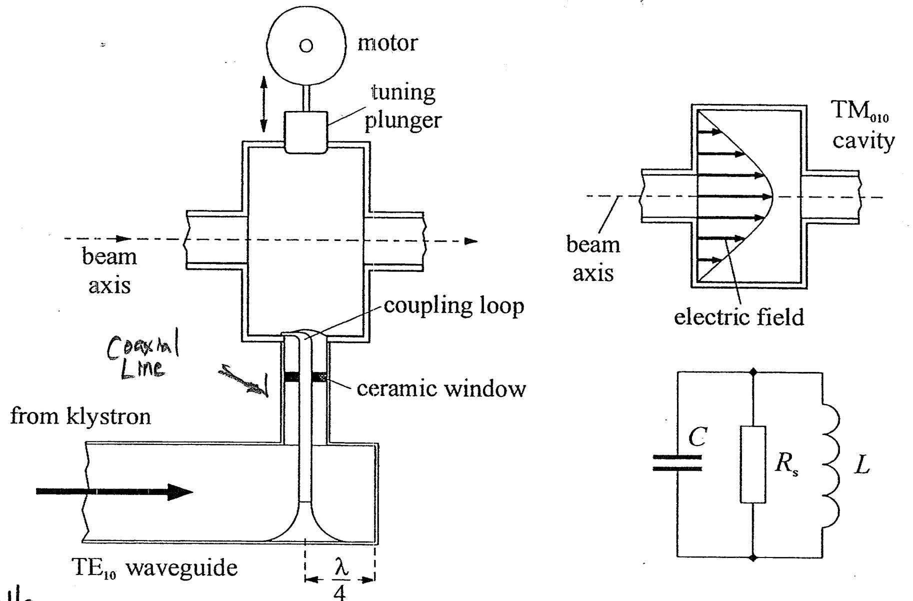
See Wille, "The Physics of Particle Accelerators," Chapter 5
Wilson, "An Introduction to Particle Accelerators," Chapter 5
Wangler, "RF Linear Accelerators" Chapter 5

Beyond scope to discuss in this class.
Many ways to couple RF power to resonant cavities.
Most common may be with a loop to couple with magnetic field of EM TM_{010} type standing wave.

- ☆ Place where magnetic field high in outer radial extent of cavity
- ☆ Field created by loop should have component in common with B_θ of TM_{010} type mode (or whatever mode) desired to excite.

Coupling of klystron to waveguide + coaxial cable also an issue. Much to consider.

Magnetic Coupling Loop at end of Coaxial Transmission Cable



Wille

Fig. 5.4 Design of a single-cell accelerating structure using the TM_{010} mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

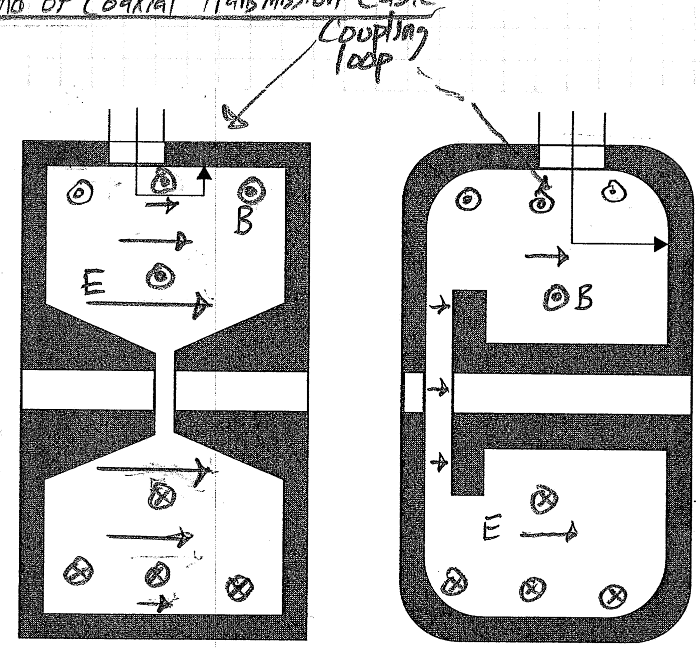


Fig. 10.15 Two examples of loop coupling.
Wilson

TM_{010} type mode

Common Methods Coupling.

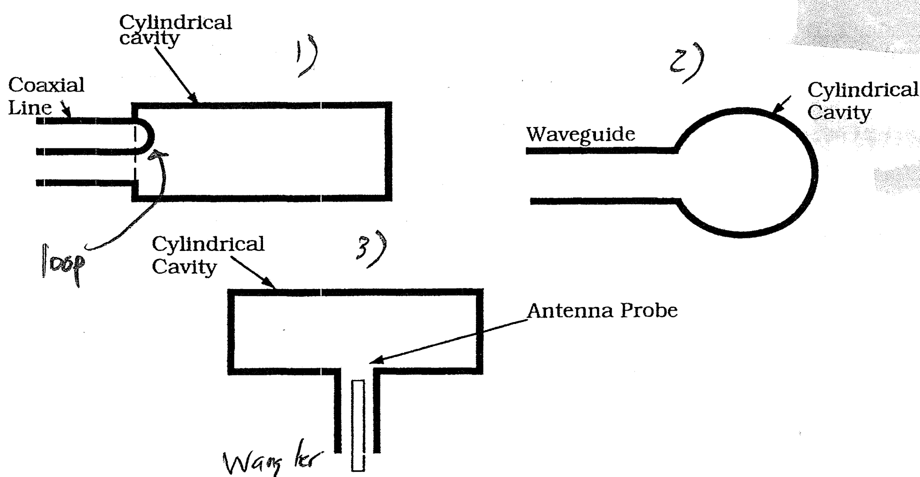


Figure 5.2 Methods of coupling to cavities.

1) Magnetic Loop at end of coaxial transmission line connected to cavity

2) Hole or Apeture in cavity wall connected to a wave guide

3) Electric Coupling Probe or Antenna using the central conductor of a coaxial transmission line.

Comments:

* Want structure using low order mode to make easy to excite and avoid coupling to higher order modes.
  - Preclude coupling to Higher order modes by frequency choice.

* Couplers have much difficult engineering
  - Heat leak for SRF structures.

RF Sources

See Wille, "The Physics of Particle Accelerators," Chapter 5
Wilson, "An Introduction to Particle Accelerators," Chapter 5

Harmonically varying RF power needed for accelerating structures ranging from a few kW to MW power levels. Pulses may be short, long, or continuous wave (CW).

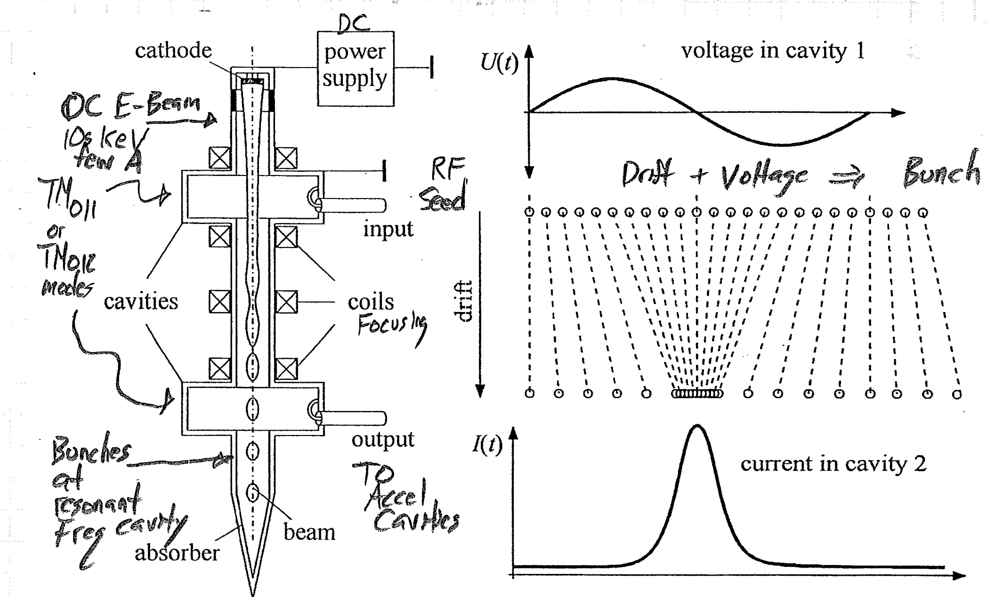
1) Triode / Tetrode: few MHz → few 100 MHz ; high power broad band
2) klystron: few 100 MHz + — Most common in accelerator systems. Narrow band tuned, very high power

Most → common for Accel. Applications

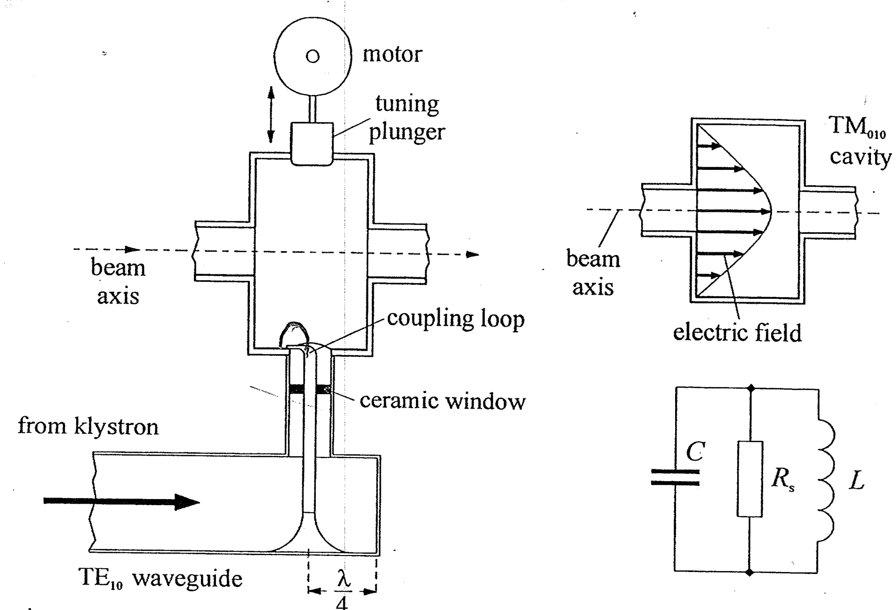
3) Also: Traveling Wave Tubes, Magnetrons, Cross-Field Amplifiers, Gyrotrons, ...

Klystron

Drift long enough to bunch.
Using TM_{011} or TM_{012}



Wille
Fig. 5.11 The classical microwave klystron, operating in the ten centimetre region.



Wille
Fig. 5.4 Design of a single-cell accelerating structure using the TM_{010} mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

Power delivered by Klystron

e^- beam source large: $I_{beam} \sim 10\,A$ typical
$V \sim$ 10's kV Source Voltage typical.

$$P_{Klystron} = \eta V I_{beam} \qquad \eta = \text{Efficiency} \quad 45\% \rightarrow 65\% \text{ typical}$$

~ 1.2 MW per tube now achieved in CW operation.
@ 350 - 500 MHz

~ 250 kW typical CW values.

Real klystrons may use several resonators to extract more energy and increase efficiency.
Many variants including relativistic klystrons using hther (MeV) energy e-beams.

* Numerous topics on RF cavity design, SRF specific issues, RF sources, couplers, cavity measurements, and engineering issues.

* Many texts exist on topic.
  - Often older books and manuscripts.

* USPAS classes cover specifics.

  - Microwave Measurements Laboratory
  - RF Power Engineering
  - Applied Electromagnetism: Magnet & Cavity Design
  - Two SRF classes.
  - Microwave Sources
  - RF Linear Accelerators

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* Many additional important topics not covered.

  - Microwave coupling to cavities and waveguides
  - Slater perturbation theorem
    - Basis of bead pull of small metallic structures to measure cavity frequency
    - Tuning RF cavities via mechanical deformation.

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