18. Solenoid Focusing*

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Transverse Particle Equations of Motion

\[ x'' + \left( \frac{\gamma_b \beta_b'}{\beta_b} \right)' \frac{\gamma_b \beta_b'}{\beta_b} x' = -\frac{q}{m \gamma_b \beta_b c} B_0^a \frac{\partial}{\partial x} - \frac{q}{m \gamma_b \beta_b c} B_0^a \frac{\partial}{\partial y} \]

\[ y'' + \left( \frac{\gamma_b \beta_b'}{\beta_b} \right)' \frac{\gamma_b \beta_b'}{\beta_b} y' = -\frac{q}{m \gamma_b \beta_b c} B_0^a \frac{\partial}{\partial x} - \frac{q}{m \gamma_b \beta_b c} B_0^a \frac{\partial}{\partial y} \]

Imply \( B^a \) can be expressed in terms of on-axis field \( B_0^a \)

S2E: Solenoidal Focusing

The field of an ideal magnetic solenoid is invariant under transverse rotations about its axis of symmetry (z) can be expanded in terms of the on-axis field as:

\[ E^a = 0 \]

\[ B_0^a = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1} B_0(z)}{\partial z^{2\nu-1}} \left( \frac{|x|}{2} \right)^{2\nu-2} \]

\[ B_x^a = B_0(z) + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu} B_0(z)}{\partial z^{2\nu}} \left( \frac{|x|}{2} \right)^{2\nu} \]

\[ B_0(z) \equiv B_0^a(x = 0, z) = \text{On-Axis Field} \]
Writing out explicitly the terms of this expansion:

\[
B^a(r, z) = iB^{x0}(r, z) + \hat{y}B^{y0}(r, z)
\]

where

\[
B^{x0}(r, z) = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(\nu-1)!} \frac{B^{2(\nu-1)}_0(z)}{2^{2\nu-1}} r^{2\nu-1} - \frac{B^{(1)}_0(z)}{2} r + \frac{B^{(3)}_0(z)}{384} r^3 + \frac{B^{(5)}_0(z)}{18432} r^5 - \frac{B^{(7)}_0(z)}{147456} r^7 + \ldots
\]

and

\[
B^{y0}(r, z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu)\nu^2} B^{4\nu}_0(z) \left( \frac{r}{2} \right)^{2\nu}
\]

are the linear terms.

For modeling, we truncate the expansion using only leading-order terms to obtain:

\[
B^a = \frac{\partial B^{x0}(z)}{\partial z} x + \frac{\partial B^{y0}(z)}{\partial z} y
\]

Note that this truncated expansion is divergence free:

\[
\nabla \cdot B^a = 0
\]

but not curl free within the vacuum aperture:

\[
\nabla \times B^a = \frac{1}{2} \nabla \times \left[ \frac{\partial^2 B^{x0}(z)}{\partial z^2} \right] (-\hat{x} y + \hat{y} x)
\]

It can be shown (see: Appendix B) that the linear cross-coupling in the applied field can be removed by an s-varying transformation to a rotating 

"Larmor" frame:

\[
\begin{align*}
\tilde{x} &= x \cos \psi(s) + y \sin \psi(s) \\
\tilde{y} &= -x \sin \psi(s) + y \cos \psi(s)
\end{align*}
\]

Equations are linearly cross-coupled in the applied field terms - x equation depends on y, y' - y equation depends on x, x'

\[
\omega_c(s) = \frac{qB^{x0}(s)}{m} = \text{Cyclotron Frequency (in applied axial magnetic field)}
\]

\[
\omega_L = \frac{\gamma \beta c}{2} = \text{Larmor wave number}
\]

\[
s = s_0 \text{ defines initial condition}
\]
If the beam space-charge is axisymmetric:
\[
\frac{\partial \phi}{\partial x_\perp} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x_\perp} = \frac{\partial \phi}{\partial r} \frac{x_\perp}{r}
\]
then the space-charge term also decouples under the Larmor transformation and the equations of motion can be expressed in fully uncoupled form:

\[
\ddot{x}_\perp + \frac{\gamma_b \beta_b}{\gamma_b \beta_b} \dot{y}' + \kappa(s) \ddot{x} = - \frac{q}{m \gamma_b^2 \beta_b^2} \frac{\partial \phi}{\partial r} \frac{x_\perp}{r}
\]
\[
\ddot{y}' + \frac{\gamma_b \beta_b}{\gamma_b \beta_b} \dot{x}' + \kappa(s) \ddot{y} = - \frac{q}{m \gamma_b^2 \beta_b^2} \frac{\partial \phi}{\partial r} \frac{y_\perp}{r}
\]
\[
\kappa(s) = k^2(s) = \left[ \frac{B_\perp(s)}{2 |B_\perp|} \right]^2 = \left[ \frac{\omega_c(s)}{2 \gamma_b \beta_b} \right]^2
\]

Because Larmor frame equations are in the same form as continuous and quadrupole focusing with a different, for solenoidal focusing we implicitly work in the Larmor frame and simplify notation by dropping the tildes:

\[
\dot{x}_\perp \rightarrow x_\perp
\]

---

Solenoid periodic lattices can be formed similarly to the quadrupole case
- Drifts placed between solenoids of finite axial length
- Allows space for diagnostics, pumping, acceleration cells, etc.
- Analogous equivalence cases to quadrupole
- Piecewise constant \( \kappa \) often used
- Fringe can be more important for solenoids

Simple hard-edge solenoid lattice with piecewise constant \( \kappa \)

\[
\kappa(s) = \begin{cases} 
0 & s < \frac{d}{2} \\
\tilde{k} & \frac{d}{2} < s < \frac{d}{2} + \ell \\
0 & \frac{d}{2} + \ell < s 
\end{cases}
\]

Calculate using transfer matrices in Appendix C

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// Example: Larmor Frame Particle Orbits in a Periodic Solenoidal Focusing Lattice: \( \ddot{x} - \dot{x}' \) phase-space for hard edge elements and applied fields

\( L_p = 0.5 \text{ m} \quad \kappa = 20 \text{ rad/m}^2 \) in Solenoids \( \ddot{x}(0) = 1 \text{ mm} \quad \ddot{y}(0) = 0 \)

\( \eta = 0.5 \quad \phi \simeq 0 \quad \gamma_b \beta_b = \text{const} \quad \ddot{x}(0) = 0 \quad \ddot{y}(0) = 0 \)

Contrast of Larmor-Frame and Lab-Frame Orbits
- Same initial condition

Larmor-Frame Coordinate: Orbit in transformed x-plane only

Lab-Frame Coordinate: Orbit in both x- and y-planes

Calculate using transfer matrices in Appendix C
Contrast of Larmor-Frame and Lab-Frame Orbits

- Same initial condition

Larmor-Frame Angle

![Larmor-Frame Angle Diagram]

Lab-Frame Angle

![Lab-Frame Angle Diagram]

Calculate using transfer matrices in Appendix C

Additional perspectives of particle orbit in solenoid transport channel

- Same initial condition

Radius evolution (Lab or Larmor Frame: radius same)

![Radius Evolution Diagram]

Side- (2 view points) and End-View Projections of 3D Lab-Frame Orbit

![Side- and End-View Projections Diagram]

Calculate using transfer matrices in Appendix C

Comments on Orbits:

- See Appendix C for details on calculation
  - Discontinuous fringe of hard-edge model must be treated carefully if integrating in the laboratory-frame.
  - Larmor-frame orbits strongly deviate from simple harmonic form due to periodic focusing
    - Multiple harmonics present
    - Less complicated than quadrupole AG focusing case when interpreted in the Larmor frame due to the optic being focusing in both planes
  - Orbits transformed back into the Laboratory frame using Larmor transform (see: Appendix B and Appendix C)
    - Laboratory frame orbit exhibits more complicated x-y plane coupled oscillatory structure
  - Will find later that if the focusing is sufficiently strong, the orbit can become unstable (see: S5)
  - Larmor frame y-orbits have same properties as the x-orbits due to the equations being decoupled and identical in form in each plane
    - In example, Larmor y-orbit is zero due to simple initial condition in x-plane
    - Lab y-orbit is nozero due to x-y coupling
Comments on Orbits (continued):
- Larmor angle advances continuously even for hard-edge focusing
- Mechanical angular momentum jumps discontinuously going into and out of the solenoid
  - Particle spins up and down going into and out of the solenoid
  - No mechanical angular momentum outside of solenoid due to the choice of initial condition in this example (initial $x$-plane motion)
- Canonical angular momentum $P_\theta$ is conserved in the 3D orbit evolution
  - As expected from analysis in S2G
  - Invariance provides a good check on dynamics
  - $P_\theta$ in example has zero value due to the specific ($x$-plane) choice of initial condition. Other choices can give nonzero values and finite mechanical angular momentum in drifts.

Some properties of particle orbits in solenoids with piecewise $\kappa = \text{const}$ will be analyzed in the problem sets.

S2F: Summary of Transverse Particle Equations of Motion

In linear applied focusing channels, without momentum spread or radiation, the particle equations of motion in both the $x$- and $y$-planes expressed as:

$$x'' + \left(\frac{\gamma_0 \beta_0}{\gamma_0 \beta_0} \right) x' + \kappa_x(s) x = -\frac{q}{m n c^2} \frac{\partial}{\partial \phi}$$

$$y'' + \left(\frac{\gamma_0 \beta_0}{\gamma_0 \beta_0} \right) y' + \kappa_y(s) y = -\frac{q}{m n c^2} \frac{\partial}{\partial \phi}$$

$$\kappa_x(s) = x\text{-focusing function of lattice}$$
$$\kappa_y(s) = y\text{-focusing function of lattice}$$

Common focusing functions:
- Continuous: $\kappa_x(s) = \kappa_y(s) = k_{20}^2 = \text{const}$
- Quadrupole (Electric or Magnetic): $\kappa_x(s) = -\kappa_y(s) = \kappa(s)$
- Solenoidal (equations must be interpreted in Larmor Frame: see Appendix B): $\kappa_x(s) = \kappa_y(s) = \kappa(s)$

Although the equations have the same form, the couplings to the fields are different which leads to different regimes of applicability for the various focusing technologies with their associated technology limits:

**Focusing:**
- Continuous: $\kappa_x(s) = \kappa_y(s) = k_{20}^2 = \text{const}$
  - Good qualitative guide (see later material/lecture)
  - BUT not physically realizable (see S2B)
- Quadrupole:
  - $\kappa_x(s) = -\kappa_y(s) = \frac{G(s)}{\beta_0 c \beta_0} [B_\rho] = \frac{m \gamma_0 \beta_0 c}{q}$
  - Electric
  - Magnetic

$G(s)$ is the field gradient which for linear applied fields is:

$$G(s) = \begin{cases} \frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = \frac{2V_0}{r_p^2}, & \text{Electric} \\
\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \frac{B_x}{r_p}, & \text{Magnetic} \end{cases}$$

**Solenoid:**

$$\kappa_x(s) = \kappa_y(s) = k_L^2(s) = \left[ \frac{B_{z0}(s)}{2|B_\rho|} \right]^2 = \left[ \frac{\omega_c(s)}{2 \gamma_0 \beta_0 c} \right]^2 = \frac{q B_{z0}(s)}{m}$$

It is instructive to review the structure of solutions of the transverse particle equations of motion in the absence of:

- Space-charge: $\frac{\partial \phi}{\partial x} \sim \frac{\partial \phi}{\partial y} \sim 0$
- Acceleration: $\gamma \beta_0 \simeq \text{const} \quad \Rightarrow \left(\frac{\gamma_0 \beta_0}{\gamma_0 \beta_0} \right) \simeq 0$

In this simple limit, the $x$ and $y$-equations are of the same Hill’s Equation form:

$$x'' + \kappa_x(s) x = 0$$
$$y'' + \kappa_y(s) y = 0$$

- These equations are central to transverse dynamics in conventional accelerator physics (weak space-charge and acceleration)
  - Will study how solutions change with space-charge in later lectures

In many cases beam transport lattices are designed where the applied focusing functions are periodic:

$$\kappa_x(s + L_p) = \kappa_x(s)$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

$L_p = \text{Lattice Period}$
Common, simple examples of periodic lattices:

**Periodic Solenoid**

\[ \kappa_z(\eta) = \kappa_\eta \]

\[ s \]

\[ d/2 \]

\[ \ell \]

\[ d/2 \]

\[ d = (1 - \eta)L_p \]

\[ \ell = \eta L_p \]

**Periodic FODO Quadrupole**

\[ \kappa_z(\eta) = -\kappa_\eta \]

\[ s \]

\[ d \]

\[ \ell \]

\[ d \]

\[ d = (1 - \eta)L_p/2 \]

\[ \ell = \eta L_p/2 \]

However, the focusing functions need not be periodic:

- Often take periodic or continuous in this class for simplicity of interpretation
- Focusing functions can vary strongly in many common situations:
  - Matching and transition sections
  - Strong acceleration
  - Significantly different elements can occur within periods of lattices in rings
    - “Panofsky” type (wide aperture along one plane) quadrupoles for beam insertion and extraction in a ring

**Example of Non-Periodic Focusing Functions: Beam Matching Section**

Maintains alternating-gradient structure but not quasi-periodic

For continuous, electric or magnetic quadrupole focusing without acceleration

\( \gamma_0 = \text{const} \), it is straightforward to verify that \( x, x', y, y' \) are canonical coordinates and that the correct equations of motion are generated by the Hamiltonian:

\[
H_\perp = \frac{1}{2} x'^2 + \frac{1}{2} y'^2 + \frac{1}{2} \kappa_x x'^2 + \frac{1}{2} \kappa_y y'^2 + \frac{q\phi}{\gamma_0^2 \beta_0^2 c^3}
\]

\[
\frac{d}{ds} x = \frac{\partial H_\perp}{\partial x'} \quad \frac{d}{ds} x' = -\frac{\partial H_\perp}{\partial x}
\]

\[
\frac{d}{ds} y = \frac{\partial H_\perp}{\partial y'} \quad \frac{d}{ds} y' = -\frac{\partial H_\perp}{\partial y}
\]

Giving the familiar equations of motion:

\[
x'' + \kappa_x x = -\frac{q}{\gamma_0^2 \beta_0^2 c^2} \frac{\partial \phi}{\partial x}
\]

\[
y'' + \kappa_y y = -\frac{q}{\gamma_0^2 \beta_0^2 c^2} \frac{\partial \phi}{\partial y}
\]
For solenoidal magnetic focusing without acceleration, it can be verified that we can take (tilde) canonical variables:

- Tildes do not denote Larmor transform variables here!

\[
\begin{align*}
\tilde{x} &= x \\
\tilde{y} &= y \\
\tilde{x}' &= \frac{B_{x0}}{2[B\rho]} y \\
\tilde{y}' &= \frac{B_{x0}}{2[B\rho]} x \\
[\rho] &= \frac{m\gamma\beta c}{q}
\end{align*}
\]

With Hamiltonian:

\[
\tilde{H}_\perp = \frac{1}{2} \left[ \left( \tilde{x}' + \frac{B_{x0}}{2[B\rho]} \tilde{y} \right)^2 + \left( \tilde{y}' - \frac{B_{x0}}{2[B\rho]} \tilde{x} \right)^2 \right] + \frac{q\phi}{m\gamma^2\beta^2 c^3}
\]

\[
\begin{align*}
\frac{d}{ds} \tilde{x} &= \frac{\partial \tilde{H}_\perp}{\partial \tilde{y}} \\
\frac{d}{ds} \tilde{y} &= -\frac{\partial \tilde{H}_\perp}{\partial \tilde{x}} \\
\frac{d}{ds} \tilde{x}' &= -\frac{q\phi}{m\gamma^2\beta^2 c^3} \\
\frac{d}{ds} \tilde{y}' &= \frac{q\phi}{m\gamma^2\beta^2 c^3}
\end{align*}
\]

Caution: Primes do not mean \(\frac{d}{ds}\) in tilde variables here: just notation to distinguish "momentum" variable!

Inserting the vector potential components consistent with linear approximation solenoid focusing in the paraxial expression gives:

- Applies to (superimposed or separately) to continuous, magnetic or electric quadrupole, or solenoidal focusing since \(A_\theta \neq 0\) only for solenoidal focusing

\[
P_\theta \simeq m\gamma\beta c(xy' - yx') + \frac{qB_{x0}}{2}(x^2 + y^2)
\]

For a coasting beam \((\gamma\beta = \text{const.})\), it is often convenient to analyze:

- Later we will find this is analogous to use of "unnormalized" variables used in calculation of ordinary emittance rather than normalized emittance

\[
P_\theta = \frac{m\gamma\beta c}{\rho} \left( \begin{array}{c}
s \frac{\partial}{\partial s} \\
\frac{\partial}{\partial s}
\end{array} \right) + \frac{k^2 \rho}{\gamma \beta c} \kappa_\rho = \text{const.}
\]

Conservation of canonical angular momentum

One expects from general considerations (Noether’s Theorem in dynamics) that systems with a symmetry have a conservation constraint associated with the generator of the symmetry. So for systems with azimuthal symmetry \((\partial/\partial \theta = 0)\), one expects there to be a conserved canonical angular momentum (generator of rotations). Based on the Hamiltonian dynamics structure, examine:

\[
P_\theta = [x \times \mathbf{P}] \cdot \dot{z} = [x \times (\mathbf{p} + qA)] \cdot \dot{z}
\]

This is exactly equivalent to

- Here \(\gamma\) factor is exact (not paraxial)

\[
P_\theta = r(p_\theta + qA_\theta) = m\gamma\tau^2\theta + qrA_\theta
\]

Or employing the usual paraxial approximation steps:

\[
P_\theta \simeq m\gamma\beta c(xz' - yz') + q(xA_y - yA_x)
\]

\[
= m\gamma\beta c r^2 \theta' + qrA_\theta
\]

Equations of Motion:

Including acceleration effects again, we summarize the equations of motion as:

- Applies to continuous, quadrupole (electric + magnetic), and solenoid focusing as expressed

- Several types of focusing can also be superimposed

- Show for superimposed solenoid

\[
\begin{align*}
x'' &= \frac{\gamma\beta cy'}{(\gamma\beta c)} \\
y' &= \frac{\gamma\beta cy'}{(\gamma\beta c)} + \kappa_x x + \frac{B_{x0}(s)}{2[B\rho]} y - \frac{B_{x0}(s)}{[B\rho]} x \\
y'' &= \frac{\gamma\beta cy'}{(\gamma\beta c)} + \kappa_y y + \frac{B_{x0}(s)}{2[B\rho]} x + \frac{B_{x0}(s)}{[B\rho]} x' \\
[\rho] &= \frac{m\gamma\beta c}{q} \kappa_\rho (s) = \left( \begin{array}{c}
k^2 \rho = \text{const.} \\
\frac{\gamma\beta c}{\beta c (\gamma\beta c)} \\
\frac{\gamma\beta c}{\beta c (\gamma\beta c)} \\
\end{array} \right)
\]

Continuous Focus \((\kappa_\rho = \kappa_x)\),

Electric Quadrupole Focus \((\kappa_\rho = -\kappa_x)\),

Magnetic Quadrupole Focus \((\kappa_\rho = -\kappa_x)\)
Employ the paraxial form of $P_\theta$ consistent with the possible existence of a solenoid magnetic field:

- Formula also applies as expressed to continuous and quadrupole focusing
- $P_\theta = m\gamma_0\beta_c(x'y' - y'z') + \frac{qB_0}{2}(x'^2 + y'^2)$

Differentiate and apply equations of motion:

- Intermediate algebraic steps not shown

\[
\frac{d}{ds}P_\theta = mc(\gamma_0\beta_c)(xy' - yx') + mc(\gamma_0\beta_c)(xy'' - yx'') + \frac{qB_0}{2}(x'^2 + y'^2) + qB_{z0}(xx' + yy')
\]

So $\frac{d}{ds}P_\theta = 0 \implies P_\theta = \text{const}$

Examples: solenoidal focusing channel

Employ the solenoid focusing channel example in S2E and plot:

- Mechanical angular momentum $\propto xy' - yx'$
- Vector potential contribution to canonical angular momentum $\propto B_{z0}(x'^2 + y'^2)$
- Canonical angular momentum (constant) $P_\theta$

Comments on Orbits (see also info in S2E on 3D orbit):

- Mechanical angular momentum jumps discontinuously going into and out of the solenoid
- No mechanical angular momentum outside of solenoid due to the choice of initial condition
- Canonical angular momentum $P_\theta$ is conserved in the 3D orbit evolution

For:

- Continuous focusing
- Linear optics solenoid magnetic focusing
- Other axisymmetric electric optics not covered such as Einzel lenses ...
Alternative expressions of canonical angular momentum

It is insightful to express the canonical angular momentum in (denoted tilde here) in the solenoid focusing canonical variables used earlier in this section and rotating Larmor frame variables:

- See Appendix B for Larmor frame transform
- Might expect simpler form of expressions given the relative simplicity of the formulation in canonical and Larmor frame variables

**Canonical Variables:**

\[
\tilde{x} = x \\
\tilde{y} = y \\
\tilde{z}' = x' - \frac{B_{s0}}{2[B\rho]} y \\
\tilde{y}' = y' + \frac{B_{s0}}{2[B\rho]} x
\]

\[
\implies \frac{P_\theta}{m\gamma_b\beta_b c} = xy' - yx' + \frac{B_{s0}}{2[B\rho]} (x^2 + y^2)
\]

- Applies to acceleration also since just employing transform as a definition here

**Larmor (Rotating) Frame Variables:**

Larmor transform following formulation in Appendix B:

- Here tildes denote Larmor frame variables

\[
\begin{bmatrix}
    x' \\
    y' \\
    z'
\end{bmatrix} =
\begin{bmatrix}
    \cos \psi & 0 & -\sin \psi & 0 \\
    k_L \sin \psi & \cos \psi & k_L \cos \psi & -\sin \psi \\
    0 & \sin \psi & 0 & \cos \psi
\end{bmatrix}
\begin{bmatrix}
    \tilde{x} \\
    \tilde{y} \\
    \tilde{z}'
\end{bmatrix} \\
\implies \tilde{r}(s) = -\int_x^s ds k_L(s)
\]

\[
k_L(s) = \frac{B_{s0}(s)}{2[B\rho]}
\]

This gives after some algebra:

\[
x'^2 + y'^2 = \tilde{x}'^2 + \tilde{y}'^2
\]

\[
x'y' - y'x' = \tilde{x}'\tilde{y}' - \tilde{y}'\tilde{x}' - \frac{B_{s0}}{2[B\rho]} (\tilde{x}'^2 + \tilde{y}'^2)
\]

**Bush's Theorem expression of canonical angular momentum conservation**

Take:

\[
B^a = \nabla \times A
\]

and apply Stokes Theorem to calculate the magnetic flux \( \Psi \) through a circle of radius \( r \):

\[
\Psi = \int_r d^2 x B^a \cdot \hat{z} = \int_r d^2 x (\nabla \times A) \cdot \hat{z} = \oint_r A \cdot d\ell
\]

For a nonlinear, but axisymmetric solenoid, one can always take:

- Also applies to linear field component case

\[
A = \hat{\theta} A_\theta(r, z)
\]

\[
\implies B^a = -r \frac{\partial A_\theta}{\partial z} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta)
\]

Thus:

\[
\Psi = 2\pi r A_\theta
\]

// Aside: Nonlinear Application of Vector Potential

Given the magnetic field components

\[
B^a_r(r, z) \quad B^a_z(r, z)
\]

the equations

\[
B^a_r(r, z) = -\frac{\partial}{\partial z} A_\theta(r, z) \\
B^a_z(r, z) = \frac{1}{r} \frac{\partial}{\partial r} [r A_\theta(r, z)]
\]

can be integrated for a single isolated magnet to obtain equivalent expressions for \( A_\theta \)

\[
A_\theta(r, z) = -\int_r^s ds B^a_r(r, z) \\
A_\theta(r, z) = \frac{1}{r} \int_0^s d\tilde{r} B^a_r(\tilde{r}, z)
\]

- Resulting \( A_\theta \) contains consistent nonlinear terms with magnetic field

//
Then the exact form of the canonical angular momentum for solenoid focusing can be expressed as:

\[ P_θ = mγr^2 ̇θ + qrAθ \]

\[ = mγr^2 ̇θ + \frac{qΨ}{2π} \]

This form is often applied in solenoidal focusing and is known as “Bush’s Theorem” with

\[ P_θ = mγr^2 ̇θ + \frac{qΨ}{2π} = \text{const} \]

- Here \( γ \) factor is exact (not paraxial)
- In a static applied magnetic field, \( γ = \text{const} \), further simplifying use of eqn
- Exact as expressed, but easily modified using familiar steps for paraxial form and/or linear field components
- Expresses how a particle “spins up” when entering a solenoidal magnetic field

Appendix B: The Larmor Transform to Express Solenoidal Focused Particle Equations of Motion in Uncoupled Form

Solenoid equations of motion:

\[
\begin{align*}
\tau'' &+ \left( \frac{γ_θβ_b^l}{γ_b} \right) \tau' - \frac{B_{z0}^l(s)}{2[Bρ]} \tau - \frac{B_{x0}(s)}{[Bρ]} = -\frac{q}{mγ_b^lβ_b^l c^2} \frac{∂φ}{∂τ} \\
y'' &+ \left( \frac{γ_θβ_b^l}{γ_b} \right) y' + \frac{B_{z0}^l(s)}{2[Bρ]} y + \frac{B_{x0}(s)}{[Bρ]} x' = -\frac{q}{mγ_b^lβ_b^l c^2} \frac{∂φ}{∂y} \\
B_{z0}(s) & = B_{z0}^l(r = 0, z = s) = \text{On-Axis Field} \\
[Bρ] & = \frac{γ_θβ_b^l mc}{q} = \text{Rigidity}
\end{align*}
\]

To simplify algebra, introduce the complex coordinate

\[ \tilde{z} ≡ x + iy \quad i \equiv \sqrt{-1} \]

Note* context clarifies use of \( i \) (particle index, initial cond, complex \( i \))

Then the two equations can be expressed as a single complex equation

\[ \tilde{z}'' + \left( \frac{γ_θβ_b^l}{γ_b} \right) \tilde{z}' + i \frac{B_{z0}^l(s)}{2[Bρ]} \tilde{z} + i \frac{B_{x0}(s)}{[Bρ]} \tilde{x}' = -\frac{q}{mγ_b^lβ_b^l c^2} \left( \frac{∂φ}{∂x} + i \frac{∂φ}{∂y} \right) \]

If the potential is axisymmetric with \( φ = φ(r) \):

\[ \frac{∂φ}{∂x} + i \frac{∂φ}{∂y} = \frac{∂φ}{∂r} r \quad r ≡ \sqrt{x^2 + y^2} \]

then the complex form equation of motion reduces to:

\[ \tilde{z}'' + \left( \frac{γ_θβ_b^l}{γ_b} \right) \tilde{z}' + i \frac{B_{z0}^l(s)}{2[Bρ]} \tilde{z} + i \frac{B_{x0}(s)}{[Bρ]} \tilde{x}' = -\frac{q}{mγ_b^lβ_b^l c^2} \frac{∂φ}{∂r} r \]

Following Wiedemann, Vol II, pg 82, introduce a transformed complex variable that is a local (s-varying) rotation:

\[ \tilde{z} ≡ z e^{-iψ(s)} = \tilde{x} + i \tilde{y} \]

\[ \tilde{ψ}(s) = \text{phase-function (real-valued)} \]

Then:

\[ \tilde{z}'' + i \left( 2 \tilde{ψ}' + \frac{B_{z0}}{[Bρ]} \right) \tilde{z} + i \left( \tilde{ψ}'' + \frac{B_{x0}}{2[Bρ]} + \left( \frac{γ_θβ_b^l}{γ_b} \right) \tilde{ψ}' \right) \tilde{z} = \]

\[ = -\frac{q}{mγ_b^lβ_b^l c^2} \frac{∂φ}{∂r} r \]

Free to choose the form of \( \tilde{ψ} \) Can choose to eliminate imaginary terms in \( i (....) \) in equation by taking:

\[ \tilde{ψ}' = -\frac{B_{z0}}{2[Bρ]} \quad \text{implies} \quad \tilde{ψ}'' = -\frac{B_{z0}'}{2[Bρ]} - \frac{B_{z0}}{2[Bρ]} \left( \frac{γ_θβ_b^l}{γ_b} \right) \tilde{ψ}' \]

\[ B3 \]
Using these results, the complex form equations of motion reduce to:

\[
\ddot{\tilde{z}} + \left( \frac{\gamma_0 \beta_0}{\lambda \beta_0} \right) \dot{\tilde{z}} + \frac{B_{z0}}{2|B_\rho|} \tilde{z} = -\frac{q}{m \gamma_0^2 \beta_0^2 c^2} \frac{\partial \phi}{\partial r} \tilde{z}
\]

Or using \( \tilde{z} = \tilde{x} + i\tilde{y} \), the equations can be expressed in decoupled \( \tilde{x}, \tilde{y} \) variables in the Larmor Frame as:

\[
\ddot{\tilde{x}} + \left( \frac{\gamma_0 \beta_0}{\lambda \beta_0} \right) \dot{\tilde{x}} + \kappa(s)\tilde{x} = -\frac{q}{m \gamma_0^2 \beta_0^2 c^2} \frac{\partial \phi}{\partial r} \tilde{x}
\]

\[
\ddot{\tilde{y}} + \left( \frac{\gamma_0 \beta_0}{\lambda \beta_0} \right) \dot{\tilde{y}} + \kappa(s)\tilde{y} = -\frac{q}{m \gamma_0^2 \beta_0^2 c^2} \frac{\partial \phi}{\partial r} \tilde{y}
\]

\[\kappa(s) \equiv k^2_L(s) \quad k_L(s) = \frac{B_{z0}(s)}{2|B_\rho|} = \frac{\omega_c(s)}{2\gamma_0 \beta_0 c} = \text{Larmor Wave-Number}\]

Equations of motion are uncoupled but must be interpreted in the rotating Larmor frame.

- Same form as quadrupoles but with focusing function same sign in each plane.

The rotational transformation to the Larmor Frame can be effected by integrating the equation for \( \tilde{y}' = -\frac{B_{z0}}{2|B_\rho|} \)

\[\tilde{\psi}(s) = -\int_{s_i}^{s} \frac{B_{z0}(\tilde{s})}{2|B_\rho|} \, d\tilde{s} = -\int_{s_i}^{s} k_L(\tilde{s}) \]

Here, \( s_i \) is some value of \( s \) where the initial conditions are taken.

- Take \( s = s_i \) where axial field is zero for simplest interpretation (see pg B6)

Because

\[\tilde{\psi}' = -\frac{B_{z0}}{2|B_\rho|} = \frac{\omega_c}{2\gamma_0 \beta_0 c}\]

the local \( \tilde{x} - \tilde{y} \) Larmor frame is rotating at \( \frac{1}{2} \) of the local s-varying cyclotron frequency.

- If \( B_{z0} = \text{const} \), then the Larmor frame is uniformly rotating as is well known from elementary textbooks (see problem sets).

The complex form phase-space transformation and inverse transformations are:

\[
\begin{align*}
\tilde{z} &= \tilde{z} e^{i\tilde{\psi}} \\
\tilde{z}' &= \left( \tilde{x}' + i\tilde{y}' \right) e^{i\tilde{\psi}} \\
\tilde{z} &= \tilde{x} + i\tilde{y} \\
\tilde{z}' &= \tilde{x}' + i\tilde{y}'
\end{align*}
\]

\[\tilde{z} = \tilde{x} + i\tilde{y} \quad \tilde{z}' = \tilde{x}' + i\tilde{y}' \quad \tilde{\psi}' = -k_L
\]

Apply to:

- Project initial conditions from lab-frame when integrating equations
- Project integrated solution back to lab-frame to interpret solution

If the initial condition \( s = s_i \) is taken outside of the magnetic field where \( B_{z0}(s_i) = 0 \), then:

\[
\begin{align*}
\tilde{x}(s = s_i) &= x(s = s_i) \\
\tilde{y}(s = s_i) &= y(s = s_i) \\
\tilde{z}(s = s_i) &= \tilde{z}(s = s_i) \\
\tilde{z}'(s = s_i) &= \tilde{z}'(s = s_i)
\end{align*}
\]

The transform and inverse transform between the laboratory and rotating frames can then be applied to project initial conditions into the rotating frame for integration and then the rotating frame solution back into the laboratory frame.

Using the real and imaginary parts of the complex-valued transformations:

\[
\begin{bmatrix}
    x \\
    y \\
    z \\
\end{bmatrix} = M_r(s|s_i) \cdot \begin{bmatrix}
    \tilde{x} \\
    \tilde{y} \\
    \tilde{z} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x' \\
    y' \\
    z' \\
\end{bmatrix} = M_r^{-1}(s|s_i) \cdot \begin{bmatrix}
    \tilde{x}' \\
    \tilde{y}' \\
    \tilde{z}' \\
\end{bmatrix}
\]

Here we used:

\[\tilde{\psi}' = -k_L \quad M_r^{-1} = \text{Inverse}[M_r]
\]
Appendix C: Transfer Matrices for Hard-Edge Solenoidal Focusing

Using results and notation from Appendix B, derive transfer matrix for single particle orbit with:
- No space-charge
- No momentum spread
- No acceleration

First, the solution to the Larmor-frame equations of motion:
\[ \dot{x}'' + \left(\frac{\gamma \beta_b}{\gamma_k \beta_k}\right) \dot{y}' + \kappa(s) \dot{x} = 0 \]
\[ \dot{y}'' + \left(\frac{\gamma \beta_b}{\gamma_k \beta_k}\right) \dot{y}' + \kappa(s) \dot{y} = 0 \]

\[ \kappa = k_L^2 = \left(\frac{B_{\text{So}}}{2|B|}\right)^2 \]

Can be expressed as:
\[
\begin{bmatrix}
\frac{\ddot{x}}{x'} \\
\frac{\ddot{y}}{y'}
\end{bmatrix}
= \hat{M}_L(z|z_i) \cdot
\begin{bmatrix}
\frac{\dot{x}}{x'} \\
\frac{\dot{y}}{y'}
\end{bmatrix}
\]

- In this appendix we use \( z \) rather than \( s \) for the axial coordinate since there are not usually bends in a solenoid

Transforming the solution back to the laboratory frame:
\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
= M_r(z|z_i) \cdot \hat{M}_L(z|z_i) \cdot
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

\[ z = z_i \]

- Here we assume the initial condition is outside the magnetic field so that there is no adjustment to the Larmor frame angles, i.e., \( \hat{M}_r^{-1}(z|z_i) = I \)

For project of initial conditions to Larmor Frame
\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
= M(z|z_i) \cdot
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

\[ z = z_i \]

\[
M(z|z_i) = \hat{M}_r(z|z_i) \cdot \hat{M}_L(z|z_i)
\]

- Care must be taken when applying to discontinuous (hard-edge) field models of solenoids to correctly calculate transfer matrices
- Fringe field influences beam “spin-up” and “spin-down” entering and exiting the magnet

Apply formulation to a hard-edge solenoid with no acceleration \( \gamma \beta_b = 0 \):
\[
B_{\text{So}}(z) = \frac{B_z}{1 - \frac{1}{\gamma_k \beta_k}} \left[ \Theta(z) - \Theta(z - \ell) \right]
\]
\[ B_z = \text{const} = \text{Hard-Edge Field} \]
\[ \ell = \text{const} = \text{Hard-Edge Magnet Length} \]

Note coordinate choice: \( z=0 \) is start of magnet

Calculate the Larmor-frame transfer matrix in \( 0 \leq z \leq \ell \):
\[ \dot{x}'' + k_L^2 \dot{x} = 0 \]
\[ \dot{y}'' + k_L^2 \dot{y} = 0 \]

\[ k_L = \frac{qB_{\text{So}}}{2\gamma_k \beta_k m c} = \frac{B_{\text{So}}}{2|B|} = \text{const} \]

Subtle Point:
Larmor frame transfer matrix is valid both sides of discontinuity in focusing entering and exiting solenoid.

The Larmor-frame transfer matrix can be decomposed as:
- Useful for later constructs

\[
\hat{M}_L(z|0^-) = \begin{bmatrix}
C & S/k_L & 0 \\
-k_L S & C & 0 \\
0 & 0 & C
\end{bmatrix}
\]

\[
\hat{M}_L(z|0^+) = \begin{bmatrix}
C & 0 & 0 \\
0 & S/k_L & 0 \\
-k_L S & C & 0
\end{bmatrix}
\]

\[
\hat{P}(z) = \begin{bmatrix}
C(z) & S(z)/k_L \\
-k_L S(z) & C(z)
\end{bmatrix}
\]

with
\[
0 \equiv \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Using results from Appendix E, \( F \) can be further decomposed as:
\[
\hat{F}(z) = \begin{bmatrix}
C(z) & S(z)/k_L \\
-k_L S(z) & C(z)
\end{bmatrix}
\]

= \[ 1 \frac{k_L}{k_e} \tan \left( \frac{k_L s}{2} \right) \] \cdot \[ 1 \frac{1}{k_e} \tan \left( \frac{k_L s}{2} \right) \]

= \[ M_{\text{init}}(z) \cdot M_{\text{thm-lens}}(z) \cdot M_{\text{init}}(z) \]

C1

C2

C3

C4
Applying these results and the formulation of Appendix B, we obtain the rotation matrix within the magnet, $0 < z < \ell$:

- Here we apply $\mathbf{M}_L$ formula with $\tilde{\psi} = - k_L z$ for the hard-edge solenoid

$$
\mathbf{M}_r(z|0^-) = \begin{bmatrix}
C & 0 & S & 0 \\
-k_L S & C & k_L C & S \\
-S & 0 & C & 0 \\
-k_L C & -S & -k_L S & C
\end{bmatrix}
$$

Comment: Careful with minus signs!

Here, C and S here have positive arguments as defined.

With special magnet end-forms:

- Here we exploit continuity of $\mathbf{M}_L$ in Larmor frame

\[ \text{Entering solenoid} \]

$$
\mathbf{M}_r(0^+|0^-) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & k_L & 0 \\
0 & 0 & 1 & 0 \\
-k_L & 0 & 0 & 1
\end{bmatrix}
$$

\[ \text{Exiting solenoid} \]

$$
\mathbf{M}_r(\ell^+|\ell^-) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -k_L & 0 \\
0 & 0 & 1 & 0 \\
k_L & 0 & 0 & 1
\end{bmatrix}
$$

The rotation matrix through the full solenoid is (plug in to previous formula for $\mathbf{M}_L(z|0^-)$):

$$
\mathbf{M}_r(\ell^+|0^-) = \begin{bmatrix}
\cos \Phi & 0 & \sin \Phi & 0 \\
-\sin \Phi & 0 & \cos \Phi & 0 \\
0 & -\sin \Phi & 0 & \cos \Phi \\
-\sin \Phi & 0 & -\cos \Phi & 0
\end{bmatrix} = \begin{bmatrix}
I & \cos \Phi & I & \sin \Phi \\
0 & -\sin \Phi & 0 & I & \cos \Phi \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$\Phi \equiv k_L \ell$

and the rotation matrix within the solenoid is (plug into formula for $\mathbf{M}_L(z|0^-)$ and apply algebra to resolve sub-forms):

The lab-frame advance matrices are then (after expanding matrix products):

\[ \text{Inside Solenoid} \]

$$
M(z|0^-) = M_r(z|0^-)M_L(z|0^-)
$$

$$
= \begin{bmatrix}
\cos^2 \phi & \frac{1}{2k_L} \sin(2\phi) & \frac{1}{2} \sin(2\phi) & \frac{1}{2} \sin^2 \phi \\
-k_L \sin(2\phi) & \cos(2\phi) & k_L \cos(2\phi) & \sin(2\phi) \\
-\frac{1}{2} \sin(2\phi) & -k_L \sin^2 \phi & cos^2 \phi & \frac{1}{2} \sin(2\phi) \\
-k_L \cos(2\phi) & -\sin(2\phi) & -k_L \sin(2\phi) & \cos(2\phi)
\end{bmatrix}
$$

$\phi \equiv k_L z$

$$
= \begin{bmatrix}
C(z) & I & S(z)I \\
-S(z)I & C(z) & \begin{bmatrix} I & K \\ -K & I \end{bmatrix} & \begin{bmatrix} F(z) \\ 0 \end{bmatrix}
\end{bmatrix}
$$

\[ \text{Through entire Solenoid} \]

$$
M(\ell^+|0^-) = M_r(\ell^+|0^-)M_L(\ell^+|0^-)
$$

$$
= \begin{bmatrix}
\cos \Phi & 0 & \sin \Phi & 0 \\
-\sin \Phi & 0 & -\cos \Phi & 0 \\
0 & -\sin \Phi & 0 & \cos \Phi \\
-\sin \Phi & 0 & -\cos \Phi & 0
\end{bmatrix} = \begin{bmatrix}
I & \cos \Phi & I & \sin \Phi \\
0 & -\sin \Phi & 0 & I & \cos \Phi \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$\Phi \equiv k_L \ell$

$$
= \begin{bmatrix}
\cos \Phi I & \sin \Phi I \\
-\sin \Phi I & \cos \Phi I
\end{bmatrix} \begin{bmatrix} F(\ell) \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix}
\cos \Phi F(\ell) & \sin \Phi F(\ell) \\
-\sin \Phi F(\ell) & \cos \Phi F(\ell)
\end{bmatrix}
$$

\[ \text{2nd forms useful to see structure of transfer matrix} \]

Note that due to discontinuous fringe field:

$$
M(0^+|0^-) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & k_L & 0 \\
0 & 0 & 1 & 0 \\
-k_L & 0 & 0 & 1
\end{bmatrix} \neq I
$$

Fringe going in kicks angles of beam
M(\ell^-|0^-) \neq M(\ell^+|0^-) \quad \text{Due to fringe exiting kicking angles of beam}

In more realistic model with a continuously varying fringe to zero, all transfer matrix components will vary continuously across boundaries
- Still important to get this right in idealized designs often taken as a first step!

Focusing kicks on particles entering/exiting the solenoid can be calculated as:

Entering:
\[ x(0^+) = x(0^-) \quad x'(0^+) = x'(0^-) + k_L y(0^-) \]
\[ y(0^+) = y(0^-) \quad y'(0^+) = y'(0^-) - k_L x(0^-) \]

Exiting:
\[ x(\ell^-) = x(\ell^+) \quad x'(\ell^-) = x'(\ell^+) - k_L y(\ell^-) \]
\[ y(\ell^-) = y(\ell^+) \quad y'(\ell^-) = y'(\ell^+) + k_L x(\ell^-) \]

- Beam spins up/down on entering/exiting the (abrupt) magnetic fringe field
- Sense of rotation changes with entry/exit of hard-edge field.

The transfer matrix for a hard-edge solenoid can be resolved into thin-lens kicks entering and exiting the optic and an rotation in the central region of the optic as:

\[
M(\ell^+|0^-) = \tilde{M}_L(\ell^+|0^-)M_L(\ell^+|0^-)
\]
\[
= \begin{bmatrix}
\cos^2 \Phi & \frac{1}{2} \sin(2\Phi) & \frac{1}{2} \sin(2\Phi) & \frac{1}{2} \sin^2 \Phi \\
\frac{1}{2} \sin(2\Phi) & \cos^2 \Phi & -k_L \sin^2 \Phi & \frac{1}{2} \sin(2\Phi) \\
-k_L \sin^2 \Phi & -\frac{1}{2} \sin(2\Phi) & \cos^2 \Phi & \frac{1}{2} \sin(2\Phi) \\
\frac{1}{2} \sin(2\Phi) & -\frac{1}{2} \sin(2\Phi) & \frac{1}{2} \sin^2 \Phi & \cos^2 \Phi
\end{bmatrix}
\]

where \( \Phi = k_L \ell \)

- Focusing effect effectively from thin lens kicks at entrance/exit of solenoid as particle traverses the (abrupt here) fringe field

The transfer matrix for the hard-edge solenoid is exact within the context of linear optics. However, real solenoid magnets have an axial fringe field. An obvious need is how to best set the hard-edge parameters \( B_z, \ell \) from the real fringe field.

Simple physical motivated prescription by requiring:

1) Equivalent Linear Focus Impulse \( \propto \int dz \ k_L^2 \propto \int dz B_z^2 \)
\[ \int_{-\infty}^{\infty} dz \ B_{z0}^2(z) = \ell \widetilde{B}_z^2 \]

2) Equivalent Net Larmor Rotation Angle \( \propto \int dz \ k_L \propto \int dz B_z \)
\[ \int_{-\infty}^{\infty} dz \ B_{z0}(z) = \ell \widetilde{B}_z \]
Appendix D: Axisymmetric Applied Magnetic or Electric Field Expansion

Static, rationally symmetric static applied fields $E^a$, $B^a$ satisfy the vacuum Maxwell equations in the beam aperture:
\[
\nabla \cdot E^a = 0 \quad \nabla \times E^a = 0 \quad \nabla \cdot B^a = 0 \quad \nabla \times B^a = 0
\]
This implies we can take for some electric potential $\phi^e$ and magnetic potential $\phi^m$:
\[
E^a = -\nabla \phi^e \quad B^a = -\nabla \phi^m
\]
which in the vacuum aperture satisfies the Laplace equations:
\[
\nabla^2 \phi^e = 0 \quad \nabla^2 \phi^m = 0
\]
We will analyze the magnetic case and the electric case is analogous. In axisymmetric ($\partial / \partial \theta = 0$) geometry we express Laplace's equation as:
\[
\nabla^2 \phi^m(r, z) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi^m}{\partial r} \right) + \frac{\partial^2 \phi^m}{\partial z^2} = 0
\]
\[\phi^m(r, z)\] can be expanded as (odd terms in $r$ would imply nonzero $B_z = -\frac{\partial \phi^m}{\partial r}$ at $r = 0$):
\[
\phi^m(r, z) = \sum_{\nu=0}^{\infty} f_{2\nu}(z) r^{2\nu} = f_0 + f_2 r^2 + f_4 r^4 + \ldots
\]
where $f_0 = \phi^m(r = 0, z)$ is the on-axis potential

Plugging $\phi^m$ into Laplace's equation yields the recursion relation for $f_{2\nu}$
\[
(2\nu + 2)^2 f_{2\nu+2} + f_{2\nu}^m = 0
\]
Iteration then shows that
\[
\phi^m(r, z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} f(0, z)}{\partial z^{2\nu}} \left( \frac{r}{2} \right)^{2\nu}
\]
Using $B_z^a(r = 0, z) \equiv B_{z0}(z) = -\frac{\partial \phi^m(0, z)}{\partial z}$ and differentiating yields:
\[
B_r^a(r, z) = -\frac{\partial \phi^m}{\partial z} = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu-1} B_{z0}(z)}{\partial z^{2\nu-1}} \left( \frac{r}{2} \right)^{2\nu-1}
\]
\[
B_z^a(r, z) = -\frac{\partial \phi^m}{\partial z} = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_{z0}(z)}{\partial z^{2\nu}} \left( \frac{r}{2} \right)^{2\nu}
\]

- Electric case immediately analogous and can arise in electrostatic Einzel lens focusing systems often employed near injectors
- Electric case can also be applied to RF and induction gap structures in the quasistatic (long RF wavelength relative to gap) limit.

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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