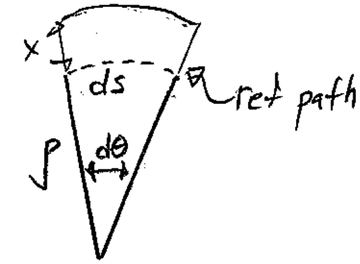
Momentum Compaction

The path length that a particle travels in a lattice between reference orbit coordinates s_1 and s_2 will vary with particle momentum p.

$$L = \int_{s_1}^{s_2} \sqrt{\underbrace{(1 + x/\rho)^2}_{\text{due to bend}} + \underbrace{x'^2 + y'^2}_{\text{from particle offset}}}\, ds$$



Expand for small orbit angles (paraxial approximation)

$$\boxed{L \simeq \int_{s_1}^{s_2} \left(1 + \frac{x}{\rho}\right) ds}$$

Neglect the relatively small contribution to x from betatron motion, but account for the larger contribution due to dispersive effects from off momentum then:

Essentially measures shift in x of orbit center with δ.

$$\boxed{\begin{aligned} x &\simeq D(s)\frac{\delta p}{p_0} = D(s)\,\delta \\ L &\simeq \int_{s_1}^{s_2} ds + \delta \int_{s_1}^{s_2} \frac{D(s)}{\rho(s)}\, ds \end{aligned}}$$

$$\boxed{\begin{aligned} L_0 &\equiv \int_{s_1}^{s_2} ds = \text{Reference orbit path length} \\ &= s_2 - s_1 \end{aligned}}$$

$$L \simeq L_0 + \delta \int_{s_1}^{s_2} \frac{D(s)}{\rho(s)}\, ds$$

Denote $\Delta L = L - L_0$ = change in path length due to off-momentum δ.

$$\Rightarrow \boxed{\Delta L = \delta \int_{s_1}^{s_2} \frac{D(s)}{\rho(s)}\, ds}$$

Define the <u>momentum compaction factor α_c</u> by

$$\boxed{\frac{\Delta L}{L_0} \equiv \alpha_c \frac{\delta p}{p_0} \qquad \alpha_c \equiv \frac{\int_{s_1}^{s_2} \frac{D(s)}{\rho(s)}\, ds}{\int_{s_1}^{s_2} ds} = \text{"Momentum Compaction" factor}}$$

* α_c is average of D/ρ over ideal (reference) path.
* α_c is a property of the lattice.
* For a ring, typical to take path over full ring lap reference path.

$$\boxed{\alpha_c = \frac{\oint_{ring} \frac{D(s)}{\rho(s)}\, ds}{\oint_{ring} ds}}$$

In this context, we <u>define</u> (will become clear soon why we do this)

$$\boxed{\alpha_c \equiv \frac{1}{\gamma_t^2} \qquad \gamma_t \equiv \text{Transition gamma.}}$$

For simple lattices: $\gamma_t \simeq$ number betatron oscillations in ring. See SY Lee pg 121

To prepare for needs to analyze longitudinal phase-focusing in RF cavities in rings. (we will cover later) need to also analyze the time it takes for a particle to travel along the path of length L.

$$\boxed{\tau = \frac{L}{c\beta} = \text{Transit time. (assume no accel).}}$$

$\beta = \frac{v}{c}$ not to be confused with C.S. betatron function.

$$\ln \tau = \ln \frac{L}{c\beta}$$

$$\Rightarrow \boxed{\frac{\Delta\tau}{\tau_0} = \frac{\Delta L}{L_0} - \frac{\Delta\beta}{\beta_0}}$$

$$\frac{\Delta L}{L_0} \equiv \alpha_c \frac{\delta p}{p_0}$$

subscript "0" denotes reference

need to calculate $\frac{\Delta\beta}{\beta_0}$ in terms of $\frac{\delta p}{p}$ to calculate $\frac{\Delta\tau}{\tau_0}$ in terms of $\frac{\delta p}{p_0}$.

Examine:

Expressions also true for reference particle (subscript 0)

$$E \equiv \gamma m c^2 \quad \text{Total Energy}$$
$$p = m\gamma\beta c \quad \text{Momentum}$$

$$\Rightarrow cp = \beta(\gamma m c^2) = \beta E \quad \Rightarrow \quad \frac{\delta p}{p_0} = \frac{\Delta\beta}{\beta_0} + \frac{\Delta E}{E_0}$$

Here we use Δ for differentials to avoid confusion with $\delta \equiv \frac{\delta p}{p_0}$

$$E^2 = c^2p^2 + m^2c^4$$
$$\delta E E = c^2 p \delta p \quad \rightarrow \quad \frac{\Delta E}{E_0} = \frac{c^2 p_0 \delta p}{E_0^2} = \frac{c(cp_0)\delta p}{E_0^2} = \frac{c\beta_0 E_0 \delta p}{E_0^2} = \frac{c\beta_0 \delta p}{(E_0)} = \frac{c\beta_0}{(cp_0/\beta_0)}\delta p$$
$$= \beta^2 \frac{\delta p}{p}$$

Thus

$$\frac{\Delta\beta}{\beta_0} = \frac{\delta p}{p_0} - \frac{\Delta E}{E_0} = (1-\beta_0^2)\frac{\delta p}{p} = \frac{1}{\gamma_0^2}\frac{\delta p}{p_0}$$

$$\boxed{\frac{\Delta\beta}{\beta_0} = \frac{1}{\gamma_0^2}\frac{\delta p}{p_0}}$$

Using these results,

$$\frac{\Delta\tau}{\tau_0} = \frac{\Delta L}{L_0} - \frac{\Delta\beta}{\beta_0}$$

$$= \alpha_c \frac{\delta p}{p_0} - \frac{1}{\gamma^2}\frac{\delta p}{p_0} = \left(\alpha_c - \frac{1}{\gamma_0^2}\right)\frac{\delta p}{p_0}$$

Define:

$$\boxed{\eta_s = \frac{1}{\gamma_0^2} - \alpha_c = \text{"Slip Factor"}}$$

Then,

$$\boxed{\frac{\Delta\tau}{\tau} = -\eta_s \cdot \frac{\delta p}{p_0}}$$

Notice that at $\gamma_0 = \gamma_t = \frac{1}{\sqrt{\alpha_c}}$ $\eta_s = 0$

corresponding to the "transition energy" of a ring.

The formula for $\Delta\tilde{T}/\tilde{T}$ is important when examining conditions to maintain synchronism with RF cavities in a ring.

Denote:

ω_r = Angular frequency to complete revolution in ring $= \dfrac{2\pi}{\tilde{T}_r}$

$\tilde{T}_r$ = revolution period in ring

$\tilde{T}_{r0}$ = design revolution period in ring

Then

$$\Delta\omega_r = \frac{-2\pi}{\tilde{T}_{r0}^2}\Delta\tilde{T}_r \quad \Rightarrow \quad \frac{\Delta\omega_r}{\omega_{r0}} = \frac{-\Delta\tilde{T}_r}{\tilde{T}_{r0}} = \eta_s\frac{\delta p}{p_0}$$

$$\boxed{\frac{\Delta\tilde{T}_r}{\tilde{T}_{r0}} = -\frac{\Delta\omega_r}{\omega_{r0}} = -\eta_s\frac{\delta p}{p_0}}$$

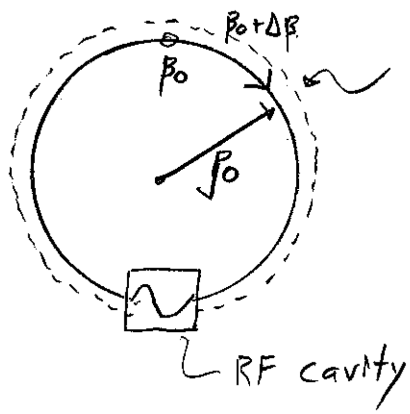
Case		Particle Energy		Slip Factor	Consequence
$\gamma > \gamma_t$	$\Leftrightarrow$	Above Transition	$\Leftrightarrow$	$\eta_s < 0$	high momentum particle takes <u>longer</u> time to make lap.
$\gamma < \gamma_t$	$\Leftrightarrow$	Below Transition	$\Leftrightarrow$	$\eta_s > 0$	high momentum particle takes <u>shorter</u> time to make lap.

$\therefore$ RF cavity phase control in a ring is different above and below transition energy.

★ Not surprising: Going through transition in an acceleration cycle can be a problem!

- When $\gamma = \gamma_t$, $\eta_s = 0$ does not really imply $\dfrac{\Delta\tilde{T}}{\Delta\tilde{T}_{r0}} = 0$. Higher order terms will matter in this case and analysis will be more difficult.

For a ring, conceptually:



Momentum deviation makes path length deviation.

We will take this into account using

$$\frac{\Delta \tilde{T}_r}{\tilde{T}_{r0}} = -\eta_s \frac{\delta p}{p_0}$$

when studying RF acceleration in a ring.

Note $\eta_s = \frac{1}{\gamma_0^2} - \alpha_c$

with α_c a property of the lattice.