USPAS Accelerator Physics
Problem Set 7 - 115 pts.

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Problem 1

P020 Motion Near Synchronous Particle: Difference Equations 25 pts.

In class we derived the longitudinal difference equations:

$$\begin{aligned}
\Delta\phi_n - \Delta\phi_{n-1} &= -\frac{2\pi N}{\gamma_{s,n-1}^3 \beta_{s,n-1}^2} \frac{\Delta W_{n-1}}{mc^2} \\
\Delta W_n - \Delta W_{n-1} &= qE_{0,n} L_n T_n(\beta_{s,n})[\cos(\phi_{s,n} + \Delta\phi_n) - \cos(\phi_{s,n})] \\
\Delta\phi_n = \phi_n - \phi_{s,n}; &\qquad \Delta W_n = W_n - W_{s,n}
\end{aligned}$$

a) 2 pts: Which term generates the nonlinearity? Why?

b) 8 pts: Following the steps in class, linearize the difference equations for small phase excursions about the synchronous particle and express the result as a 2×2 transfer matrix, $\mathbf{M}_s$ defined by:

$$\begin{bmatrix} \Delta\phi \\ \Delta W \end{bmatrix}_n = \mathbf{M}_s \cdot \begin{bmatrix} \Delta\phi \\ \Delta W \end{bmatrix}_{n-1}$$

Show that $\det\mathbf{M}_s = 1$ and resolve $\mathbf{M}_s$ as the product of a thin lens and a drift as:

$$\mathbf{M}_s = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

and identify the drift length, d, and the inverse focal length, $1/f$. Is it a problem if $d < 0$? Will the system still focus? Why?

c) 10 pts: Assume negligible synchronous particle energy gain and a regular periodic lattice with:

$$\begin{aligned}
&\beta_{s,n} = \beta_s = \text{const}; \implies \gamma_{s,n} = \gamma_s = \text{const} \\
&E_{0,n} = E_0 = \text{const}; \\
&T_n = T = \text{const}; \\
&L_n = L = \text{const} \\
&\phi_{s,n} = \phi_s = \text{const}
\end{aligned}$$

Define a synchronous phase advance using $\mathbf{M}_s$ in part (b) and calculate the synchronous phase advance, σ_s, per cell, L. Compare the result to the synchrotron wavenumber, k_s, calculated in class for small phase advance per cell. Should you expect the relationship obtained? Why?

d) 5 pts: Suppose we apply the linear equations in the limit of small acceleration within the continuous approximation derived in class. For an orbit with max phase extent $\Delta\phi_0$, find an expression for the longitudinal emittance in $\Delta\phi$-ΔW phase-space with:

$$\pi\epsilon_s = \text{Area of ellipse in } \Delta\phi\text{-}\Delta W$$

The units of ϵ_s will be radians-eV (energy). How should we scale this result to measure ϵ_s in Δt-ΔW phase-space to measure area in eV-sec?

Problem 2

P021 Hamiltonian Form of Synchrotron Equations of Motion 30 pts.

In class we showed in the continuous approximation that the longitudinal equations of motion about the synchronous particle are:

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}s} &= -Aw \\
\frac{\mathrm{d}w}{\mathrm{d}s} &= B(\cos\phi - \cos\phi_{s,n}) \\
w &= \frac{\Delta W}{mc^2}; \quad A = \frac{2\pi}{\lambda_{\mathrm{rf}}(\gamma_s\beta_s)^3}; \quad B = \frac{qE_0T}{mc^2}
\end{aligned}$$

assuming that $\gamma_s\beta_s$ varies slowly.

a) 10 pts: Find a Hamiltonian, $H(\phi, p_\phi)$, and conjugate 'momentum' variable, p_ϕ, such that the equations of motion are given by:

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}s} &= \frac{\partial H}{\partial p_\phi} \\
\frac{\mathrm{d}p_\phi}{\mathrm{d}s} &= -\frac{\partial H}{\partial \phi}
\end{aligned}$$

Compare H to H_ϕ constructed in class.

b) 10 pts: Consider a distribution of particles evolving according to H in longitudinal phase-space. Neglect particle-particle interactions (not in formulation). A smooth distribution $f(\phi, p_\phi, s) \geq 0$ must satisfy:

$$\frac{\partial f}{\partial s} + \frac{\partial}{\partial\phi}\left(f\frac{\mathrm{d}\phi}{\mathrm{d}s}\right) + \frac{\partial}{\partial p_\phi}\left(f\frac{\mathrm{d}p_\phi}{\mathrm{d}s}\right) = 0$$

since 'probability' must flow somewhere. Show for non-linear longitudinal dynamics that:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{\text{particle trajectory}} = 0$$

Explain how this implies the total phase-space weight of the particles at a given density is constant in the non-linear evolution. You may want to read about Liouville's Theorem of non-interacting particles in statistical mechanics if you need help.

c) 5 pts: If $\gamma_s\beta_s \neq$ constant, but varies slowly to maintain validity of the continuous formulation, will H be constant? Why?

Keep all other factors T, ϕ_s, E_0, λ_{rf} constant in s.

d) 5 pts: If the phase excursion is small ($\phi = \phi_s + \Delta\phi$ where $\Delta\phi$ is small with $\gamma_s\beta_s$ slowly varying, derive a second order differential equation for the evolution of $\Delta\phi$. Do you expect this equation to have a conserved longitudinal emittance? Why?

For this part, start from the continuous formulation with:

$$(\gamma_s\beta_s)^3 \frac{\mathrm{d}}{\mathrm{d}s}(\phi - \phi_s) = -\frac{2\pi}{\lambda_{\mathrm{rf}}} \frac{\Delta W}{mc^2}$$
$$\frac{\mathrm{d}\Delta W}{\mathrm{d}s} = qE_0 T(\cos\phi - \cos\phi_s)$$

Take λ_{rf}, E_0, T, ϕ_s, to be constants. Write results using:

$$k_s^2 = \frac{2\pi}{\lambda_{\mathrm{rf}}} \frac{qE_0 T \sin(-\phi_s)}{\gamma_s^3\beta_s^3 mc^2}$$

Problem 3

P022 RF Phase Choice 10 pts.

In class, for the continuous model with $qE_0 > 0$, we showed that where:

$$qE_z(r=0, z=0, t=0) = qE_0\cos(\phi_s) > 0 \implies \text{accel}$$
$$qE_z(r=0, z=0, t=0) = qE_0\cos(\phi_s) < 0 \implies \text{deccel}$$

and where

$$V(\phi) = B[\sin(\phi) - \phi\cos(\psi_s)]; \quad B = \frac{qE_0T}{mc^2} > 0$$

has concavity

$$\left.\frac{\mathrm{d}^2V(\phi)}{\mathrm{d}\phi^2}\right|_{\phi=\phi_s} > 0 \implies \text{stability (focusing)}$$
$$\left.\frac{\mathrm{d}^2V(\phi)}{\mathrm{d}\phi^2}\right|_{\phi=\phi_s} < 0 \implies \text{instability (defocusing)}$$

locally about the synchronous particle.
Use these to argue:

a) 2 pts: Range of ϕ_s for deceleration *and* focusing?

b) 2 pts: Range of ϕ_s for acceleration *and* defocusing?

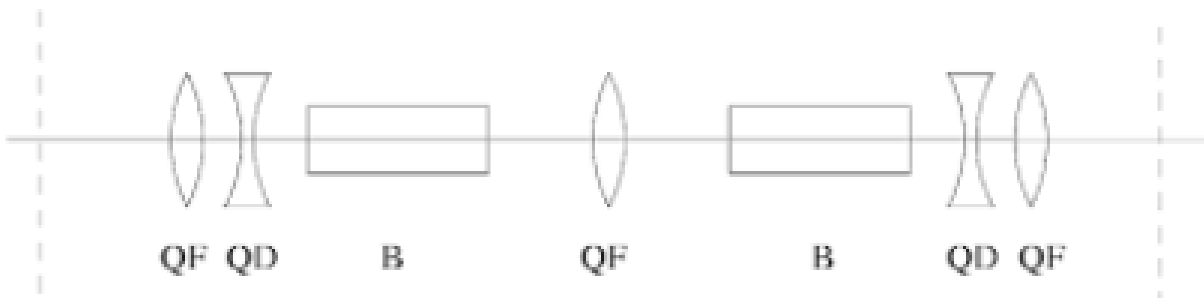
c) 2 pts: Separately from parts a) and b), what value of ϕ_s will provide maximum longitudinal focusing and acceptance using the continuous model? Why? Does this allow acceleration?

d) 4 pts: "Fast Rotation" Consider a bunch with weak or no acceleration in the continuous model filling a small phase-width of the bucket. If E_0 suddenly jumps, argue what will happen to the longitudinal phase-space ellipse. At what propagation distance will the bunch have the shortest phase width? Use the synchrotron wavenumber, k_s, to estimate the distance. What value of ϕ_s should be chosen to minimize the distance?

Problem 4

P042 NSLS II Longitudinal Parameters 50 pts.

NSLS II adopts a double bent achromat, DBA, lattice.



Parameters	Values
Energy [GeV]	3.0
Circumference [m]	780
Number of dipoles	60
Dipole field [T]	0.4
Beam current [A]	0.5
RF frequency [MHz]	499.68
Harmonic number	1320

From the design parameters of NSLS II shown in the table above, calculate the following:

1) Calculate the axial length of the dipoles assuming all dipoles are the same.

2) In the DBA lattice, the dispersion, D, and dispersion slope, D', are zero at one end of the dipoles and non-zero at the other end. Find the dispersion inside the dipole magnet.

3) Calculate the energy loss due to the dipole field.

4) If the synchronous accelerating phase of the RF cavity is $\pi/6$, what is the minimum RF voltage required? How much power is required?

5) The actual RF voltage is about 3 MV. Using this value, find the longitudinal tune of NSLC II.

6) What is the critical radiation frequency of the dipole radiation?

7) Find the partition number, $\overline{D}$, due to the synchrotron radiation in the dipole.

8) Find the longitudinal damping rate, α_E, and compare with the period of longitudinal oscillations.

9) Find the equilibrium energy spread in NSLS II.