Physics 905

Accelerator Physics

Problem Set #4

Tuesday Jan 28, 2020 due Tues Feb. 4, 2020

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Problem 1 FRIB Solenoids 15 pts

FRIB will use superconducting solenoids for beam focusing in the 3 linac segments. Consider U_{238}^{78+} (post stripper) accelerated to a kinetic energy of 128 MeV/u. Solenoids have axial length ~ 1/2 m and a field strength of 7 Tesla is employed.

5 pts a) Estimate $(B\rho)$ for the ion in Tesla-meters.

5 pts. b) Estimate the focal length f of the solenoid assuming $B_z \simeq 7$ Tesla over the axial length of 1/2 m. and is zero outside (Hard Edge). Is the thin lens approximation reasonable? Why?

5 pts c) Suppose superconducting cable with an average current density of J is wound to a thickness T around the beam pipe. Derive a formula to roughly estimate the central field B_z in terms of J and T. If the coil thickness is $T = 2$ cm what is J in Amps/mm^2 to produce $B_z = 7$ Tesla?

Problem 2 Magnetic Optics 30 pts

10 pts a) From the Lorentz force equation, show that a static magnetic field $\vec{B}(\vec{x})$ cannot change the particle kinetic energy $W = (\gamma - 1)mc^2$. Make no approximations.

$$m \frac{d}{dt}(\gamma \vec{\beta}) = q \vec{\beta} \times \vec{B}$$

$$\gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}} \quad ; \quad \vec{\beta} = \frac{1}{c}\frac{d\vec{x}}{dt}$$

5 pts b) In class, it was shown for a solenoid magnet with azimuthal symmetry ($\partial/\partial\theta = 0$), that the magnetic field can be expanded in terms of the on-axis field as

$$B_r(r,z) = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1}}{\partial z^{2\nu-1}} B_{z0}(z) \left(\frac{r}{2}\right)^{2\nu-1}$$

$$B_z(r,z) = B_{z0}(z) + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_{z0}(z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu}$$

$$B_{z0}(z) \equiv B_z(r=0, z)$$

Take $W = (\gamma-1)mc^2 \simeq$ const and employ standard paraxial approximations to show that if nonlinear applied force terms are dropped ($\propto x^2, xy, xy'$, etc.) that the equations of motion are:

$$x'' - \frac{B_{z0}'}{2(B\rho)} y - \frac{B_{z0}}{(B\rho)} y' = 0 \qquad (B\rho) = \frac{\gamma\beta mc}{q}$$

$$y'' + \frac{B_{z0}'}{2(B\rho)} x + \frac{B_{z0}}{(B\rho)} x' = 0 \qquad B_{z0}' = \frac{\partial B_{z0}(z)}{\partial z}$$

3 pts c) Qualitative answer only: Are the results of part c) inconsistent with part a)? If so, could they still be OK to use? Why?

2pts d) If we take $\vec{B} = \nabla \times \vec{A}$, show we can generate the linear field components of the solenoid as

$$B_r = -\frac{1}{2} \frac{\partial B_{z0}}{\partial z} r$$

$$B_z = B_{z0}$$

from $\vec{A} = \frac{1}{2} B_{z0} r \hat{\theta}$

10pts e) Use the paraxial approximation and the results from d) to show for a solenoid that

$$\underline{P}_\theta = [\vec{x} \times (\vec{p} + q\vec{A})] \cdot \hat{z}$$
$$\simeq m\gamma\beta c (xy' - yx') + \frac{qB_{z0}}{2}(x^2+y^2)$$

Show that the equations of motion in b) imply that

$$\underline{P}_\theta = \text{const.}$$

Problem #3

Transfer Matrix Elements Expressed in Phase-Amplitude Form

Show that the principal functions of the transfer matrix solution of the particle orbit — 20 pts

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s|s_i)\begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix} = \begin{pmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{pmatrix}\begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

are:

$$C(s|s_i) = \frac{W(s)}{W_i}\cos\Delta\Psi(s) - W_i' W(s)\sin\Delta\Psi(s)$$

$$S(s|s_i) = W_i W(s)\sin\Delta\Psi(s)$$

$$C'(s|s_i) = \left(\frac{W'(s)}{W_i} - \frac{W_i'}{W(s)}\right)\cos\Delta\Psi(s) - \left(\frac{1}{W_i W(s)} + W_i' W'(s)\right)\sin\Delta\Psi(s)$$

$$S'(s|s_i) = \frac{W_i}{W(s)}\cos\Delta\Psi(s) + W_i W'(s)\sin\Delta\Psi(s)$$

$$\Delta\Psi(s) = \Psi - \Psi_i = \int_{s_i}^{s}\frac{d\bar{s}}{W^2(\bar{s})}$$

$$W_i = W(s=s_i)$$
$$W_i' = W'(s=s_i)$$

Hint use:

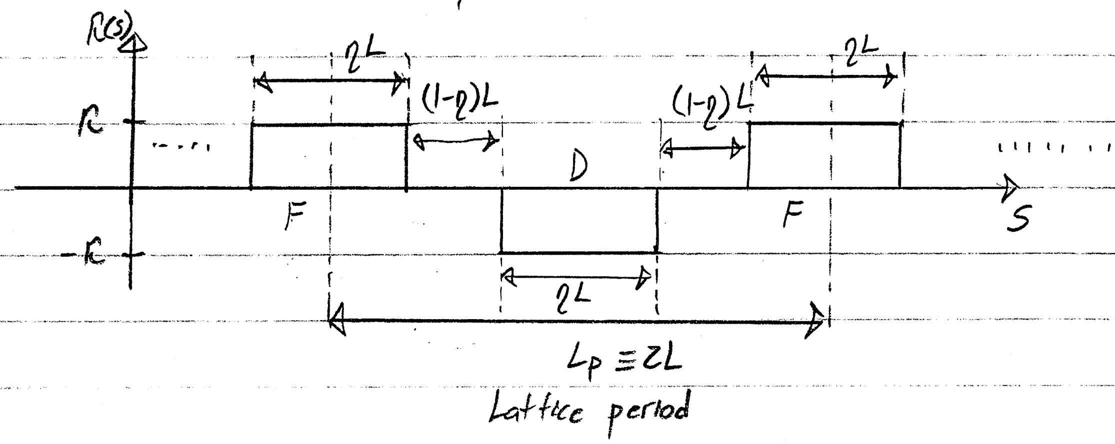
$$x = A_i\, w\cos\psi$$
$$x' = A_i\, w'\cos\psi - \frac{A_i}{w}\sin\psi$$

$$\psi = \psi_i + \Delta\psi$$

Problem #4 Phase Advance 40pts

1/

Consider a "FODO" periodic lattice:



$L_p = 2L$ = lattice period

ηL = Quadrupole lengths

$(1-\eta) L$ = drift lengths

η = Quadrupole occupancy $\quad 0 < \eta \le 1$

κ = Quadrupole strength

5 pts a) Write the transfer matrices $\overline{M}(s|s_i)$ for each section of the periodic lattice in terms of $\theta \equiv \sqrt{\kappa}\,\eta L$, d, and η. Use results from class.

- $\overline{M}_F$: Transfer through Focus Quadrupole.
- $\overline{M}_O$: " " Drift
- $\overline{M}_D$: " " Defocus Quadrupole
- $\overline{M}_O$: " " Drift.

5pts b) Write the transfer matrix $\overline{M}(s_i+L_p, s_i)$ through one lattice period starting from the left side of a focus quadrupole. No need to fully expand!

20pts c) Show that the phase advance δ_0 of a particle through this lattice period

$$\cos\delta_0 = \frac{1}{2}\,\mathrm{Trace}\; M(s_i + L_p | s_i)$$

can be expressed as:

$$\cos\delta_0 = \cos\theta\cosh\theta + \frac{(1-\eta)}{\eta}\,\theta\,(\cos\theta\sinh\theta - \sin\theta\cosh\theta)$$
$$- \frac{1}{2}\frac{(1-\eta)^2}{\eta^2}\,\theta^2\sin\theta\sinh\theta$$

Hint: only calculate simplify elements of M that you need.

3pts d) Will it matter where the lattice period is started in the calculation of δ_0 in part c)? Why?

5pts e) For $\theta \ll 1$ (thin lens limit), show that

$$\cos\delta_0 \simeq 1 - \frac{1}{2}\left(1 - \frac{2}{3}\eta\right)\frac{\theta^4}{\eta^2}$$

f) If $\delta_0 \ll 1$, and $\eta \ll 1$, show that

$$\delta_0 \simeq \eta |\hat{R}| L^2$$

2pts g) If one wanted to model a "FODO" focusing lattice by a continuous focusing channel with $\kappa(s) = k_{\beta 0}^2 = \mathrm{const.}$, how could one choose $k_{\beta 0}^2$ based on part f)?

Optional Problem

JPD Problem 1 - Larmor Frame

1/ For a uniform solenoidal channel:

$$B_z^a(s) = B_0 = \text{const}$$

with no acceleration

$$\gamma_b \beta_b = \text{const}$$

and an axisymmetric ($\partial/\partial\theta = 0$) beam with

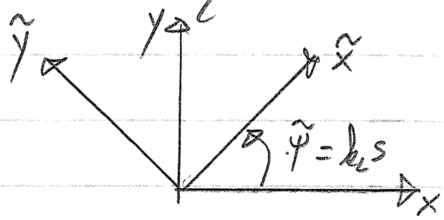
$$\frac{\partial \phi}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\vec{x}_\perp}{r} \qquad r = \sqrt{x^2+y^2}$$

the particle equations of motion reduce to:

$$x'' = \frac{qB_0}{m\gamma_b\beta_b c}\, y' - \frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial r}\frac{x}{r}$$

$$y'' = -\frac{qB_0}{m\gamma_b\beta_b c}\, x' - \frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial r}\frac{y}{r}$$

a) Parallel steps taken in the class notes to transform the eqations of motion to a co-rotating frame:



$k_L = \text{const} =$ Larmor wavenumber

$$\tilde{x} = x\cos(k_L s) + y\sin(k_L s)$$
$$\tilde{y} = -x\sin(k_L s) + y\cos(k_L s)$$

Find an expression for k_L to reduce the equations of motion to the decoupled form:

TPD Problem 1

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P1a/

$$\tilde{x}'' + \kappa \tilde{x} = \frac{-q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{x}}{r}$$

$$\tilde{y}'' + \kappa \tilde{y} = \frac{-q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{y}}{r}$$

and identify $\kappa = \text{const.}$

Hint:
The transformation can be carried out directly. But you may find the algebra simpler using complex coordinates as in the class notes:

$$z = x + iy \qquad i = \sqrt{-1} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$
$$\tilde{z} = \tilde{x} + i\tilde{y}$$

b) If the direction of the magnetic field is reversed:

$$B_0 \rightarrow -B_0$$

how will the dynamics be influenced?

c) Neglect space-charge:

$$\phi = 0$$

and sketch a typical orbit in the rotating Larmor frame. Will this orbit appear more complicated in the Laboratory Frame? Why?

Bonus: Sketch the orbit taking advantage of simple choices of initial conditions that can always be made through choice of coordinates.