Physics 905
Accelerator Physics
Problem Set #5
Tuesday Feb 4, 2020 due Tuesday Feb 11, 2020
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Problem # 1 — Thin lens transfer matrix.

Most steps in this problem can be found in 03.sup..... notes.

Consider $\kappa_x = \frac{1}{f}\delta(s-s_0)$

$f =$ const $=$ focal length
$s_0 =$ axial location optic

and the equation of motion

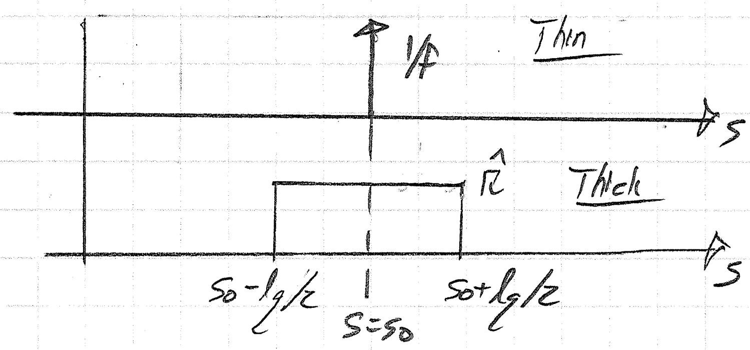
$$x'' + \frac{1}{f}\delta(s-s_0)\,x = 0$$

a) Derive the 2x2 transfer matrix $\bar{M}$ for the optic:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0^+} = \bar{M}\cdot\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0^-}$$

where $s = s_0^{\pm}$ are the coordinates infinitesimally to the left (s_0^-) and right (s_0^+) of the optic at $s=s_0$. $\bar{M}$ is the thin-lens transfer matrix.

b) Derive a constraint for the thin lens "kick" to give the same impulse $\int ds\, \kappa_x(s)$ to a particle as a thick quadrupole lens with $\hat{\kappa} = \hat{\kappa} =$ const over an axial length l_q:



c) Using the result in b) as a constraint, show that the thick lens quadrupole transfer matrices have thin lens form when $l_q \to 0$. Show for both focus and defocusing quadrupoles.

$$\bar{M}_{Focus} = \begin{pmatrix} \cos(\sqrt{\hat{\kappa}}\, l_q) & \frac{1}{\sqrt{\hat{\kappa}}}\sin(\sqrt{\hat{\kappa}}\, l_q) \\ -\sqrt{\hat{\kappa}}\sin(\sqrt{\hat{\kappa}}\, l_q) & \cos(\sqrt{\hat{\kappa}}\, l_q) \end{pmatrix}$$

$\hat{\kappa} > 0$

$\kappa_x = \hat{\kappa}$ in quad

$$\bar{M}_{deFocus} = \begin{pmatrix} \cosh(\sqrt{\hat{\kappa}}\, l_q) & \frac{1}{\sqrt{\hat{\kappa}}}\sinh(\sqrt{\hat{\kappa}}\, l_q) \\ \sqrt{\hat{\kappa}}\sinh(\sqrt{\hat{\kappa}}\, l_q) & \cosh(\sqrt{\hat{\kappa}}\, l_q) \end{pmatrix}$$

$\kappa_x = -\hat{\kappa}$ in quad.

d) A 2x2 Transfer matrix

$$\bar{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

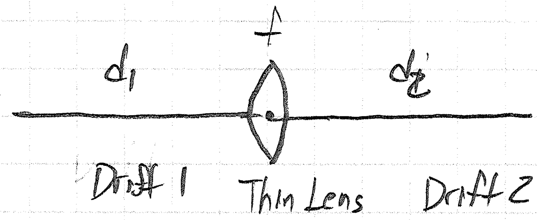
gives the solution to Hills' equation

$$x'' + K_x(s)\, x = 0$$

through some advance. Due to the Wronskian symmetry

$$\det \bar{M} = M_{11} M_{22} - M_{12} M_{21} = 1$$

always holds. Show that $\bar{M}$ can always be replaced by two drifts and a thin lens kick as



$$\bar{M} = \bar{M}_{\text{Drift 2}} \circ \bar{M}_{\text{Thin Lens}} \circ \bar{M}_{\text{Drift 1}}$$

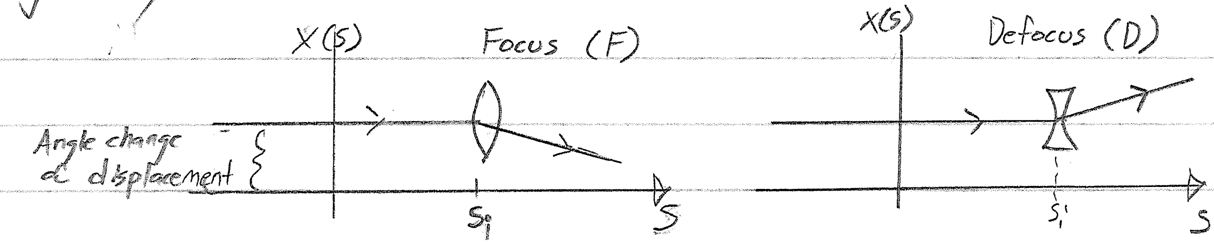
Find d_1, d_2, f in terms of M_{11}, M_{22}, M_{21} for equivalance.

Problem #2

TPD Problem 4 — 20 points

S.M. Lund — P4/

4/ A thin lense changes the angle of a particle trajectory but not the coordinate:



This action can be specified by transfer matrices applied at $s = s_i$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M(s|s_i) \begin{pmatrix} x_i \\ x_i' \end{pmatrix}$$

Focusing:

$$M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$f > 0$

Defocusing:

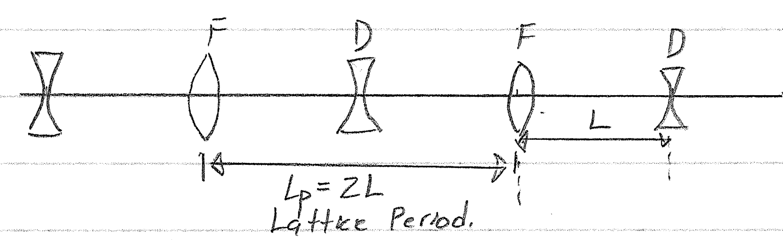
$$M_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$f > 0$

A free space drift of length L has a transfer matrix:

$$M_O = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Consider a lattice of period $2L$ made up of equally spaced F and D lenses with equal values of f.



This is the simplist "FODO" alternating gradient lattice!

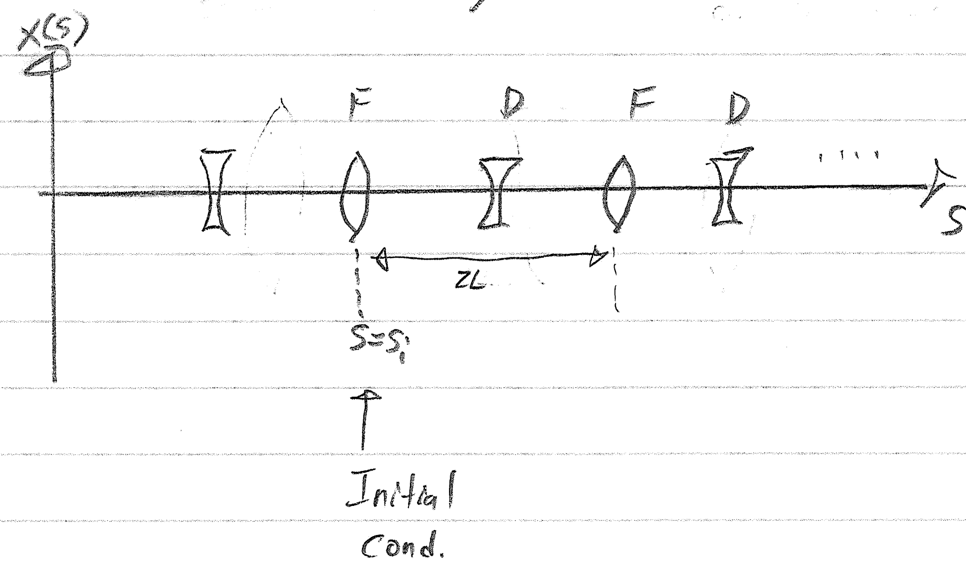
a) Use the transfer matrices above to find the range of L/f for which the particle orbit is stable.

b) Calculate $\cos\sigma_0$ where σ_0 is the particle phase advance in terms of L/f.

c) For the case of of L/f chosen to correspond to the stability limit, sketch the motion of a particle with initial condition

$$\lim_{s \to s_i^-} x(s) = x_0$$
$$\lim_{s \to s_i^-} x'(s) = x_0/L$$

where $s = s_i$ is the axial location of a focusing thin lens kick, and $s \to s_i^-$ is just before the kick. Sketch the particle orbit for focusing strength slightly larger than the stability limit. Superimpose the orbit sketch on a diagram of the lattice (see below):



Problem #3

Critical Points: Courant-Snyder Ellipse 30 pts

$\checkmark$

In class we derived the single-particle Courant-Snyder Invariant:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \mathcal{E} = \text{const.}$$

where:

$$\beta(s) = w^2(s)$$

$$\alpha(s) = -w(s)w'(s)$$

$$\gamma(s) = \frac{1}{w^2(s)} + w'(s)^2 = \frac{1+\alpha^2(s)}{\beta(s)}$$

Derive the critical values of the ellipse indicated on the figure below:

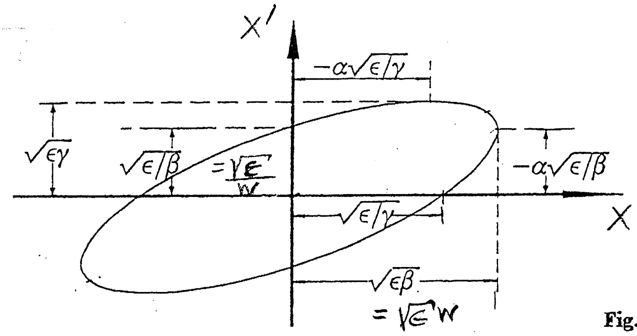


Fig. 5.22. Phase space ellipse

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From Wiedemann

Hint: to avoid messy algebra, take a differential of the constraint equation $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \text{const}$ and use this result to find turning points.

$$\Rightarrow \quad 2\gamma x\,dx + 2\alpha x\,dx' + 2\alpha x'\,dx + 2\beta x'\,dx' = 0$$

Problem #4 Normalized Emittance 20 pts

Consider a distribution of particles evolving according to the particle equation of motion

$$x'' + \frac{(\gamma\beta)'}{(\gamma\beta)} x' + \kappa(s) x = 0 , \qquad ' \equiv \frac{d}{ds}$$

Denote an average over the distribution as

$$\langle \cdots \rangle$$

A statistical measure of beam phase-space area is provided by the normalized rms emittance.

$$\varepsilon_{nx} \equiv (\gamma\beta)\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right]^{1/2}$$

Show directly using the equation of motion that:

$$\varepsilon_{nx} = \text{const.}$$

Would you expect ε_{nx} to be conserved if the equation of motion had nonlinear terms?

$$x'' + \frac{(\gamma\beta)'}{(\gamma\beta)} x' + \kappa(s) x = F_{nl} \qquad F_{nl} \text{ some function of } x, \text{ not } \propto x.$$

Explain why. Be specific.

Hint: Easier algebra to show

$$\frac{d\,\varepsilon_{nx}^2}{ds} = 0$$

$$\Rightarrow \varepsilon_{nx}^2 = \text{const} \quad \Rightarrow \quad \varepsilon_{nx} = \text{const.}$$