MSU Physics 905
Spring Semester 2020
Accelerator Physics
Problem Set 8 - 125 points

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Problem 1

P013 Resonances 40 pts.

Consider the driven harmonic oscillator equation for $U(\varphi)$:

$$\begin{aligned}
\frac{\mathrm{d}^2 U(\varphi)}{\mathrm{d}\varphi^2} + \nu_0^2 U(\varphi) &= A\cos(\nu\varphi) + B\sin(\nu\varphi) \\
\nu_0 &= \text{constant restoring frequency} \\
\nu &= \text{constant driving frequency} \\
A,\ B &= \text{constant amplitudes} \\
A\cos(\nu\varphi) + B\sin(\nu\varphi) &= \text{driving terms}
\end{aligned}$$

The general solution for $U(\varphi)$ can be expanded as:

$$U(\varphi) = U_h(\varphi) + U_p(\varphi)$$

where $U_h(\varphi)$ is the general solution to the homogeneous equation:

$$\begin{aligned}
\frac{\mathrm{d}^2 U_h(\varphi)}{\mathrm{d}\varphi^2} + \nu_0^2 U_h(\varphi) &= 0 \\
\Longrightarrow U_h(\varphi) &= C_1\cos(\nu_0\varphi) + C_2\sin(\nu_0\varphi) \\
C_1,\ C_2 &\ \text{constants}
\end{aligned}$$

and $U_p(\varphi)$ is the particular solution to:

$$\frac{\mathrm{d}^2 U(\varphi)}{\mathrm{d}\varphi^2} + \nu_0^2 U(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi)$$

a) 5 pts: For $\nu \neq \nu_0$, show that a solution $U_p(\varphi)$ exists proportional to the driving term and find the constant of proportionality.

b) 5 pts: Use the results of part (a) to construct the solution ($\nu \neq \nu_0$) for $U(\varphi)$ satisfying the initial conditions at $\varphi = 0$:

$$U(\varphi = 0) = U_0$$
$$\left.\frac{\mathrm{d}U}{\mathrm{d}\varphi}\right|_{\varphi=0} = \dot{U}_0; \quad \frac{\mathrm{d}U}{\mathrm{d}\varphi} \equiv \dot{U}(\varphi)$$

c) 10 pts: Set $\nu = \nu_0 + \delta\nu$, and find the leading order form of the solution valid for $|\delta\nu|/\nu_0 \ll 1$ and $|\varphi\delta\nu| \ll 1$. What does this limit imply on the amplitude of the particle oscillation as $\nu \to \nu_0$?

d) 5 pts: What do these results imply for a general periodic forcing function:

$$\frac{d^2}{d\varphi^2}U(\varphi) + \nu_0^2 U(\varphi) = f(\varphi)$$
$$f(\varphi) = \text{periodic forcing function with } f(\varphi + 2\pi) = f(\varphi)$$

How does this fit in with the analysis of machine tunes carried out in the class notes?

e) 5 pts: Suppose the drive frequency is exactly equal to the resonant frequency (i.e., $\nu = \nu_0$):

$$\frac{d^2}{d\varphi^2}U(\varphi) + \nu_0^2 U(\varphi) = A\cos(\nu_0\varphi) + B\sin(\nu_0\varphi)$$

Motivated by part c), show that a particular solution exists

$$U_p(\varphi) = \frac{A}{2\nu_0}\varphi\sin(\nu_0\varphi) - \frac{B}{2\nu_0}\varphi\cos(\nu_0\varphi)$$

with no approximations. Write down the general solution. Does this agrees with (c)? Should it?

f) 10 pts: For the case of $\nu \neq \nu_0$, estimate the deviation in $\delta\nu/\nu_0$ to wash out the resonance. Please keep arguments simple.

Hint: Look at the second order deviations in $\delta\nu/\nu_0$.

Problem 2

P014 Resonance Driving Perturbations 15 pts.

In class we derived the perturbed Hill's equationfor transverse magnetic field perturbations:

$$x'' + \kappa_x x = \mathcal{P}_x \qquad \kappa_x = \frac{G}{[B\rho]}$$

where

$$\mathcal{P}_x = \mathcal{P}_x(x, y) = \text{perturbation in x-plane}$$

Use the results from class to explicitly identify $\mathcal{P}_x$ for the following conditions:

a) 5 pts: Normal and skew orientation dipole field perturbations.

b) 5 pts: Normal and skew orientation quadrupole field perturbations. Which of these can be included in κ_x? Which of these results in y-plane coupling?

c) 5 pts: Normal and skew orientation sextupole field perturbations. In either case, is the x-motion independent of y when $y \neq 0$? Do "normal" and "skew" orientations have clear physical distinction for sextupole perturbations? Why or why not?

Caution: You must correctly interpret the index n in the class notes to identify the appropriate multipole field term.

<u>Problem 3</u>

P016 Dispersion Function 50 pts.

The dispersion function in a periodic ring satisfies:

$$\begin{aligned} D''(s) + \kappa(s)D(s) &= \frac{1}{\rho(s)} \\ \rho(s) &= \text{bend radius, } \kappa(s) = \text{focusing function} \\ D(s+L_p) &= D(s),\ \rho(s+L_p) = \rho(s),\ \kappa(s+L_p) = \kappa(s) \\ L_p &= \text{lattice period} \end{aligned}$$

a) 5 pts: Argue the solution for D is unique. This implies that there is a unique closed orbit $x = \delta \cdot D$ for every value of off-momentum δ. This aids interpretation of D.

Hint: Let D_1 and D_2 be two independent solutions and look for a contradiction.

b) 5 pts: Argue that the solution for D can be expressed in an extended 3×3 transfer matrix from as:

$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_s = \begin{bmatrix} \mathbf{M}_{11}(s|s_i) & \mathbf{M}_{12}(s|s_i) & d(s|s_i) \\ \mathbf{M}_{21}(s|s_i) & \mathbf{M}_{22}(s|s_i) & d'(s|s_i) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s=s_i}$$

where $\mathbf{M}(s|s_i)$ is the usual 2×2 transfer matrix from Hill's equation.

Express the periodicity requirement $D(s+L_p) = D(s)$ in this 3×3 formulation. Do you expect this equaiton to have a solution? Explain your answer.

c) 15 pts: Show for ρ =constant and:

$$\begin{aligned} &\kappa = \text{const} > 0 : \\ &\qquad d(s|s_i) = \frac{1}{\rho\kappa}[1 - \cos(\sqrt{\kappa}(s-s_i))] \\ &\qquad d'(s|s_i) = \frac{1}{\rho\sqrt{\kappa}}\sin[\sqrt{\kappa}(s-s_i)] \\ &\kappa = \text{const} < 0 : \\ &\qquad d(s|s_i) = \frac{1}{\rho|\kappa|}[-1 + \cosh(\sqrt{|\kappa|}(s-s_i))] \\ &\qquad d'(s|s_i) = \frac{1}{\rho\sqrt{|\kappa|}}\sinh[\sqrt{\kappa}(s-s_i)] \end{aligned}$$

Use the Green's functions results from class and forms derived for $\mathbf{M}$.

d) 5 pts: Use $\kappa = 1/\rho^2$ in part (c) to show for a sector dipole that the 3×3 transfer matrix through a bend of length ℓ can be expressed as:

$$\mathbf{M} = \begin{bmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ \frac{-\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_s = \mathbf{M} \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s=s_i}$$

$$\ell = \rho\theta, \qquad \theta = \text{bend angle}$$

Show for a small angle bend ($\theta \ll 1$) that:

$$\mathbf{M} = \begin{bmatrix} 1 & \ell & \frac{\ell\theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{bmatrix}$$

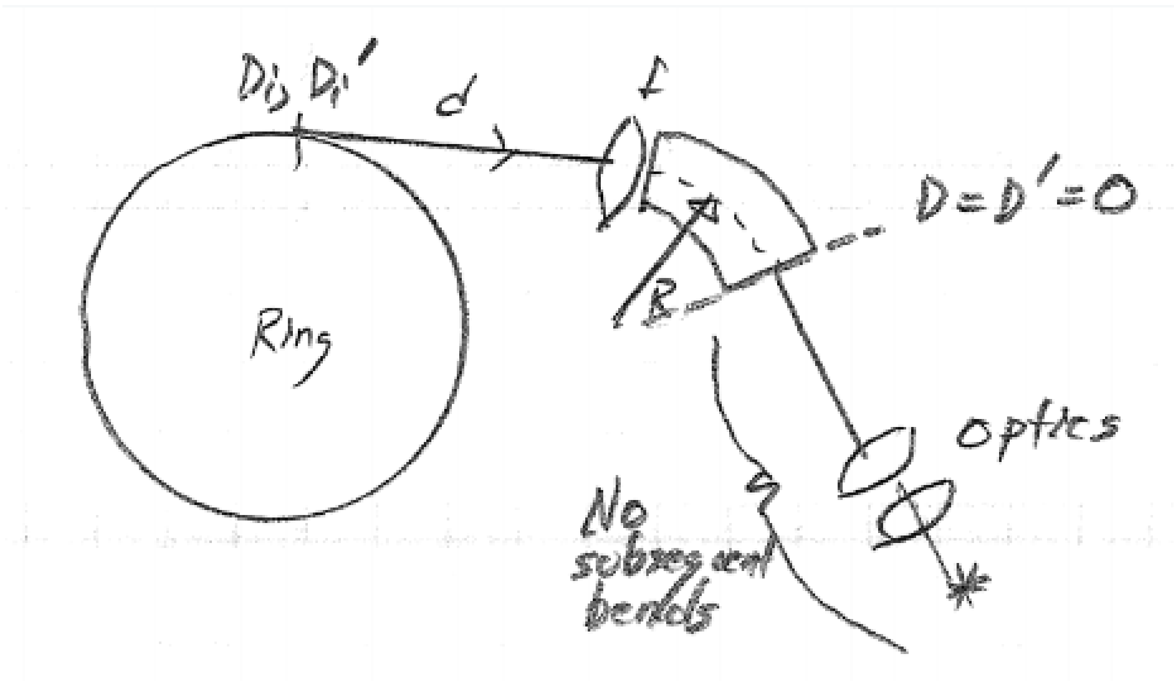
e) 10 pts: Derive the 3×3 transfer matrix for D for:

1) A drift with $\kappa = 0$ and $\rho \to \infty$.
2) A thin lens at $s = s_i$ with $\kappa = \frac{\delta(s-s_i)}{f}$ where $f =$ constant and $\rho \to \infty$. Here, $\delta(x)$ is a Dirac-delta function.
3) Within a uniform sector bend with large bend radius ρ where we take $\kappa \approx 0$ and $\rho =$ constant.

First use direct methods as opposed to Green's functions.

Then show that the results agree with the Green's functions for 1) and 2) and for the small angle bend result derived from the Green's functions for 3).

f) 10 pts: A particle is kicked out of a ring with dispersion $D = D_i$ and $D' = D_i'$ just after the kick. The particle is then transported through an extraction line with a drift length d, a thin lens focusing kick with focal length f, and then a sector bend of radius $\rho = R$ and length ℓ, and finally though an uspecified series of optics to the target.



Using the results from part e), derive constraints on the lattice parameters d, f, R, and ℓ that can be enforced to ensure that $D = D' = 0$ after the bending magnet to have zero dispersion in the straight transport and focusing line to the target?

Are these constraints practical to implement in the lab? Why? Qualitative answer only.

<u>Problem 4</u>

P097 Chromaticity Correction 20 pts.

Two normal orientation sextupoles are added to a linear quadrupole focusing lattice ring with natural chromaticities:

$$\begin{aligned}
\xi_{x0} &= -\frac{1}{4\pi}\oint_{\text{ring}} ds\ \beta_{x0}\kappa_x &= -\frac{1}{4\pi}\oint_{\text{ring}} ds\ \beta_{x0}\kappa \\
\xi_{y0} &= -\frac{1}{4\pi}\oint_{\text{ring}} ds\ \beta_{y0}\kappa_y &= \frac{1}{4\pi}\oint_{\text{ring}} ds\ \beta_{y0}\kappa
\end{aligned}$$

where $\kappa_x = -\kappa_y = \kappa$. Two "thin" sextupoles with equal effective axial lengths ℓ_s strengths and strengths

$$\int_{\text{sextupole 1}} ds\, \mathcal{S} = \hat{\mathcal{S}}_1\ell_s$$

$$\int_{\text{sextupole 2}} ds\, \mathcal{S} = \hat{\mathcal{S}}_2\ell_s$$

are placed in the lattice at $s = s_1, s_2$. Here, $\hat{\mathcal{S}}_{1,2}$ denote the effective sextupole strengths at $s = s_{1,2}$. D is the periodic dispersion function of the ring.

a) 10 pts: Derive thin lens formulas for ξ_x and ξ_y, the chromaticity in x and y using formulas derived in class in terms of ℓ_s, $\hat{\mathcal{S}}_1$, $[B\rho]_0$, ξ_{x0}, ξ_{y0} and β_{x01}, β_{y01}, β_{x02}, β_{y02}, D_1, D_2. Here, $\beta_{x01} = \beta_{0x}(s = s_1)$, etc.

b) 5 pts: Solve for $\hat{\mathcal{S}}_1$, $\hat{\mathcal{S}}_2$ for $\xi_x = 0 = \xi_y$ for zero chromaticity.

c) 5 pts: From part b), can you correct for chromaticity when $D = 0$? Explain where the sextupoles should be placed in the lattice in terms of $\beta x0$, β_{y0} amplitudes to allow correction with minimal sextupole strengths.