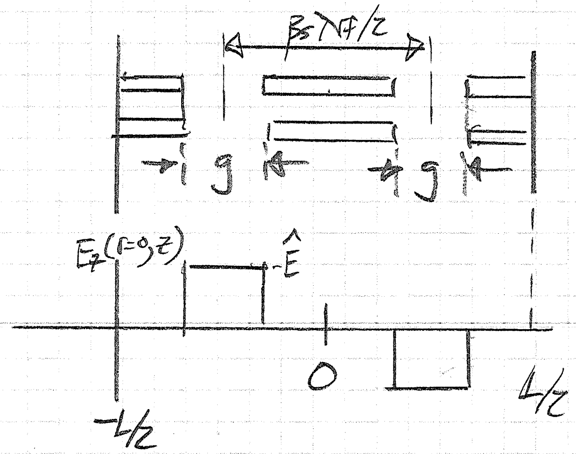
Physics 905
Accelerator Physics
Problem Set #10
Tuesday, March 17, due Tuesday March 24, 2020
Steve Lund and Yue Hao

Problem #1 Transit Time Factor 50 pts

Many cavities are multi-gap — Including at FRIB. They can be modeled by the usual Panofsky equation if an appropriate transit time factor T is employed

2 Gap Cavity



$E_z = E_z(r,z)\cos(\omega t + \phi)$

$\sim$ Approx E_z - uniform in gaps.

The energy gain of a particle traversing the cavity is

$$\Delta W = q\int_{-L/2}^{L/2} E(0,z)\sin\left(\frac{2\pi z}{\beta\lambda_{rf}} + \phi\right) dz$$

Approximate $\beta \simeq$ const in the cavity.

15pts a) For this structure derive a transit-time factor T to show that

$$\Delta W = q E_0 L T \cos\phi$$

with $E_0 = \frac{1}{L}\int_{-L/2}^{L/2} |E(0,z)|\, dz$ = Avg. Field over cell (Note absolute value)

$$T = \frac{\sin[\pi g/(\beta\lambda_{rf})]}{\pi g/(\beta\lambda_{rf})}\sin\left(\frac{\pi\beta_s}{2\beta}\right)$$

30pts b) Assume that the length of each gap is $g = \frac{\beta_s\lambda_{rf}}{8}$, Plot T vs β for the following 4 cases:

1) $\lambda_{rf} \Leftrightarrow$ 80.5 MHz i) $\beta_s = 0.041$ ii) $\beta_s = 0.085$

2) $\lambda_{rf} \Leftrightarrow$ 322 MHz i) $\beta_s = 0.29$ ii) $\beta_s = 0.53$

For each case estimate the approximate range of β for $T > 0.65$ corresponding to efficient RF.

Use any graphics package you want to make plots. Please no hand plots!

5pts

c) Explain why (qualitative only) why this two gap transit time factor shows more variation in β than a 1 gap model. Why can T be zero for some values of β?

25 points

Problem #2 Motion Near Synchronous Particle: Difference Eqns

In class we derived longitudinal difference equations

$$\Delta\phi_n - \Delta\phi_{n-1} = \frac{-2\pi N}{\gamma_{s,n-1}^3 \beta_{s,n-1}^2} \frac{\Delta \bar{W}_{n-1}}{mc^2}$$

$$\Delta\bar{W}_n - \Delta\bar{W}_{n-1} = q E_{0,n} L_n T_n(\beta_{sn}) \left[\cos(\phi_s + \Delta\phi_n) - \cos\phi_{s,n} \right]$$

$$\Delta\phi_n = \phi_n - \phi_{s,n}$$

$$\Delta\bar{W}_n = \bar{W}_n - \bar{W}_{s,n}$$

2pts a) What term generates the nonlinearity? Why?

8pts b) Following steps in class, linearize the difference equations for small phase excursions about the synchronous particle and express the result as a 2×2 transfer matrix

$$\begin{bmatrix} \Delta\phi \\ \Delta\bar{W} \end{bmatrix}_n = \begin{bmatrix} & M_{2\times2} & \end{bmatrix} \cdot \begin{bmatrix} \Delta\phi \\ \Delta\bar{W} \end{bmatrix}_{n-1} \equiv \underline{M}_s \cdot \begin{bmatrix} \Delta\phi \\ \Delta\bar{W} \end{bmatrix}_{n-1}$$

Show that $\det \underline{M}_s = 1$ and resolve $\underline{M}_s$ as

$$\underline{M}_s = \underbrace{\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}}_{\text{Thin lens}} \underbrace{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}_{\text{drift}}$$

and identify

d = drift length

$1/f$ = inverse focal length.

If $d < 0$ is this a problem? Will the system still focus? Why?

10pts c) Assume negligible synchronous particle energy gain and a regular periodic lattice:

$$\beta_{s,n} = \beta_s = \text{const} \quad \Rightarrow \quad \gamma_{s,n} = \gamma_s = \text{const}$$

$$E_{0n} = E_0 = \text{const}$$

$$L_n = L = \text{const} \qquad T_n = T = \text{const}$$

$$\phi_{s,n} = \phi_s = \text{const}$$

Define a synchronous phase advance using $\underline{M}_s$ in part b) and calculate the synchronous phase advance σ_s = phase advance per cell L. Compare the result to the synchrotron wavenumber k_s calculated in class for small phase advance per cell. Should you expect the relationship obtained? Why?

15 pts

d) Suppose we apply the linear equations in the limit of small acceleration within the continuous approximation derived in class. For an orbit with max phase extent $\Delta\phi_0$ find an expression for the longitudinal emittance in $\Delta\phi$, $\Delta\bar{W}$ phase-space with

$$\pi \epsilon_s = \text{Area ellipse in } \Delta\phi - \Delta\bar{W}$$

The units of ϵ_s will be radians - eV (energy). How should we scale this result to measure ϵ_s in Δt, $\Delta\bar{W}$ phase-space to measure area in eV-sec.?

1

Problem #3 30 points
Hamiltonian form of Synchrotron Equations of Motion

In class we showed that in the continuous approximation that the longitudinal equations of motion about the synchronous particle are

$$\frac{d\phi}{ds} = -AW$$

$$\frac{dW}{ds} = B\left[\cos\phi - \cos\phi_s\right]$$

$W = \Delta W/(mc^2)$

$A = \frac{2\pi}{\lambda_{rf}(\gamma_s\beta_s)^3}$

$B = \frac{q E_0 T}{mc^2}$

$\gamma_s\beta_s$ varies slowly

10pts

a) Find a Hamiltonian $H(\phi, P_\phi)$ and conjugate "momentum" variable P_ϕ such that the equations of motion are given by:

$$\frac{d\phi}{ds} = \frac{\partial H}{\partial P_\phi}$$

$$\frac{d}{ds}P_\phi = -\frac{\partial H}{\partial \phi}$$

Compare H to H_ϕ constructed in class.

10pts

b) Consider a distribution of particles evolving according to H in longitudinal phase-space. Neglect particle-particle interactions (not in formulation). A smooth distribution $f(\phi, P_\phi, s) \geq 0$ must satisfy

$$\frac{\partial f}{\partial s} + \frac{\partial}{\partial \phi}\left(\frac{d}{ds}\phi f\right) + \frac{\partial}{\partial P_\phi}\left(\frac{d}{ds}P_\phi f\right) = 0$$

since "probability" must flow somewhere. Show for our nonlinear longitudinal dynamics that

$$\left.\frac{d}{ds} f\right|_{\text{particle trajectory}} = 0$$

Explain how this implies that the total phase-space weight of particles at a given density is constant in the nonlinear evolution. You may want to read about Liouville's Theorem of noninteracting particles in statistical mechanics if you need help.

5 pts

c) If $\gamma_s\beta_s \neq$ const but vary slowly to maintain validity of the continuous formulation will H be constant? Why? [Keep all other factors constant in s $T, \phi_s, E_0, \lambda_{rf}, \phi_s$]

5 pts

d) If the phase excursion is small ($\phi = \phi_s + \Delta\phi$; $\Delta\phi$ small) with $\gamma_s\beta_s$ slowly varying, derive a 2nd order differential equation for the evolution of $\Delta\phi$. Do you expect this equation to have a conserved longitudinal emittance? Why?

For this part start from the continuous formulation with

$$(\gamma_s\beta_s)^3 \frac{d}{ds}(\phi - \phi_s) = \frac{-2\pi}{\lambda_{rf}} \frac{\Delta W}{mc^2}$$

$$\frac{d}{ds}\Delta W = qE_0T(\cos\phi - \cos\phi_s)$$

Take $\lambda_{rf}, E_0, T, \phi_s$ to be constants.

Write results using $k_s^2 = \frac{2\pi}{\lambda_{rf}} \frac{qE_0T\sin(-\phi_s)}{\gamma_s^3\beta_s^3 mc^2}$

Problem #4 RF Phase Choices. 10 points

In class, for the continuous model we showed that where

$$E_z(r=0, z=0, t=0) = E_0 \cos\phi_s \quad \begin{array}{l} > 0 \Rightarrow \text{accel} \\ < 0 \Rightarrow \text{deaccel.} \end{array} \qquad E_0 > 0$$

and where

$$V(\phi) = B(\sin\phi - \phi\cos\phi_s) \qquad B = \frac{qE_0T}{mc^2} > 0$$

has concavity

$$\left.\frac{d^2V(\phi)}{d\phi^2}\right|_{\phi=\phi_s} \begin{array}{l} > 0 \Rightarrow \text{stability (focusing)} \\ < 0 \Rightarrow \text{instability (defocusing)} \end{array}$$

locally about the synchronous particle

Use these to argue:

2pts a) Range of ϕ_s for deacceleration and focusing?

2pts b) Range of ϕ_s for acceleration and defocusing?

Seperately, from continuous model

2pts c) What value of ϕ_s will provide max longitudinal focusing and acceptance? Why? Does this allow acceleration?

"Fast Rotation"

4pts d) Consider a bunch with weak or no accel in the continuous model filling a small phase-width of the bucket.
If E_0 suddenly jumps argue what will happen to the longitudinal ~~phase~~ phase-spase ellipse. At what propagation length will the bunch have shortest phase width? Use the synchrotron wavenumber k_s to estimate. What vale of ϕ_s should be chosen