USPAS Accelerator Physics
Problem Set 5 - 120 pts.

S. M. Lund, Y. Hao
Graders: C. Richard and C. Y. Wong

June 8, 2018

Problem 1

P013 Resonances 40 pts.

Consider the driven harmonic oscillator equation for $U(\varphi)$:

$$\frac{\mathrm{d}^2U(\varphi)}{\mathrm{d}\varphi^2} + \nu_0^2 U(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi)$$

$$\begin{aligned}
\nu_0 &= \text{constant restoring frequency} \\
\nu &= \text{constant driving frequency} \\
A,\ B &= \text{constant amplitudes} \\
A\cos(\nu\varphi) + B\sin(\nu\varphi) &= \text{driving terms}
\end{aligned}$$

The general solution for $U(\varphi)$ can be expanded as:

$$U(\varphi) = U_h(\varphi) + U_p(\varphi)$$

where $U_h(\varphi)$ is the general solution to the homogeneous equation:

$$\frac{\mathrm{d}^2U_h(\varphi)}{\mathrm{d}\varphi^2} + \nu_0^2 U_h(\varphi) = 0$$

$$\Longrightarrow U_h(\varphi) = C_1\cos(\nu_0\varphi) + C_2\sin(\nu_0\varphi)$$

$$C_1,\ C_2 \text{ constants}$$

and $U_p(\varphi)$ is the particular solution to:

$$\frac{\mathrm{d}^2U(\varphi)}{\mathrm{d}\varphi^2} + \nu_0^2 U(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi)$$

a) 5 pts: For $\nu \neq \nu_0$, show that a solution $U_p(\varphi)$ exists proportional to the driving term and find the constant of proportionality.

b) 5 pts: Use the results of part (a) to construct the solution ($\nu \neq \nu_0$) for $U(\varphi)$ satisfying the initial conditions at $\varphi = 0$:

$$U(\varphi = 0) = U_0$$

$$\left.\frac{\mathrm{d}U}{\mathrm{d}\varphi}\right|_{\varphi=0} = \dot{U}_0; \quad \frac{\mathrm{d}U}{\mathrm{d}\varphi} \equiv \dot{U}(\varphi)$$

c) 10 pts: Set $\nu = \nu_0 + \delta\nu$, and find the leading order form of the solution valid for $|\delta\nu|/\nu_0 \ll 1$ and $|\varphi\delta\nu| \ll 1$. What does this limit imply on the amplitude of the particle oscillation as $\nu \to \nu_0$?

d) 5 pts: What do these results imply for a general periodic forcing function:

$$\frac{d^2}{d\varphi^2}U(\varphi) + \nu_0^2 U(\varphi) = f(\varphi)$$
$$f(\varphi) = \text{periodic forcing function with } f(\varphi + 2\pi) = f(\varphi)$$

How does this fit in with the analysis of machine tunes carried out in the class notes?

e) 5 pts: Suppose the drive frequency is exactly equal to the resonant frequency (i.e., $\nu = \nu_0$):

$$\frac{d^2}{d\varphi^2}U(\varphi) + \nu_0^2 U(\varphi) = A\cos(\nu_0\varphi) + B\sin(\nu_0\varphi)$$

Motivated by part c), show that a particular solution exists

$$U_p(\varphi) = \frac{A}{2\nu_0}\varphi\sin(\nu_0\varphi) - \frac{B}{2\nu_0}\varphi\cos(\nu_0\varphi)$$

with no approximations. Write down the general solution. Does this agrees with (c)? Should it?

f) 10 pts: For the case of $\nu \neq \nu_0$, estimate the deviation in $\delta\nu/\nu_0$ to wash out the resonance. Please keep arguments simple.

Hint: Look at the second order deviations in $\delta\nu/\nu_0$.

Problem 2

P014 Resonance Driving Perturbations 15 pts.

In class we derived the perturbed Hill's equationfor transverse magnetic field perturbations:

$$x'' + \kappa_x x = \mathcal{P}_x \qquad \kappa_x = \frac{G}{[B\rho]}$$

where

$$\mathcal{P}_x = \mathcal{P}_x(x, y) = \text{perturbation in x-plane}$$

Use the results from class to explicitly identify $\mathcal{P}_x$ for the following conditions:

a) 5 pts: Normal and skew orientation dipole field perturbations.

b) 5 pts: Normal and skew orientation quadrupole field perturbations. Which of these can be included in κ_x? Which of these results in y-plane coupling?

c) 5 pts: Normal and skew orientation sextupole field perturbations. In either case, is the x-motion independent of y when $y \neq 0$? Do "normal" and "skew" orientations have clear physical distinction for sextupole perturbations? Why or why not?

Caution: You must correctly interpret the index n in the class notes to identify the appropriate multipole field term.

Problem 3

P033 Symplectic Matrices 20 pts.

Prove the following statements for a symplectic matrix, $\mathbf{M}$:

1) If $\mathbf{M}$ is symplectic, then $\mathbf{M}^{-1}$ is symplectic.
2) If $\mathbf{M}$ and another matrix $\mathbf{N}$ are both symplectic, then $\mathbf{M} \cdot \mathbf{N}$ is also symplectic.
3) For the 1-D case, the symplectic condition becomes $\det(\mathbf{M}) = 1$.
4) If λ is an eigenvalue of $\mathbf{M}$, then λ^{-1} is also an eigenvalue of $\mathbf{M}$.

Problem 4

P034 Transfer Map Free Parameters 20 pts.

Consider a 2-D Hamiltonian system with phase-space (x, x', y, y'). From the symplectic condition, find how many free parameters the 4-D transfer map can have. In general, for an N-dimensional Hamiltonian system, how many free parameters are there in the transfer map?

Problem 5

P037 FODO Dispersion Suppression 25 pts.

For a FODO cell with dipole and axially short quadrupole magnets $(Q_f/2, B, Q_d, B, Q_f/2)$, the betatron and dispersion functions at the middle plane of the focusing quadrupole are β_f and d_f from the periodic boundary condition. The bending angle of each dipole is θ and the phase advance of the cell is ψ. Here the bending angle is small so that small angle approximation can be utilized.

1) Find the 3×3 transfer matrix, $\mathbf{M}$ of the cell.
2) To match the cell's dispersion function to zero, a dispersion suppressor needs to be be attached to the end. Show that using n FODO cells with zero bending angle cannot suppress the dispersion.
3) To design a proper dispersion suppressor, we can use two FODO cells with reduced bending angles. The dipoles in the first cell have bending angle θ_1 and the dipoles in the second cell have bending angle θ_2. Find θ_1 and θ_2 to create the dispersion suppressor.