USPAS Accelerator Physics
Problem Set 2 - 95 pts.

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Problem 1

P005a Magnetic Optics 15 pts.

From the Lorentz force equation, show that a static magnetic field $\vec{B}\,(\vec{x})$ cannot change the particle kinetic energy, $\mathcal{E} = (\gamma - 1)\, mc^2$. Make no approximations.

$$
\begin{aligned}
m\frac{d}{dt}\left(\gamma\vec{\beta}\right) &= q\vec{\beta}\times\vec{B} \\
\gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}}; &\qquad \vec{\beta} = \frac{1}{c}\frac{d\vec{x}}{dt}
\end{aligned}
$$

Problem 2

P009 Thin lens transfer matrix for a single particle 30 pts.

Consider the thin-lens focusing function $\kappa_x = \frac{1}{f}\delta(s - s_0)$ and the equation of motion:

$$
\begin{aligned}
x'' + \frac{1}{f}\delta(s-s_0)x &= 0 \\
f &= \text{constant focal length} \\
s_0 &= \text{axial location of the optic}
\end{aligned}
$$

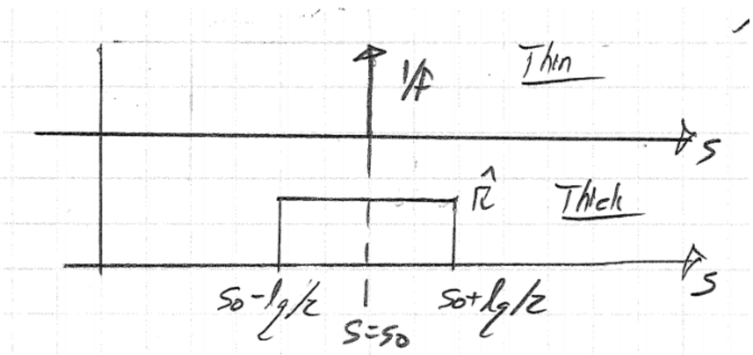
a) 10 pts: Derive the 2×2 transfer matrix $\mathbf{M}$ for the optic:

$$
\begin{bmatrix} x \\ x' \end{bmatrix}_{s_0^+} = \mathbf{M}\cdot\begin{bmatrix} x \\ x' \end{bmatrix}_{s_0^-}
$$

where $s_0^{\pm}$ are the coordinates infinitesimally to the left and right of the optic at $s = s_0$. $\mathbf{M}$ is the thin-lens transfer matrix.

b) 10 pts: Construct a thin lens limit, $\lim_{\ell \to 0}$, for 2×2 thick lens transfer matrices for focusing and defocusing quadrupoles with $\kappa = \hat{\kappa}_q = \text{const}$.

$$\mathbf{M}_{\text{thick focus}} = \begin{bmatrix} \cos(\sqrt{\kappa}\ell) & \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\ell) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}\ell) & \cos(\sqrt{\kappa}\ell) \end{bmatrix}$$
$$\mathbf{M}_{\text{thick defocus}} = \begin{bmatrix} \cosh(\sqrt{\kappa}\ell) & \frac{1}{\sqrt{\kappa}}\sinh(\sqrt{\kappa}\ell) \\ \sqrt{\kappa}\sinh(\sqrt{\kappa}\ell) & \cosh(\sqrt{\kappa}\ell) \end{bmatrix}$$



Hint: Require the same "impulse" $\int \kappa_x(s)\, ds$ be applied to a particle going through the thick and thin lens to relate κl and f. Use this constraint when taking the limit keeping $\kappa\ell$ finite.

c) 10 pts: A 2×2 transfer matrix

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

gives the solution to Hills' equation $x'' + \kappa(s)x = 0$ through some advance. Using that the Wronskian symmetry

$$\det \mathbf{M} = M_{11}M_{22} - M_{21}M_{12} = 1$$

always holds for any physical solution, show that $\mathbf{M}$ can always be replaced by two drifts and a thin lens kick as

$$\mathbf{M} = \mathbf{M}_{\text{drift2}} \cdot \mathbf{M}_{\text{thin lens}} \cdot \mathbf{M}_{\text{drift1}}$$
$$\mathbf{M}_{\text{drift}} = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \qquad \mathbf{M}_{\text{thin lens}} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Find d_1, d_2, and f in terms of M_{11}, M_{22}, M_{12}, and M_{21} for the equivalence to hold.

In this equivalence will the axial lengths of the physical and thin lens systems be the same? Does it matter if the axial lengths are unequal?

Problem 3

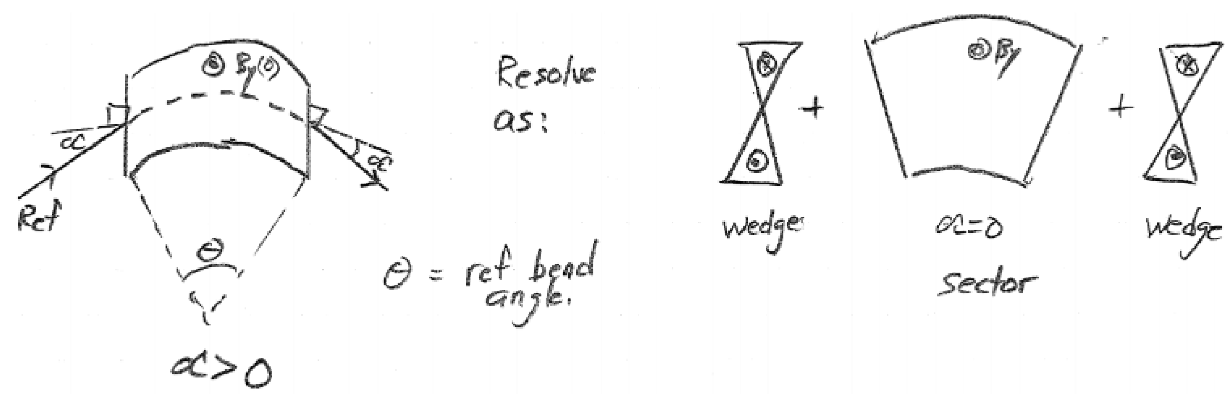
P030 Accelerator Bending Parameters 10 pts.

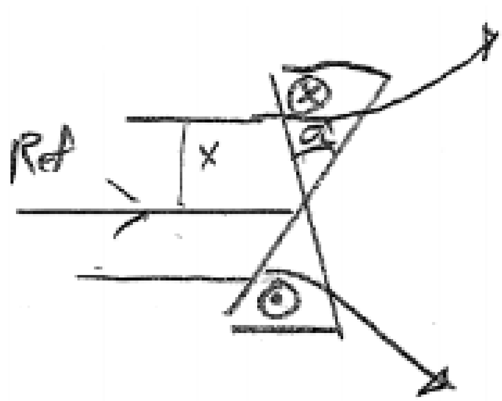
Fill in the following table for different accelerators.

	NSLS II	LHC	FRIB
Type	Ring	Ring	Folding segment (π rotation)
Energy	3 GeV	7 TeV	$E_k = 150$ MeV/u
Species	Electron	Proton	${}^{238}_{78+}U$
Number of dipoles	60	1200	4
Bending Field [T]	0.4	8.3	
Dipole Length [m]			9

Problem 4

P047 Dipole Edge Corrections 30 pts.





a) 10 pts: Horizontal correction:

A dipole with entry and exit angles α (defined relative to reference orbit, see figure above) can be divided into three sections: a sector dipole in the center with a wedge magnet on both ends. The sector dipole has the transfer matrix derived in class.

For a short sector, argue from the Lorentz force equation that the particle experiences the following corrections for the orbit displacement x through the wedge:

$$\Delta x' = \frac{B_y(0)}{[B\rho]} \tan(\alpha) x$$

$$\Delta x = 0$$

to show that:

$$\mathbf{M}_{\text{wedge}} = \begin{bmatrix} 1 & 0 \\ \frac{\tan(\alpha)}{\rho} & 1 \end{bmatrix}; \quad \frac{1}{\rho} = \frac{B_y(0)}{[B\rho]}$$

This "kick" correction will be applied entering and exiting the magnet.

b) 3 pts: Show that the x-transfer matrix of the full dipole can be expressed as:

$$\mathbf{M}_x = \mathbf{M}_{\text{wedge}} \cdot \mathbf{M}_{\text{sector}} \cdot \mathbf{M}_{\text{wedge}} = \begin{bmatrix} \frac{\cos(\theta-\alpha)}{\cos\alpha} & \rho\sin(\theta) \\ -\frac{\sin(\theta-2\alpha)}{\rho\cos^2(\alpha)} & \frac{\cos(\theta-\alpha)}{\cos\alpha} \end{bmatrix}$$

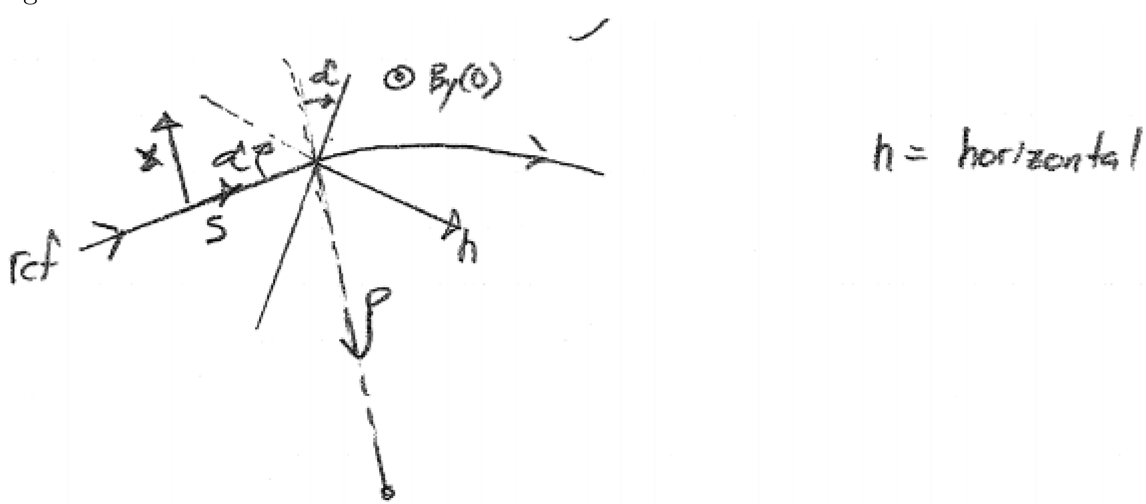
Hint:

$$\cos(\theta) + \tan(\alpha)\sin(\theta) = \frac{\cos(\theta-\alpha)}{\cos(\alpha)}$$

$$\sin(\theta) - \tan(\alpha)\cos(\theta) = \frac{\sin(\theta-\alpha)}{\cos(\alpha)}$$

c) 10 pts: Vertical Correction

There is also a vertical correction to the kick due to the slanted edge entry which results from fringe fields.



Using arguments analogous to part a), the y-plane angular deflection entering the magnet is:

$$\Delta y' = \frac{qv\int B_x(y)\,\mathrm{d}s}{\gamma m v^2} = \frac{1}{[B\rho]}\int_{\text{edge}} B_x(y)\,\mathrm{d}s$$

Argue that:

$$\Delta y' = -\frac{\tan(\alpha)}{[B\rho]}\int_{\text{edge}} \mathbf{B}\cdot\mathrm{d}\mathbf{s}$$

$$\int_{\text{edge}} \mathbf{B}\cdot\mathrm{d}\mathbf{s} = yB_y(0)$$

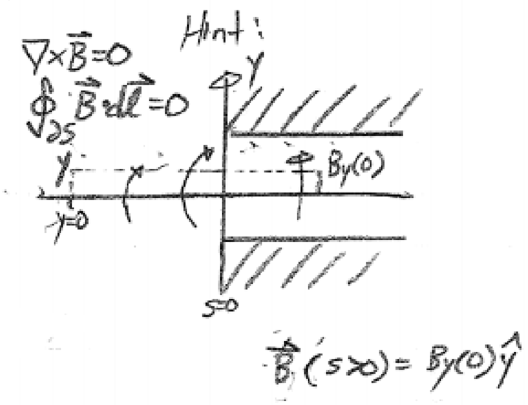
to obtain:

$$\Delta y' = -\frac{\tan(\alpha)}{\rho} y, \quad \Delta y = 0$$

$$\mathbf{M}_y|_{\text{edge}} = \begin{bmatrix} 1 & 0 \\ -\frac{\tan(\alpha)}{\rho} & 1 \end{bmatrix}$$

for the edge correction going into the dipole. The same correction applies going out of the dipole.

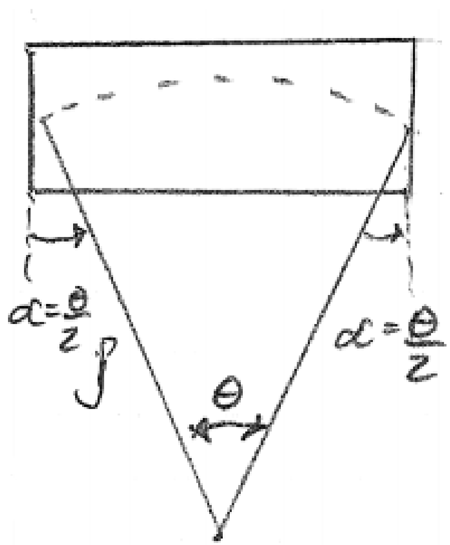
Hint: Consider the vacuum Maxwell equations and the apply to the fringe field entering the magnet as suggested in the sketch below.



d) 3 pts: Show that:

$$\mathbf{M}_y = \mathbf{M}_y|_{\text{edge}} \cdot \mathbf{M}_y|_{\text{sector}} \cdot \mathbf{M}_y|_{\text{edge}} = \begin{bmatrix} 1 - \theta \tan\alpha & \rho\theta \\ -\frac{\tan\alpha}{\rho}(2 - \theta\tan\alpha) & 1 - \theta\tan\alpha \end{bmatrix}$$

e) 4 pts: A box dipole is commonly constructed for easy fabrication.



Use results from parts b) and d) to show for a box dipole that:

$$\mathbf{M}_x = \begin{bmatrix} 1 & \rho\sin(\theta) \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_y = \begin{bmatrix} 1-\theta\tan(\theta/2) & \rho\theta \\ -\frac{1}{\rho}\tan(\theta/2)[2-\theta\tan(\theta/2)] & 1-\theta\tan(\theta/2) \end{bmatrix} \approx \left.\begin{bmatrix} \cos(\theta) & \rho\sin(\theta) \\ -\frac{\sin(\theta)}{\rho} & \cos(\theta) \end{bmatrix}\right|_{\theta\ll 1}$$

Problem 5

P031 Symmetric Lattice 10 pts.

Let $\mathbf{M}$ represent the 1-D linear betatron transfer map of a section of a lattice. Given that the lattice has mirror symmetry with respect to its mid-point, find the required conditions that this symmetry imposes on $\mathbf{M}$.