USPAS Accelerator Physics
Problem Set 4 - 90 pts.

S. M. Lund, Y. Hao
Graders: C. Richard and C. Y. Wong

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Problem 1

P016 Dispersion Function 50 pts.

The dispersion function in a periodic ring satisfies:

$$
\begin{aligned}
D''(s) + \kappa(s) D(s) &= \frac{1}{\rho(s)} \\
\rho(s) &= \text{bend radius}, \ \kappa(s) = \text{focusing function} \\
D(s + L_p) &= D(s), \ \rho(s + L_p) = \rho(s), \ \kappa(s + L_p) = \kappa(s) \\
L_p &= \text{lattice period}
\end{aligned}
$$

a) 5 pts: Argue the solution for D is unique. This implies that there is a unique closed orbit $x = \delta \cdot D$ for every value of off-momentum δ. This aids interpretation of D.

Hint: Let D_1 and D_2 be two independent solutions and look for a contradiction.

b) 5 pts: Argue that the solution for D can be expressed in an extended 3×3 transfer matrix from as:

$$
\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_s = \begin{bmatrix} \mathbf{M}_{11}(s|s_i) & \mathbf{M}_{12}(s|s_i) & d(s|s_i) \\ \mathbf{M}_{21}(s|s_i) & \mathbf{M}_{22}(s|s_i) & d'(s|s_i) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s=s_i}
$$

where $\mathbf{M}(s|s_i)$ is the usual 2×2 transfer matrix from Hill's equation.

Express the periodicity requirement $D(s + L_p) = D(s)$ in this 3×3 formulation. Do you expect this equaiton to have a solution? Explain your answer.

c) 15 pts: Show for ρ =constant and:

$$\kappa = \text{const} > 0:$$
$$d(s|s_i) = \frac{1}{\rho\kappa}[1 - \cos(\sqrt{\kappa}(s - s_i))]$$
$$d'(s|s_i) = \frac{1}{\rho\sqrt{\kappa}}\sin[\sqrt{\kappa}(s - s_i)]$$
$$\kappa = \text{const} < 0:$$
$$d(s|s_i) = \frac{1}{\rho|\kappa|}[-1 + \cosh(\sqrt{|\kappa|}(s - s_i))]$$
$$d'(s|s_i) = \frac{1}{\rho\sqrt{|\kappa|}}\sinh[\sqrt{\kappa}(s - s_i)]$$

Use the Green's functions results from class and forms derived for **M**.

d) 5 pts: Use $\kappa = 1/\rho^2$ in part (c) to show for a sector dipole that the 3×3 transfer matrix through a bend of length ℓ can be expressed as:

$$\mathbf{M} = \begin{bmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ \frac{-\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_s = \mathbf{M} \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s=s_i}$$
$$\ell = \rho\theta, \qquad \theta = \text{bend angle}$$

Show for a small angle bend ($\theta \ll 1$) that:

$$\mathbf{M} = \begin{bmatrix} 1 & \ell & \frac{\ell\theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{bmatrix}$$

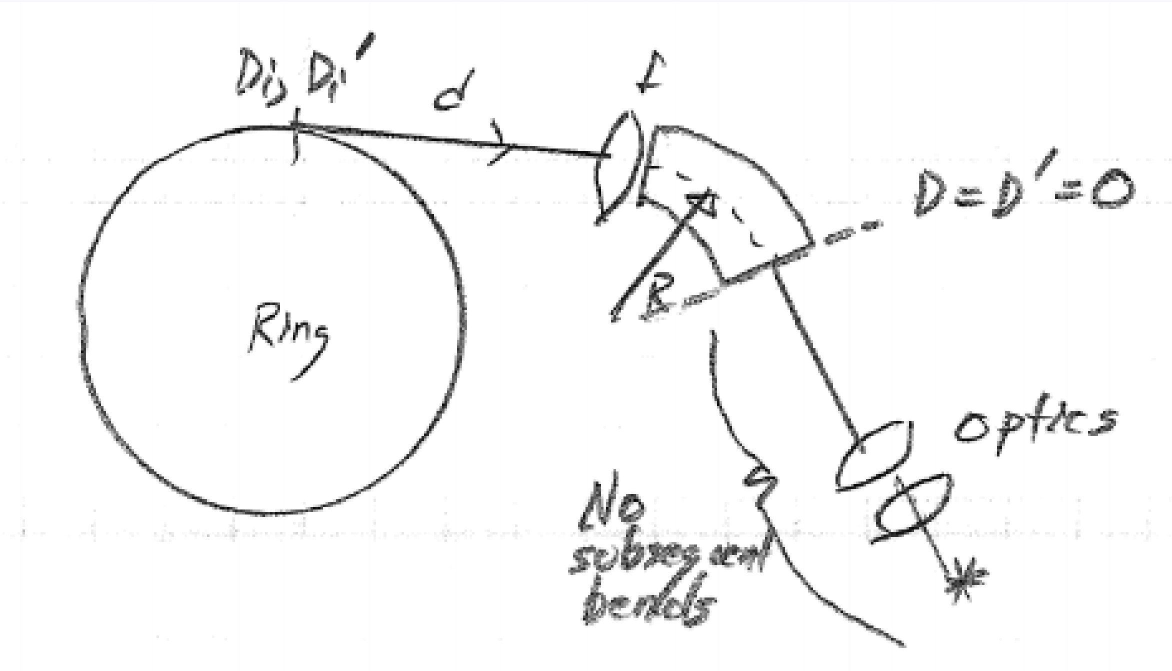
e) 10 pts: Derive the 3×3 transfer matrix for D for:

1) A drift with $\kappa = 0$ and $\rho \to \infty$.
2) A thin lens at $s = s_i$ with $\kappa = \frac{\delta(s-s_i)}{f}$ where f = constant and $\rho \to \infty$. Here, $\delta(x)$ is a Dirac-delta function.
3) Within a uniform sector bend with large bend radius ρ where we take $\kappa \approx 0$ and ρ = constant.

First use direct methods as opposed to Green's functions.

Then show that the results agree with the Green's functions for 1) and 2) and for the small angle bend result derived from the Green's functions for 3).

f) 10 pts: A particle is kicked out of a ring with dispersion $D = D_i$ and $D' = D'_i$ just after the kick. The particle is then transported through an extraction line with a drift length d, a thin lens focusing kick with focal length f, and then a sector bend of radius $\rho = R$ and length ℓ, and finally though an uspecified series of optics to the target.



Using the results from part e), derive constraints on the lattice parameters d, f, R, and ℓ that can be enforced to ensure that $D = D' = 0$ after the bending magnet to have zero dispersion in the straight transport and focusing line to the target?

Are these constraints practical to implement in the lab? Why? Qualitative answer only.

Problem 2

P017 Chromatic Effects in Solenoids 15 pts.

a) 10 pts: Use the Larmor frame equation of motion to derive an equation of motion for chromatic effects in a solenoid transport line. Follow the procedure in class to show that:

$$\tilde{\eta}'' + \kappa_0 \tilde{\eta} = 2\delta\kappa_0 \tilde{x}_0$$
$$\tilde{x}_0 = \text{Larmor frame unperturbed orbit for } \delta = 0$$
$$\kappa_0 = \left(\frac{B_{z0}(s)}{2[B\rho]_0}\right)$$
$$[B\rho]_0 = \text{Design rigidity}$$
$$\tilde{x} = \tilde{x_0} + \tilde{\eta} = \text{Larmor frame orbit}$$

b) 5 pts: Are chromatic effects expected to be larger or smaller for solenoid focusing relative to magnetic quadrupole focusing when comparing 'equivalent' focusing strengths? Why?

Problem 3

P015 Dispersion Function Estimate 25 pts.

Estimate the average value of the dispersion fuction, D, in a ring with:

$$\sigma_{0,x} = 85 \text{ deg/period}$$
$$C = 100 \text{ m circumference}$$
$$L_p = 5 \text{ m (20 periods)}$$

a) 5 pts: Use the continuous ring model results from class to estimate the average value of the periodic dispersion function D of the ring.

b) 5 pts: In a periodic focusing lattice ring with bends and focusing optics only filling part of the lattice period, how do you expect the continuous model estimate in a) will compare to the solution for D?

Let us design a ring of circumference 100m using FODO cells. You may start with the following example parameters:

Table 1: Example of lattice design

Parameter	Value
Dipole Length	1.7 m
Quad length	0.4 m
Drift between D and Q	0.2 m
Number of FODO cells	20
Quad K1 (F/D)	1.4/-1.0 m^{-2}

c) 5 pts: Use Elegant to find the axial positions (locations within lattice period) and values of the maximum/minimum beta function and dispersion function and the phase advance in both the x- and y-directions of the FODO cell using the example parameters. The x-direction is the bend plane.

d) 5 pts: Adjust the phase advance of horizontal direction to 85 degrees per lattice period. Keep the same total circumfrance of the ring fixed. Give the the value of your lattice focusing functions. Your answers only need to be approximate. So you can iterate by “guessing”.

e) 5 pts: Design a ring with same circumference using only 10 FODO cells. Specify the parameters of your design.