Accelerator Physics
Final Exam - 170 pts.

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Problem 1

P052 Emittance Evolution 40 pts.

a) 5 pts: Consider a coasting beam composed of particles evolving with an equation of motion:

$$x''(s) + \alpha(s) x^n(s) = 0$$
$$n > 0, \quad \alpha(s) = \text{non-linear function}, \quad {}' \equiv \frac{\mathrm{d}}{\mathrm{d}s}$$
$$x = \text{transverse coordinate}$$
$$s = \text{longitudinal coordinate}$$

Define an RMS emittance:

$$\varepsilon_{x,\mathrm{rms}} \equiv \left[\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \right]^{1/2}$$

where $\langle ... \rangle$ denotes statistical average over a distribution of particles.

Use the particle equation of motion to derive an evolution equation for $\frac{\mathrm{d}}{\mathrm{d}s}\varepsilon^2_{x,\mathrm{rms}}$ in terms of $\langle x^2 \rangle$, $\langle x x' \rangle$, $\langle x' x^n \rangle$, and $\langle x^{n+1} \rangle$

b) 5 pts: For what value of n is $\frac{\mathrm{d}}{\mathrm{d}s}\varepsilon^2_{x,\mathrm{rms}} = 0$? Does this make sense if $\varepsilon_{x,\mathrm{rms}}$ is a statistical measure of phase-space area? Why?

c) 20 pts: Hill's equation has $n = 1$ and $\alpha(s) = \kappa(s)$ is periodic with period L_p:

$$x''(s) + \kappa(s) x(s) = 0, \quad \kappa(s + L_p) = \kappa(s)$$

Take a phase-amplitude form of the particle orbit with

$$x = A_i w(s) \cos[\psi(s)]$$
$$A_i = \text{constant, particle amplitude}$$
$$\psi_i = \text{constant, particle initial phase}$$
$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{\mathrm{d}\tilde{s}}{w^2(\tilde{s})}$$
$$w''(s) + \kappa(s) w(s) - \frac{1}{w^3(s)} = 0$$
$$w(s + L_p) = w(s), \quad w(s) > 0$$

Show if particles within the beam are uniformly distributed in initial phase ψ_i that we have:

$$\langle x^2 \rangle = \frac{1}{2}\beta \langle A_i^2 \rangle$$
$$\langle x'^2 \rangle = \frac{1+\alpha}{2\beta} \langle A_i^2 \rangle = \frac{1}{2}\gamma \langle A_i^2 \rangle$$
$$\langle xx' \rangle = -\frac{1}{2}\alpha \langle A_i^2 \rangle$$

where $\langle ... \rangle$ denotes an average over the phase-space of the full beam:

$$\langle ... \rangle = \frac{\int_{-\pi}^{\pi} \int_0^{\infty} ... f(A_i) A_i \; \mathrm{d}A_i \mathrm{d}\psi_i}{\int_{-\pi}^{\pi} \int_0^{\infty} f(A_i) A_i \; \mathrm{d}A_i \mathrm{d}\psi_i}$$

$f(A_i)$ is the number of particles with initial amplitude A_i, there is no initial phase dependence, and α, β, and γ are the Courant-Snyder "parameters":

$$\beta \equiv w^2, \quad \gamma = \frac{1}{w^2} + w'^2 = \frac{1+\alpha^2}{\beta}, \quad \alpha = -ww'$$

You only need to show enough work on term averages to make clear that you understand which terms vanish and why they vanish.

d) 5 pts: Using results from part (c), relate the rms emittance

$$\varepsilon_{x,\mathrm{rms}} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

to the rms average particle amplitude $\langle A_i \rangle^{1/2}$.

e) 5 pts: If the initial particle amplitudes are uniformly distributed to $A_{\max}$ and we take:

$$\pi \epsilon_{x,\max} = \pi A_{\max}^2$$

with $\epsilon_{x,\max}$ the maximum *single-particle* emittance, relate $\epsilon_{x,\max}$ to $\varepsilon_{x,\mathrm{rms}}$.

Problem 2

P041 RHIC Longitudinal Parameters 25 pts.

RHIC is a heavy-ion collider which accelerates proton beams from 24 GeV to 250 GeV or the heavy-ion $^{197}_{79+}$Au from 10 GeV/u to 100 GeV/u. The ring lattice has transition gamma, $\gamma_T = 22.8$. Ignore the effect of the transit time factor of the RF cavity.

a) 5 pts: The circumference of the RHIC ring is 3833.845 m. The RF system is designed to have harmonic number $h = 360$. Find the required RF frequency.

b) 10 pts: The total voltage of the RF cavities in RHIC ring is 300 kV. The acceleration rate is designed to be $\mathrm{d}\gamma/\mathrm{d}t = 0.5 \; s^{-1}$. Find the proper synchronous phase for the ion and proton beams. Estimate the synchrotron tune of both proton and ion beams at 50 GeV/u.

c) 10 pts: The normalized RMS longitudinal emittance of the proton beam is 0.5 eV·s. Find the best estimate of the proton beam's rms bunch length and rms energy spread at maximum energy (250 GeV).

Problem 3

P008a Larmor Phase Advance 35 pts.

For a uniform solenoidal channel:

$$B_z^a(s) = B_0 = \text{const.}$$

with no acceleration

$$\gamma_b \beta_b = \text{const.}$$

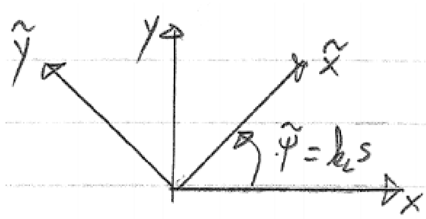
and an axisymmetric ($\partial/\partial\theta = 0$) beam with

$$\frac{\partial \phi}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\vec{x}_\perp}{r} \qquad r = \sqrt{x^2 + y^2}$$

the particle equations of motion reduce to:

$$\begin{aligned} x'' &= \frac{qB_0}{m\gamma_b\beta_b c} y' - \frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial r}\frac{x}{r} \\ y'' &= -\frac{qB_0}{m\gamma_b\beta_b c} x' - \frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial r}\frac{y}{r} \end{aligned}$$

a) 10 pts: Parallel the steps taken in the class notes to transform the equations of motion to a co-rotating frame:



k_L = const = Larmor wavenumber

$$\begin{aligned} \tilde{x} &= \ x\cos(k_L s) + y\sin(k_L s) \\ \tilde{y} &= -x\sin(k_L s) + y\cos(k_L s) \end{aligned}$$

Find an expression for k_L to reduce the equations of motion to the decoupled form:

$$\begin{aligned} \tilde{x}'' + \kappa\tilde{x} &= \frac{-q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{x}}{r} \\ \tilde{y}'' + \kappa\tilde{y} &= \frac{-q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{y}}{r} \end{aligned}$$

and identify $\kappa = \text{const.}$

Hint: The transformation can be carried out directly. But you may find the algebra simpler using complex coordinates as in the class notes:

$$\underset{\sim}{z} = x + iy \qquad i = \sqrt{-1} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$
$$\tilde{\underset{\sim}{z}} = \tilde{x} + i\tilde{y}$$

b) 5 pts: If the direction of the magnetic field is reversed:

$$B_0 \rightarrow -B_0$$

how will the dynamics be influenced?

c) 5 pts: Neglect space-charge:

$$\phi = 0$$

and sketch a typical orbit in the rotating Larmor frame. Will this orbit appear more complicated in the Laboratory frame? Why?

Hint: Sketch the orbit taking advantage of simple choices of initial conditions that can always be made through choice of coordinates.

d) 5 pts: If we take $\vec{B} = \vec{\nabla} \times \vec{A}$ where $\vec{A} = \frac{1}{2} B_{z0} r \hat{\theta}$, show we can generate the linear field components of the solenoid as:

$$B_r = -\frac{1}{2}\frac{\partial B_{z0}}{\partial z} r$$
$$B_z = B_{z0}$$

e) 10 pts: Use the paraxial approximation and the results from part d) to show for a solenoid that:

$$\begin{aligned} P_\theta &= \left[\vec{x} \times \left(\vec{p} + q\vec{A}\right)\right] \cdot \hat{z} \\ &\simeq m\gamma_b\beta_b c\left(xy' - yx'\right) + \frac{qB_{z0}}{2}\left(x^2 + y^2\right) \end{aligned}$$

Show that the equations of motion:

$$x'' - \frac{B'_{z0}}{2\,[B\rho]} y - \frac{B_{z0}}{[B\rho]} y' = 0 \qquad\qquad [B\rho] = \frac{\gamma_b\beta_b mc}{q}$$
$$y'' + \frac{B'_{z0}}{2\,[B\rho]} x + \frac{B_{z0}}{[B\rho]} x' = 0 \qquad\qquad B'_{z0} = \frac{\partial B_{z0}(z)}{\partial z}$$

imply that P_θ is constant.

Problem 4

P055 Colliders 40 pts.

a) 5 pts: As defined in lectures, the luminosity $\mathcal{L}$ is a measure of the probability of the particles encountered per unit area and time. It is defined as:

$$\mathcal{L} = 2fN_1N_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_1(x,y,s_1)\rho_2(x,y,s_2)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}(\beta ct)$$

where $s_{1,2} = s \pm \beta ct$, β is the velocity of the particle, s is the location of the collision, t is the time duration of the collision, f is the collision frequency, and $\rho_{1,2}$ are the normalized distributions of the two colliding bunches.

$$\rho_i(x,y,s,\sigma_{x,i},\sigma_{y,i},\sigma_{s,i}) = \frac{1}{(2\pi)^{3/2}\sigma_{x,i}\sigma_{y,i}\sigma_{s,i}} \exp\left(-\frac{x^2}{2\sigma_{x,i}^2} - \frac{y^2}{2\sigma_{y,i}^2} - \frac{s^2}{2\sigma_{s,i}^2}\right)$$

Prove that for two short colliding bunches, the luminosity is given by:

$$\mathcal{L} = \frac{N_1N_2f}{2\pi\sqrt{\sigma_{x,1}^2+\sigma_{x,2}^2}\sqrt{\sigma_{y,1}^2+\sigma_{y,2}^2}}$$

where the subscripts 1 and 2 denote beams 1 and 2.

b) 5 pts: The above definition of luminosity is called the 'peak' luminosity. As the particles are lost and the beam degrades over time, the luminosity also drops. The integrated luminosity $\int \mathcal{L}\ \mathrm{d}t$ is a more important value. Usually, experimental physicists would like to use the unit pb^{-1}, where pb is pico-barn or 10^{-40} m^2. Take RHIC as an example. RHIC's peak luminosity is about 1.5×10^{32} $\mathrm{cm}^{-2}\mathrm{s}^{-1}$. The integrated luminosity in one week is about 50 pb^{-1}. Calculate the average luminosity in units of $\mathrm{cm}^{-2}\mathrm{s}^{-1}$ and compare with the peak luminosity.

c) 10 pts: The arc lattice of a collider can be as simple as a FODO cell. For example, RHIC uses FODO cells in its arcs. The relevant parameters of RHIC arc cells are listed below.

Parameters	Values
Dipole bending radius	242.78 m
Dipole bending angle	0.039 rad
Cell length	29.59 m
Quad strength $\int B_1\ \mathrm{d}\ell/[B\rho]$	0.09 m^{-1}

Use the thin lens approximation to calculate the maximum and minimum betatron functions and dispersion function and also the phase advance and natural chromaticity per cell.

d) 10 pts: If we put the two thin sextupoles at the center plane of the quadrupoles, what is the required strength to cancel the natural chromaticity exactly? Do the two sextupoles have the same strength?

e) 5 pts: In the straight interaction region, the two beams collide a the interaction point, $s = 0$. The interaction region spans from $[-L, L]$, which can be considered a drift space. To maximize the luminosity, a small beta 'waist' is usually created at the interaction point, which is denoted β^*. The low beta value is created by a three quadrupole system (QF1, QD2, QF3) called a triplet. The maximum beta function in the triplet is referred to as β_{max}. Calculate the phase advance from the interaction point to the location of β_{max} and prove that it is a constant when $\beta_{\text{max}} \gg \beta^*$.

f) 5 pts: The performance of RHIC is found to be limited by an effect called intra-beam scattering. One countermeasure is to reduce the average $\mathcal{H}$ function in the arc. Propose a way this goal can be achieved.

Problem 5

P056 Synchrotron Radiation 30 pts.

a) 15 pts: The LHC accelerates proton beams to 7 TeV in a superconducting ring with a 26.7 km circumference. The superconducting bending dipoles provide a 8.3 T field. Calculate:

- The radiation energy of the 7 TeV protons per turn.
- The critical energy of photons.
- The radiation power for a beam current of 800 mA

b) 10 pts: The LHC tunnel was used for the LEP (Large Electron Project). LEP accelerates the electron beam to ~100 GeV, the highest electron energy achieved in a collider. Assuming the dipole layout of LEP is the same as for the LHC, repeat the calculations carried out in part (a) for the electron beam.

c) 5 pts: The equilibrium electron emittance at 100 GeV in LEP is aobut 0.06 mm-mrad. In order to use the LEP machine as a storage ring for a 5 GeV synchrotron light source, what will be the equilibrium electron emittance? Compare this result with the value of newer light sources with a multi-bend achromat lattice and a much shorter circumference.