Transverse Particle Resonances with Application to Circular Accelerators*

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Acknowledgments

S1: Overview

In our treatment of transverse single particle orbits of lattices with s-varying focusing, we found that Hill's Equation describes the orbits to leading-order approximation:

$$x''(s) + \kappa_x(s)x(s) = 0$$
$$y''(s) + \kappa_y(s)y(s) = 0$$

where $\kappa_x(s)$, $\kappa_y(s)$ are functions that describe linear applied focusing forces Of the lattice

- Focusing functions can also incorporate linear space-charge forces
  - Self-consistent for special case of a KV distribution

In analyzing Hill's equations we employed phase-amplitude methods

- See: S.M. Lund lectures on Transverse Particle Dynamics, S8, on the betatron form of the solution

$$x(s) = A_{xi}\sqrt{\beta_x(s)}\cos\psi_x(s) \qquad A_{xi} = \text{const}$$

$$\frac{1}{2}\beta_x(s)\beta_x''(s) - \frac{1}{4}\beta_x'^2(s) + \kappa_x(s)\beta_x^2(s) = 1 \qquad \psi_x(s) = \psi_{xi} + \int_{s_i}^{s}\frac{d\bar{s}}{\beta_x(\bar{s})}$$

$$\beta_x(s + L_p) = \beta_x(s)$$

This formulation simplified identification of the Courant-Snyder invariant:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_x^2 = \text{const} \qquad w_x = \sqrt{\beta_x}$$

which helped to interpret the dynamics.

We will now exploit this formulation to better (analytically!) understand resonant instabilities in periodic focusing lattices. This is done by choosing coordinates such that *stable* unperturbed orbits described by Hill's equation:

$$x''(s) + \kappa_x(s)x(s) = 0$$

are mapped to a continuous oscillator

$$\tilde{x}''(\tilde{s}) + \tilde{k}_{\beta 0}^2 \tilde{x}(\tilde{s}) = 0$$
$$\tilde{k}_{\beta 0}^2 = \text{const} > 0$$

$\cdot\tilde{\cdot}\cdot = \text{Transformed Coordinate}$

- Because the linear lattice is designed for single particle stability this transformation can be effected for any practical machine operating point

These transforms will help us more simply understand the action of perturbations (from applied field nonlinearities, ...) acting on the particle orbits:

$$x''(s) + \kappa_x(s)x(s) = \mathcal{P}_x(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$
$$y''(s) + \kappa_y(s)y(s) = \mathcal{P}_y(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$
$$\mathcal{P}_x,\ \mathcal{P}_y = \text{Perturbations}$$
$$\vec{\delta} = \text{Extra Coupling Variables}$$

For simplicity, we restrict analysis to:

$\gamma_b\beta_b = \text{const}$ — No Acceleration

$\delta = 0$ — No Axial Momentum Spread

$\phi = 0$ — Neglect Space-Charge

- Acceleration can be incorporated using transformations (see Transverse Particle Dynamics, S10)
- Comments on space-charge effects will be made in S7

We also take the applied focusing lattice to be periodic with:

$$\kappa_x(s + L_p) = \kappa_x(s)$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$
$$L_p = \text{Lattice Period}$$

For a ring we also always have the superperiodicity condition:

$$\mathcal{P}_x(s+\mathcal{C};\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta}) = \mathcal{P}_x(s;\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta})$$

$$\mathcal{P}_y(s+\mathcal{C};\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta}) = \mathcal{P}_y(s;\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta})$$

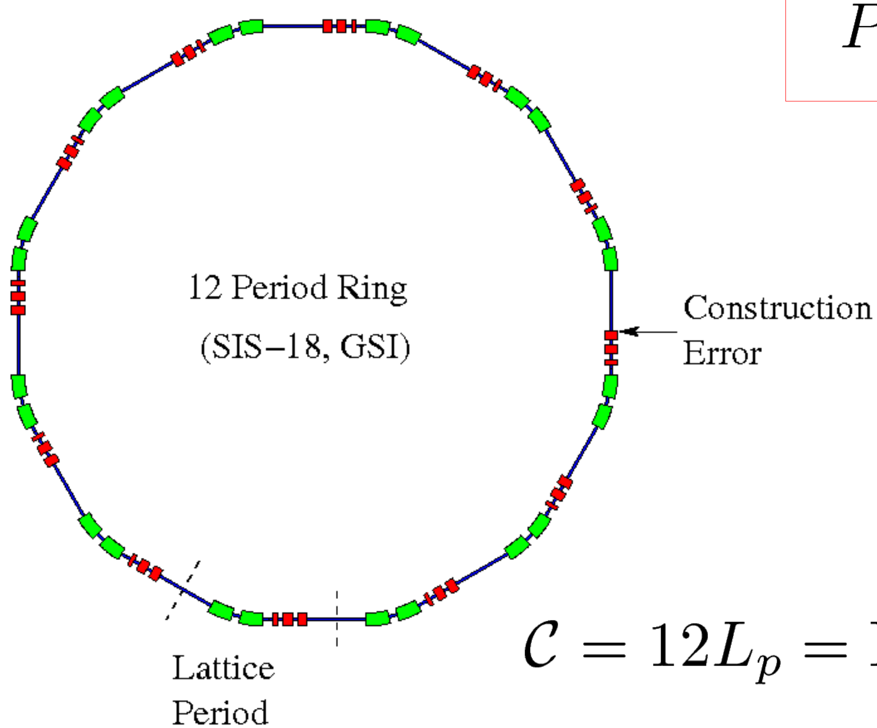
$$\mathcal{C} = \mathcal{N}L_p = \text{Circumference Ring}$$

$$\mathcal{N} \equiv \text{Superperiodicity}$$

Perturbations can be Random and/or Systematic:

Random Errors in a ring will be felt once per particle lap in the ring rather than every lattice period

$$P_{x,y}(\cdots, s+\mathcal{N}L_p) = P_{x,y}(\cdots, s)$$

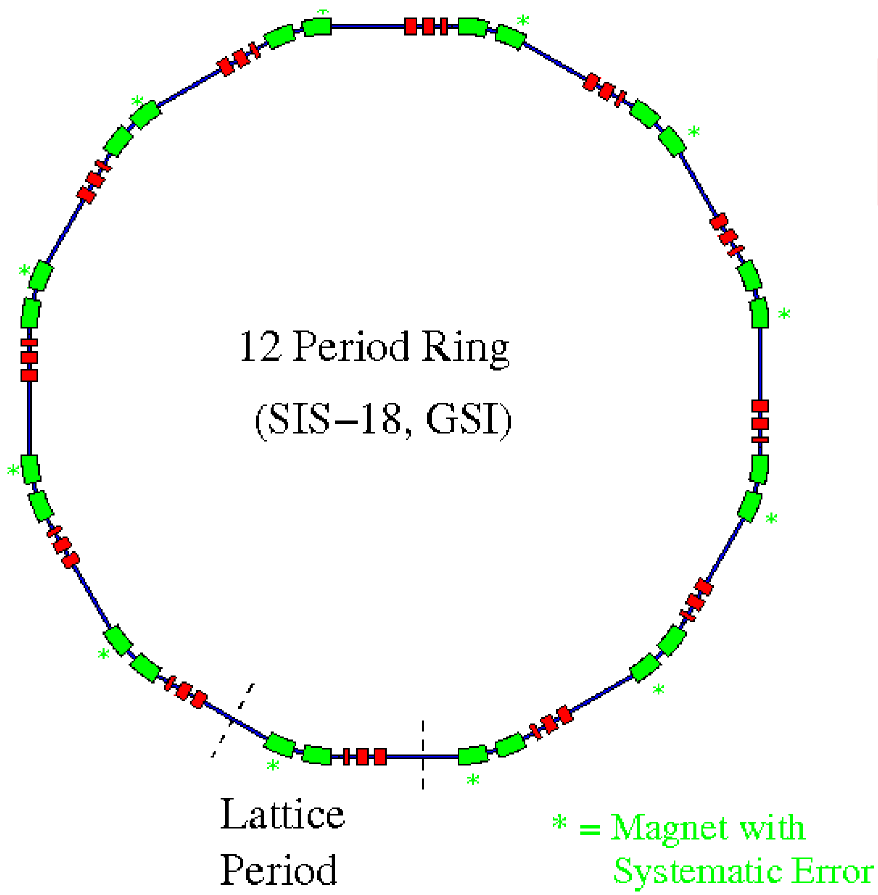


Random Error Sources:
- Fabrication
- Assembly/Construction
- Material Defects
- ....

$$\mathcal{C} = 12L_p = \text{Ring Circumference}$$

Systematic Errors can occur in both linear machines and rings and effect *every* lattice period in the same manner.

Example: FODO Lattice with the same error in each dipole of pair



$$P_{x,y}(\cdots, s + L_p) = P_{x,y}(\cdots, s)$$

Systematic Error Sources:

- Design Flaw/Limit/Ideal
- Repeated Construction
- or Material Error
- ....

We will find that perturbations arising from both random and systematic error can drive resonance phenomena that destabilize particle orbits and limit machine performance

S2: Floquet Coordinates and Hill's Equation

Define for a *stable* solution to Hill's Equation

- Drop *x* subscripts and only analyze *x*-orbit for now to simplify analysis
- Later will summarize results from coupled *x*-*y* orbit analysis

"Radial" Coordinate: $$u \equiv \frac{x}{\sqrt{\beta}}$$

"Angle" Coordinate: (dimensionless, normalized) $$\varphi \equiv \frac{1}{\nu_0} \int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \equiv \frac{\Delta\psi(s)}{\nu_0}$$

where:

$$\beta = w^2 = \text{Betatron Amplitude Function}$$

$$\nu_0 \equiv \frac{\Delta\psi(\mathcal{N}L_p)}{2\pi} = \frac{\mathcal{N}\sigma_0}{2\pi} =$$ Number underpressed *x*-betatron oscillations in ring

$$\psi = \text{Phase of } x\text{-orbit}$$

$$\Delta\psi(s) = \psi(s) - \psi(s_i)$$

Can also take $\mathcal{N} = 1$ and then ν_0 is the number (usually fraction thereof) of undepressed particle oscillations in *one* lattice period

Comment:

φ can be interpreted as a normalized angle measured in the particle betatron phase advance:

Ring: $(\mathcal{N} = \text{Superperiod } \#)$ $\Longrightarrow$ φ advances by 2π on one transit around ring for analysis of Random Errors

Linac or Ring: $(\mathcal{N} = 1)$ $\Longrightarrow$ φ advances by 2π on transit through one lattice period for analysis of Systematic Errors in a ring *or* linac

Take φ as the independent coordinate:

$$u = u(\varphi)$$

and define a new "momentum" phase-space coordinate

$$\dot{u} \equiv \frac{du}{d\varphi} \qquad\qquad \dot{} \equiv \frac{d}{d\varphi}$$

These new variables will be applied to express the unpreturbed Hill's equation in a simpler (continuously focused oscillator) form

From the definition

$$u \equiv \frac{x}{\sqrt{\beta}}$$

Rearranging this and using the chain rule:

$$x = \sqrt{\beta} u$$

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \sqrt{\beta} \frac{du}{d\varphi} \frac{d\varphi}{ds} \qquad \frac{d}{ds} = \frac{d\varphi}{ds} \frac{d}{d\varphi}$$

From:

$$\varphi \equiv \frac{1}{\nu_0} \int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \implies \boxed{\frac{d\varphi}{ds} = \frac{1}{\nu_0 \beta}}$$

we obtain

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0 \sqrt{\beta}} \dot{u}$$

0 (cancels)

$$x'' = \frac{d}{ds} x' = \frac{\beta''}{2\sqrt{\beta}} u - \frac{\beta'^2}{4\beta^{3/2}} u + \frac{\beta'}{2\nu_0 \beta^{3/2}} \dot{u} - \frac{\beta'}{2\nu_0 \beta^{3/2}} \dot{u} + \frac{1}{\nu_0^2 \beta^{3/2}} \ddot{u}$$

Summary:

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0 \sqrt{\beta}} \dot{u}$$

$$x'' = \frac{\beta''}{2\sqrt{\beta}} u - \frac{\beta'^2}{4\beta^{3/2}} u + \frac{1}{\nu_0^2 \beta^{3/2}} \ddot{u}$$

Using these results, Hill's equation:

$$x''(s) + \kappa_x(s) x(s) = 0$$

becomes

$$\ddot{u} + \nu_0^2 \left[\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \kappa\beta^2 \right] u = 0$$

But the betatron amplitude equation satisfies:

$$\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \kappa\beta^2 = 1 \qquad \beta(s + L_p) = \beta(s)$$

Thus the terms in [...] = 1 and Hill's equation reduces to simple harmonic oscillator form:

$$\ddot{u} + \nu_0^2 u = 0 \qquad \nu_0^2 = \text{const} > 0$$

Transform has mapped a stable, time dependent solution to Hill's equation to a simple harmonic oscillator!

The general solution to the simple harmonic oscillator equation can be expressed as:

$$u(\varphi) = u_i \cos(\nu_0 \varphi) + \frac{\dot{u}_i}{\nu_0} \sin(\nu_0 \varphi)$$

$$\dot{u}(\varphi) = -u_i \nu_0 \sin(\nu_0 \varphi) + \dot{u}_i \cos(\nu_0 \varphi)$$

$$u(\varphi = 0) = u_i = \text{const}$$

$$\dot{u}(\varphi = 0) = \dot{u}_i = \text{const}$$

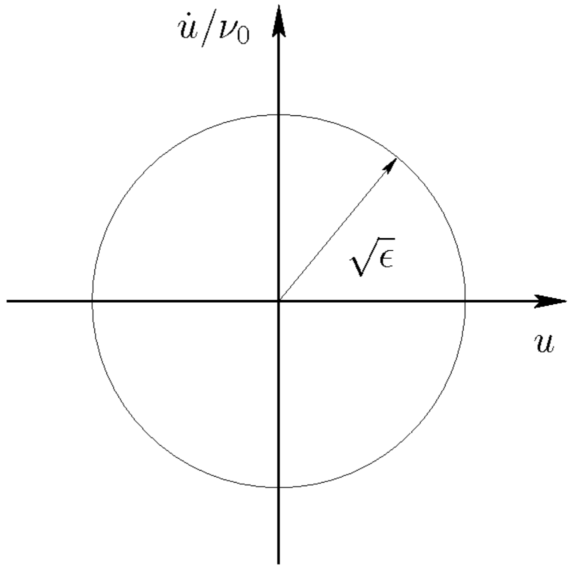
u_i and $\dot{u}_i$ set by x, x'
initial conditions at $s = s_i$
(phase choice $\varphi = 0$ at $s = s_i$)

The Floquet representation also simplifies the interpretation of the Courant-Snyder invariant:

$$u^2 + \left(\frac{\dot{u}}{\nu_0}\right)^2 = u_i^2 + \left(\frac{\dot{u}_i}{\nu_0}\right)^2 \equiv \epsilon = \text{const}$$

- Unperturbed phase-space in $u - \dot{u}/\nu_0$ is a unit circle of area $\pi\epsilon$!
- Relate this area to *x*-*x*' phase-space area shortly
  - Preview: areas are equal due to the transform being symplectic
  - Same symbols used as in Transverse Particle Dynamics is on purpose

Unperturbed phase-space ellipse:



This simple structure will also allow more simple visualization of perturbations as distortions on a unit circle, thereby clarifying symmetries:

Pic to be replaced ... bad example

The $u-\dot{u}/\nu_0$ variables also preserve phase-space area

- Feature of the transform being symplectic (Hamiltonian Dynamics)

From previous results:

$$x = \sqrt{\beta}u$$

$$x' = \frac{\beta'}{2\sqrt{\beta}}u + \sqrt{\beta}\frac{d\varphi}{ds}\dot{u} = \frac{\beta'}{2\sqrt{\beta}}u + \frac{1}{\nu_0\sqrt{\beta}}\dot{u}$$

$$\frac{d\varphi}{ds} = \frac{1}{\nu_0\beta}$$

Transform area elements by calculating the Jacobian:

$$dx \otimes dx' = |J| du \otimes d\dot{u}$$

$$J = \det\left|\begin{array}{cc} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial \dot{u}}, \\ \frac{\partial x'}{\partial u} & \frac{\partial x'}{\partial \dot{u}} \end{array}\right] = \det\left|\begin{array}{cc} \sqrt{\beta} & 0 \\ \frac{\beta'}{2\sqrt{\beta}} & \frac{1}{\nu_0\sqrt{\beta}} \end{array}\right] = \frac{1}{\nu_0}$$

$$\boxed{dx \otimes dx' = du \otimes \frac{d\dot{u}}{\nu_0}}$$

Thus the Courant-Snyder invariant ϵ is the usual single particle emittance in *x*-*x*' phase-space; see lectures on Transverse Dynamics, S7

S3: Perturbed Hill's Equation in Floquet Coordinates

Return to the perturbed Hill's equation in S1:

$$x''(s) + \kappa_x(s)x(s) = \mathcal{P}_x(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$

$$y''(s) + \kappa_y(s)y(s) = \mathcal{P}_y(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$

$$\mathcal{P}_x, \ \mathcal{P}_y = \text{Perturbations}$$

$$\vec{\delta} = \text{Extra Coupling Variables}$$

Drop the extra coupling variables and apply the Floquet transform in S2:

- Examine only *x*-equation, *y*-equation analogous
- Drop *x*-subscript in $\mathcal{P}_x$ to simplify notation

$$\ddot{u} + \nu_0^2 u = \nu_0^2 \beta^{3/2} \mathcal{P}$$

Here,

$$\mathcal{P} = \mathcal{P}(s(\varphi), \sqrt{\beta}u, y, \vec{\delta})$$

Transform *y* similarly to *x*
If analyzing general orbit with *x* and *y* motion

Expand the perturbation in a power series:

- Can be done for *all* physical applied field perturbations
- Multipole symmetries can be applied to restrict the form of the perturbations
  - See: S4 in these notes and S3 in Transverse Particle Dynamics
- Perturbations can be random (once per lap; in ring) or systematic (every lattice period; in ring or in linac)

$$\begin{aligned}\mathcal{P}(x,y,s) &= \mathcal{P}_0(y,s) + \mathcal{P}_1(y,s)x + \mathcal{P}_2(y,s)x^2 + \cdots \\ &= \sum_{n=0}^{\infty} \mathcal{P}_n(y,s)x^n\end{aligned}$$

Take:

$$x = \sqrt{\beta}u$$

to obtain:

$$\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} \mathcal{P}_n(y,s) u^n$$

A similar equation applies in the *y*-plane.

S4: Sources and Forms of Perturbation Terms

Within a 2D transverse model it was shown that applied magnetic fields can be expanded as:

- See: S3, Transverse Particle Dynamics
- Applied electric fields can be analogously expanded

$$\underline{B}^*(\underline{z}) = B_x^a(x,y) - iB_y^a(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}$$

$$\underline{b}_n = \text{const (complex)} \equiv \mathcal{A}_n - i\mathcal{B}_n \qquad \underline{z} = x + iy \qquad i = \sqrt{-1}$$

$$n = \text{Multipole Index} \qquad r_p = \text{Aperture "Pipe" Radius}$$

$$\mathcal{B}_n \Longrightarrow \text{"Normal" Multipoles}$$

$$\mathcal{A}_n \Longrightarrow \text{"Skew" Multipoles}$$

Cartesian projections: $\overline{B_x} - i\overline{B_y} = (\mathcal{A}_n - i\mathcal{B}_n)(x+iy)^{n-1}/r_p^{n-1}$

Index n	Name	Normal ($\mathcal{A}_n = 0$) $B_x r_p^{n-1}/\mathcal{B}_n$	Normal ($\mathcal{A}_n = 0$) $B_y r_p^{n-1}/\mathcal{B}_n$	Skew ($\mathcal{B}_n = 0$) $B_x r_p^{n-1}/\mathcal{A}_n$	Skew ($\mathcal{B}_n = 0$) $B_y r_p^{n-1}/\mathcal{A}_n$
1	Dipole	0	1	1	
2	Quadrupole	y	x	x	$-y$
3	Sextupole	$2xy$	$x^2 - y^2$	$x^2 - y^2$	$-2xy$
4	Octupole	$3x^2y - y^3$	$x^3 - 3xy^2$	$x^3 - 3xy^2$	$-3x^2y + y^3$
5	Decapole	$4x^3y - 4xy^3$	$x^4 - 6x^2y^2 + y^4$	$x^4 - 6x^2y^2 + y^4$	$-4x^3y + 4xy^3$

Trace back how the applied magnetic field terms enter the *x*-plane equation of motion:

- See: S2, Transverse Particle Dynamics
- Apply equation in S2 with: $\beta_b = \text{const}, \ \phi \simeq \text{const}, \ E_x^a \simeq 0, \ B_z^a \simeq 0$

$$x'' = -\frac{q}{m\gamma_b\beta_b c} B_y^a$$

Express this equation as:

$$x'' + \kappa_x(s)x = -\frac{q}{m\gamma_b\beta_b c}\left[B_y^a(x,y,s) - \left. B_y^a(x,y,s)\right|_{\text{lin } x\text{-foc}}\right]$$

Nonlinear focusing terms only in []

- "Normal" part of linear applied magnetic field contained in focus func κ_x

Compare to the form of the perturbed Hill's equation:

$$x'' + \kappa_x x = \mathcal{P}_x = \sum_{n=0}^{\infty} \mathcal{P}_n(y,s)x^n$$

Gives:

$$\Longrightarrow \quad \mathcal{P}_x = -\frac{q}{m\gamma_b\beta_b c}\left[B_y^a - B_y^a\big|_{\text{lin } x\text{-foc}}\right]$$

where the *y*-field components can be obtained from the multipole expansion as:

$$B_y^a = -\text{Im}[\underline{B}^*]$$

$$B_y^a\big|_{\text{lin } x\text{-focus}} = -\text{Im}[\underline{B}^*|_{n=1 \text{ term}}]$$

$$\underline{B}^* = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{x+iy}{r_p}\right)^{n-1}$$

- Use multipole field components of magnets to obtain explicit form of field component perturbations consistent with the Maxwell Equations
- Caution: Multipole index *n* and power series index *n* in $\mathcal{P}_x$ expansion not the same (notational overuse: wanted analogous symbol)

- Multipole: *n* = 1 Dipole; *n* = 2 Quadrupole; *n* = 3 Sextupole; *n* = ...

- Power Series for $\mathcal{P}_x$

x-plane Motion (*y=0*)		*x*-*y* plane motion
n = 0	Dipole	Depends on form of y-coupling
n = 1	Quadrupole	
n = 2	Sextupole	
....		

- Similar steps employed to identify *y*-plane perturbation terms or perturbations for applied electric field components

S6: Solution of the Perturbed Hill's Equation: Resonances

Analyze the solution of the perturbed orbit equation:

$$\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} \mathcal{P}_n(y, s) u^n$$

derived in S4.

To more simply illustrate resonances, we analyze motion in the *x*-plane with:

$$y(s) \equiv 0$$

- Essential character of general analysis illustrated most simply in one plane
- Can generalize by expanding $\mathcal{P}_n(y, s)$ in a power series in *y* and generalizing notation to distinguish between Floquet coordinates in the *x*- and *y*-planes
  - Results in coupled *x*- and *y*-equations of motion

Note that each *n*-labeled perturbation expansion coefficient is periodic with period of the ring circumference (random perturbations) or lattice period (systematic):

$$\beta(s + L_p) = \beta(s), \quad \mathcal{P}_n(y, s + \mathcal{N} L_p) = \mathcal{P}_n(y, s)$$

$$\Longrightarrow \quad \beta^{\frac{n+3}{2}}(s + \mathcal{N} L_p)\mathcal{P}_n(y, s + \mathcal{N} L_p) = \beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y, s)$$

Expand each *n*-labeled perturbation expansion coefficient in a Fourier series as:

$$\beta^{\frac{n+3}{2}} \mathcal{P}_n(y=0,s) = \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ikp\varphi}$$

$$i \equiv \sqrt{-1} \qquad p \equiv \begin{cases} 1, & \text{Random perturbation} \\ & \text{(once per lap in ring)} \\ \mathcal{N}, & \text{Systematic perturbation} \\ & \text{(every lattice period)} \end{cases}$$

$$C_{n,k} = \int_{-\pi/p}^{\pi/p} \frac{d\varphi}{2\pi}\, e^{-ikp\varphi} \beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y=0,s) \quad = \quad \text{const}$$

(complex-valued)

$$s = s(\varphi) \qquad \varphi = \int_{s_0}^{s} \frac{1}{\nu_0} \frac{d\tilde{s}}{\beta(\tilde{s})}$$

- Can apply to Rings for random perturbations (with $p = 1$) or systematic perturbations (with $p = \mathcal{N}$)
- Can apply to linacs for periodic perturbations (every lattice period) with $p = 1$
- Does not apply to random perturbations in a linac
  - In linac random perturbations will vary every lattice period and drive random walk type effects but not resonances

The perturbed equation of motion becomes:

$$\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ikp\varphi} u^n$$

Expand the solution as:

$$u = u_0 + \delta u$$

$u_0 =$ unperturbed solution

$\delta u =$ perturbation due to errors

where u_0 is the solution to the *simple harmonic oscillator* equation in the absence of perturbations:

$$\ddot{u}_0 + \nu_0^2 u_0 = 0$$

Unperturbed equation of motion

Assume small-amplitude perturbations so that

$$|u_0| \gg |\delta u|$$

Then to leading order, the equation of motion for δu is:

$$\ddot{\delta u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ipk\varphi} u_0^n$$

Perturbed equation of motion

To obtain the perturbed equation of motion, the unperturbed solution u_0 is inserted on the RHS terms

- Gives simple harmonic oscillator equation with driving terms

Solution of the unpreturbed orbit is simply expressed as:

$$u_0 = u_{0i}\cos(\nu_0\varphi + \varphi_i) = u_{0i}\frac{e^{i(\nu_0\varphi+\varphi_i)} - ie^{-i(\nu_0\varphi+\varphi_i)}}{2}$$

$u_{0i} = \text{const}$ $\quad$ Set by particle initial conditions:

$\varphi_i = \text{const}$ $\quad$ $x(s_i) = x_i, \quad x'(s_i) = x'_i$

Then binomial expand:

$$u_0^n = u_{0i}^n\left(\frac{e^{i(\nu_0\varphi+\varphi_i)} + e^{-i(\nu_0\varphi+\varphi_i)}}{2}\right)^n$$

$$= \frac{u_{oi}^n}{2^n}\sum_{m=0}^{n}\binom{n}{m}e^{i(n-m)(\nu_0\varphi+\varphi_i)}e^{-im(\nu_0\varphi+\varphi_i)}$$

$$= \frac{u_{oi}^n}{2^n}\sum_{m=0}^{n}\binom{n}{m}e^{i(n-2m)\nu_0\varphi}e^{i(n-2m)\varphi_i}$$

where $\binom{n}{m} \equiv \frac{n!}{m!(n-m)!}$ is a binomial coefficient

Using this expansion the linearized perturbed equation of motion becomes:

$$\delta\ddot{u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{C_{n,k}}{2^n} e^{i[(n-2m)\nu_0 + pk]\varphi} e^{i(n-2m)\varphi_i}$$

The solution for δu can be expanded as:

$$\delta u = \delta u_h + \delta u_p$$

$$\delta u_h = \text{homogenous solution}$$

General solution to: $\delta\ddot{u}_h + \nu_0^2 \delta u_h = 0$

$$\delta u_p = \text{particular solution}$$

Any solution with: $\delta u \to \delta u_p$

- Can drop homogeneous solution because it can be absorbed in unperturbed solution u_0
  - Exception: some classes of linear amplitude errors in adjusting magnets
- Only a particular solution need be found, take:

$$\delta u = \delta u_p$$

$$\delta\ddot{u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{C_{n,k}}{2^n} e^{i[(n-2m)\nu_0 + pk]\varphi} e^{i(n-2m)\varphi_i}$$

Equation describes a driven simple harmonic oscillator with a periodic driving terms on the RHS:

- Homework problem reviews that solution of such an equation will be unstable when the driving term has a frequency component equal to the restoring term
  - Resonant exchange and amplitude grows *linearly* in φ
  - Parameters meeting resonance condition will lead to instabilities

Resonances occur when:

$$(n - 2m)\nu_0 + pk = \pm\nu_0$$

is satisfied for the operating tune ν_0 and some values of:

$$n = 0,\ 1,\ 2,\ \cdots \qquad m = 0,\ 1,\ 2,\ \cdots,\ n$$

$$k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty$$

$$p \equiv \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

If growth rate is sufficiently large, machine operating points satisfying the resonance condition will be problematic since particles will be lost (scraped) by the machine aperture due to increasing oscillation amplitude:

- Machine operating tune (ν_0) can be adjusted to avoid
- Perturbation can be actively corrected to reduce amplitude of driving term

Low order resonance terms with smaller *n*, *k*, *m* magnitudes are expected to be more dangerous because:

- Less likely to be washed out by effects not included in model
- Amplitude coefficients expected to be stronger

In the next section we will examine how resonances restrict possible machine operating parameters.

S7: Machine Operating Points: Tune Restrictions Resulting from Resonances

Examine situations where the *x*-plane motion resonance condition:

$$(n - 2m)\nu_0 + pk = \pm\nu_0$$

is satisfied for the operating tune ν_0 and some values of:

$$n = 0,\ 1,\ 2,\ \cdots \qquad m = 0,\ 1,\ 2,\ \cdots,\ n$$

$$k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty$$

$$p \equiv \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

Resonances can be analyzed one at a time using linear superposition

- Analysis valid for small-amplitudes

Analyze resonance possibilities starting with index *n* <==> Multipole Order

$n = 0$, Dipole Perturbations:

$$n = 0, \implies m = 0$$

and the resonance condition gives:

$$\nu_0 = \pm pk \qquad pk = \text{integer} \qquad k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$$

$$p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

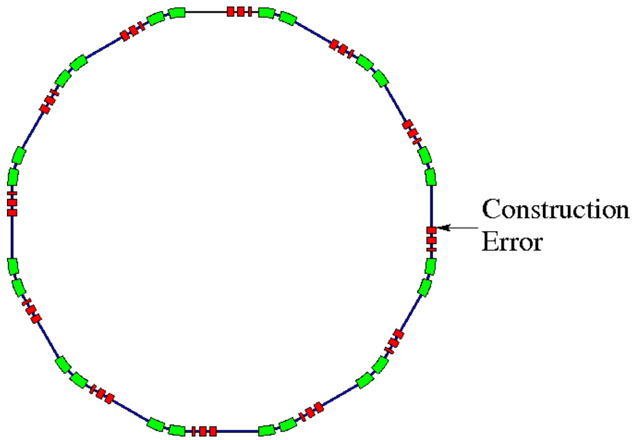
Therefore, to avoid dipole resonances *integer tunes* operating points not allowed:

$p = 1$	Random Perturbation	$\nu_0 \neq 1, 2, 3, \cdots$
$p = \mathcal{N}$	Systematic Perturbation	$\nu_0 \neq \mathcal{N}, 2\mathcal{N}, 3\mathcal{N}, \cdots$

- Systematic errors are less restrictive on machine operating points
- Multiply random perturbation tune restrictions by $\mathcal{N}$ to obtain the systematic perturbation case

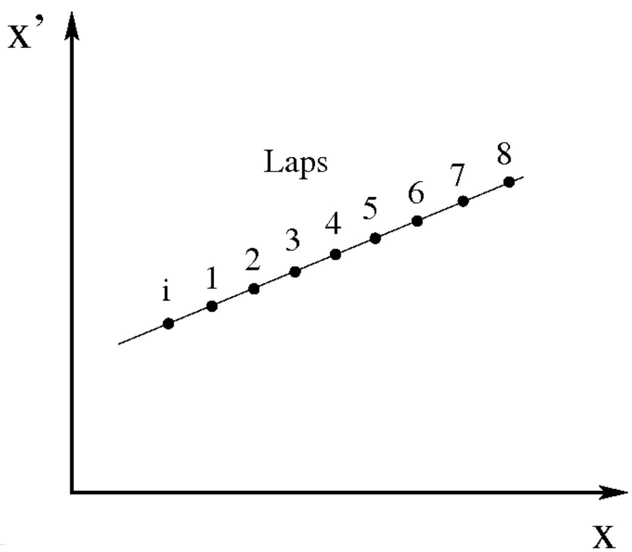
Interpretation of result:

Consider a ring with a single (random) dipole error along the reference path of the ring:



If the particle is oscillating with integer tune, then the particle experiences the dipole error on each lap in the same oscillation phase and the trajectory will "walk-off" on a lap-to-lap basis in phase-space:

- With finite machine aperture the particle will be scraped/lost



$n = 1$, Quadrupole Perturbations:

$$n = 1, \implies m = 0,\ 1$$

and the resonance conditions give:

$$\begin{aligned} n = 1,\ m = 0: &\qquad \nu_0 + pk = \pm\nu_0 \\ n = 1,\ m = 1: &\quad -\nu_0 + pk = \pm\nu_0 \end{aligned} \qquad \implies pk = 0,\ \nu_0 = \pm\frac{pk}{2}$$

Implications:

$$1)\ pk = 0 \ \Rightarrow k = 0$$

Case can be treated by "renormalizing" oscillator focusing strength and need not be considered

$$\ddot{u} + \nu_0^2 u = \nu_0^2 C_{1,0} u$$

$$2)\ \nu_0 = \pm\frac{pk}{2} \ \Rightarrow \nu_0 = \frac{|pk|}{2}$$

Therefore, to avoid quadrupole resonances, *half-integer tune* operating points not allowed:

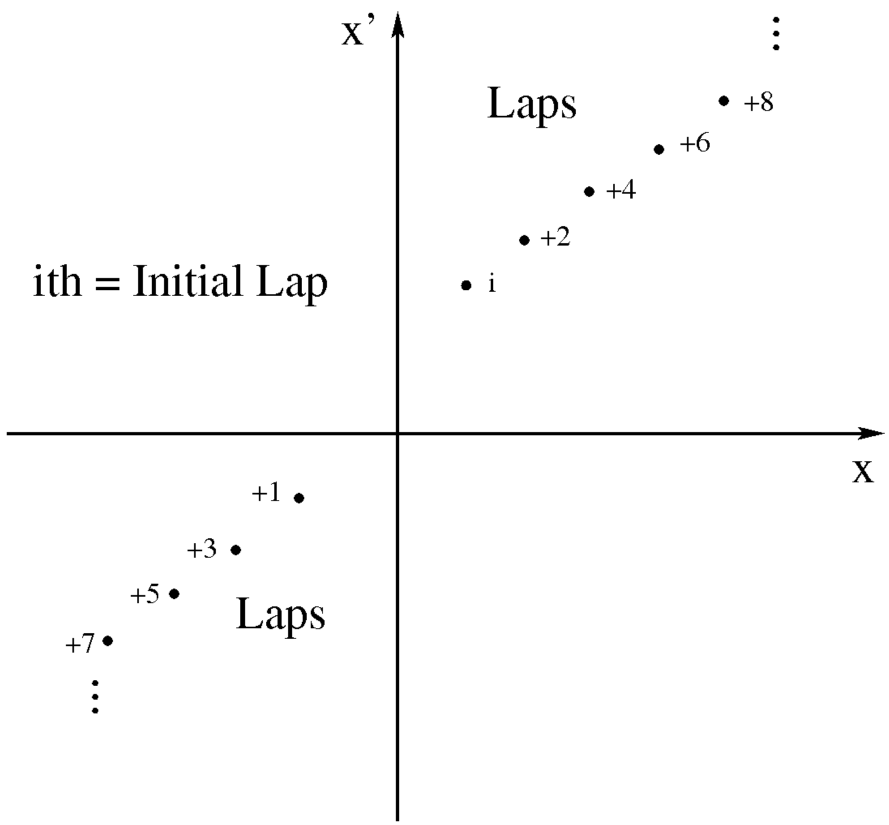
$$\nu_0 \neq \frac{|pk|}{2} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

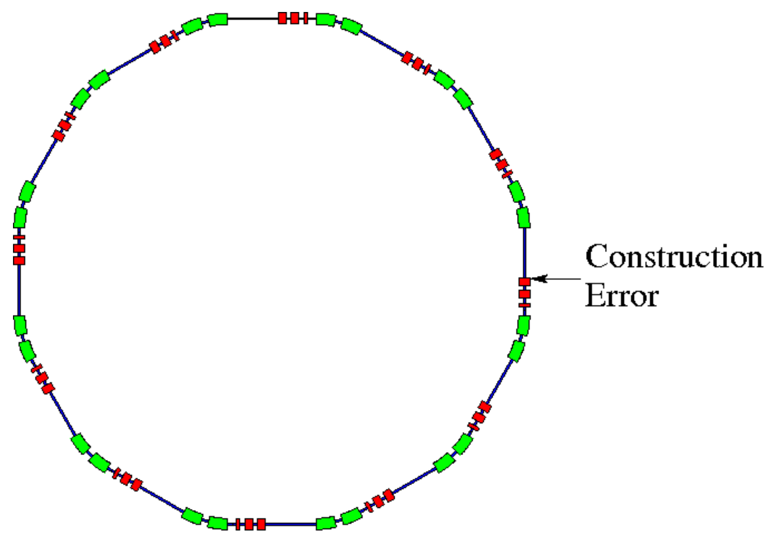
$$k = -\infty, \cdots, -1,\ 0,\ 1, \cdots, \infty$$

- New restriction on machine tunes from being half-integer values
- Integers also restricted for $p = 1$ random, but redundant with dipole case
- Some large integers restricted for $p = N$ random

Interpretation of result (new restrictions):

For a single (random) quadrupole error along the azimuth of a ring, a similar qualitative argument as presented in the dipole resonance case leads on to conclude that if a particle oscillates with ½ integer tune, then the orbit can "walk-off" on a lap-to-lap basis in phase-space:





$n = 2$, Sextupole Perturbations:

$$n = 2, \implies m = 0,\ 1,\ 2$$

and the resonance conditions give:

$$n = 2,\ m = 0: \qquad 2\nu_0 + pk = \pm\nu_0$$

$$n = 2,\ m = 1: \qquad pk = \pm\nu_0$$

$$n = 2,\ m = 2: \qquad -2\nu_0 + pk = \pm\nu_0$$

Therefore, to avoid sextupole resonances, the following tunes are not allowed:

$$\nu_0 \neq \begin{cases} |pk| & \text{integer} \\ |pk|/2 & \text{half-integer} \\ |pk|/3 & \text{third-integer} \end{cases} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

$$k = -\infty, \cdots, -1,\ 0,\ 1, \cdots, \infty$$

- Integer and ½-integer restrictions already obtained for dipole and quadrupole perturbations
- 1/3-integer restriction new

Higher-order ($n > 2$) cases analyzed analogously

- Produce more constraints but expected to be weaker as order increases

General form of resonances

The general resonance condition (all *n*-values) for *x*-plane motion can be summarized as:

$$M\nu_0 = N \qquad \begin{aligned} M,\ N &= \text{Integers of same sign} \\ |M| &= \text{"Order" of resonance} \end{aligned}$$

- Higher order numbers *M* are generally less dangerous
  - Longer coherence length for validity of theory: effects not included can "wash-out" the resonance
  - Coefficients generally smaller

Particle motion is not, in general, restricted to the *x*-plane, and a more general analysis taking into account coupled *x*- and *y*-plane motion shows that the generalized resonance condition is:

$$M_x\nu_{0x} + M_y\nu_{0y} = N \qquad \begin{aligned} M_x,\ M_y,\ N &= \text{Integers of same sign} \\ |M_x| + |M_y| &= \text{"Order" of resonance} \end{aligned}$$

$$\begin{aligned} \nu_{0x} &= x\text{-plane tune} \\ \nu_{0y} &= y\text{-plane tune} \end{aligned}$$

- Lower order resonances are generally more dangerous analogously to *x*-case

Restrictions on machine operating points

Tune restrictions are generally plotted in $\nu_{0x} - \nu_{0y}$ space order-by-order up to a max order value to find allowed tunes where the machine can safely operate

- Often 3[rd] order is chosen as a maximum to consider
- Cases for random ($p = 1$) and systematic ($p = \mathcal{N}$) perturbations considered

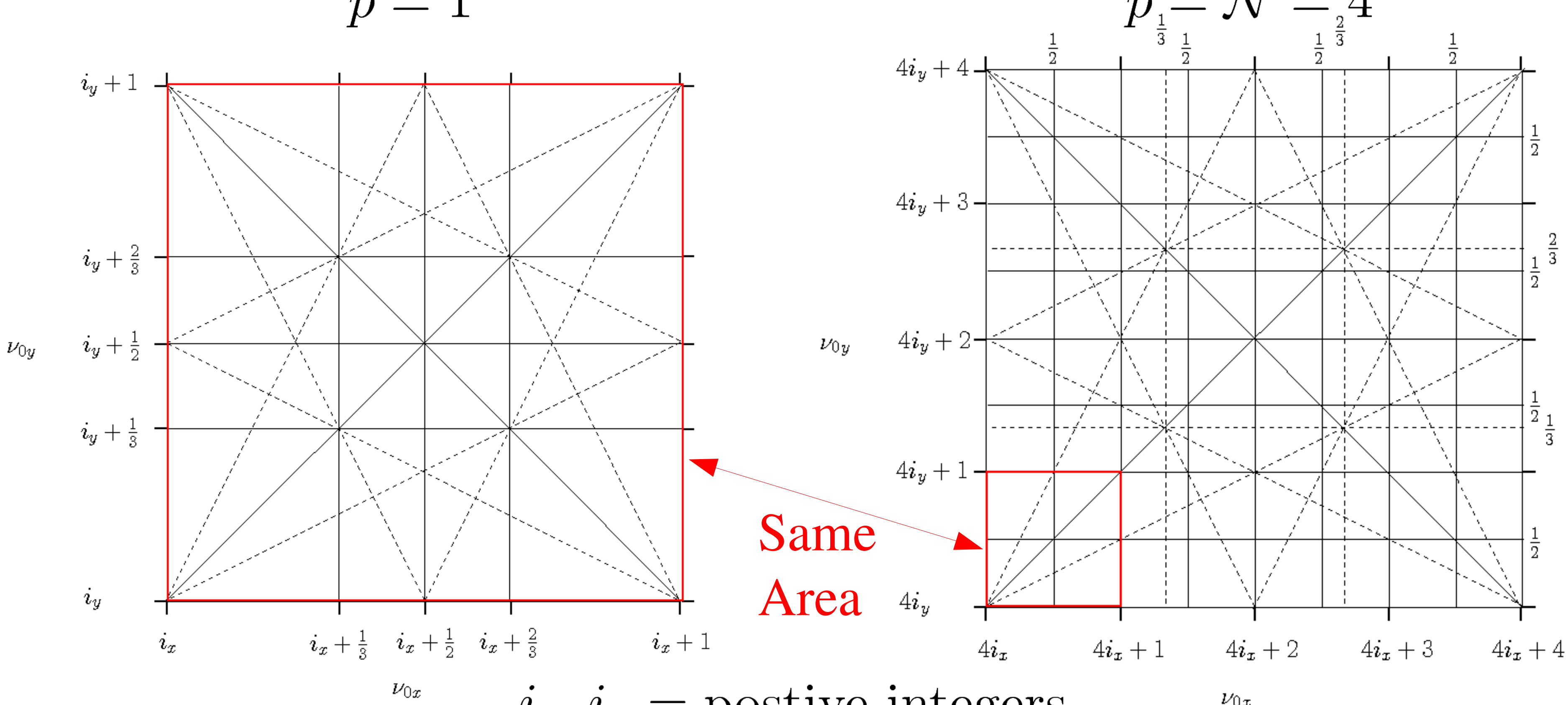
Machine operating points chosen as far as possible from low order resonance lines

Random Perturbations

$p = 1$

Systematic Perturbations

$p = \mathcal{N} = 4$



$i_x, i_y =$ postive integers

Discussion:

Random errors:

- Errors always present and give low-order resonances
- Usually have weak amplitude coefficients
  - Can be corrected to reduce effects

Systematic errors:

- Lead to higher-order resonances for large *N* and a lower density of resonance lines (see plots on previous slide comparing the equal boxed red areas)
  - Large symmetric rings with high *N* values have less operating restrictions from systematic errors
  - Practical issues such as construction cost and getting the beam into and out of the ring can lead to smaller *N* values (racetrack lattice)
- BUT systematic error Amplitude coefficients can be large
  - Systematic effects accumulate in amplitude period by period

Resonances beyond 3rd order rarely need be considered

- Effects outside of model assumed tend to wash-out higher order resonances

More detailed treatments calculate amplitudes/strengths of resonant terms

- See accelerator physics references: ....

S8: Space-Charge and Other Effects Altering Resonances

Ring operating points are generally chosen to be far from low-order resonance lines in *x*-*y* tune space. Processes that act to shift resonances closer towards the low-order lines can prove problematic:

- Oscillation amplitudes increase (spoiling beam quality and control)
- Particles can be lost

Tune shift limits of machine operation are often named "Laslett Limits" in honor of Jackson Laslett who first calculated tune shift limits for many processes:

- Image charges
- Image currents
- KV model linear self-fields internal to the beam
- ...

Processes shifting resonances can be grouped into two broad categories:

Coherent — Same for every particle in distribution
- Usually most dangerous

Incoherent — Different for particles in separate parts of the distribution
- Usually less dangerous: only effects part of beam

Laslett space-charge limit

Laslett first obtained a space-charge limit for rings by assuming that the beam space-charge is uniformly distributed as in a KV model and thereby acts as a coherent shift to previously derived resonance conditions. Denote:

$$\nu_{0x} \equiv x\text{-tune (bare) in absence of space-charge}$$

$$\nu_x \equiv x\text{-tune (depressed) with uniform density beam}$$

$$\Delta\nu_x \equiv \nu_{0x} - \nu_x = \text{Space-charge tune shift} \qquad \Delta\nu_x \geq 0$$

Assume that dipole (integer) and quadrupole (half-integer) tunes only need be excluded when space-charge effects are included.

- Space-charge likely induces more washing-out of higher-order resonances

If the bare tune operating point is chosen as far as possible from ½ -integer resonance lines, the maximum space-charge induced tune shift allowed is ¼-integer, giving:

$$\Delta\nu_x|_{\max} = \frac{1}{4} \implies$$

Establishes maximum current (use KV results in lectures on Transverse Equilibrium Distributions)

- Analogous equation applies in the *y*-plane
  - Identical restriction in lattices with equal *x*- and *y*-focusing strengths

Simple estimate of maximum perveance allowed under the Laslett limit:

Consider a ring with:

$$\mathcal{N} = \text{Lattice Periods}$$

$$L_p = \text{Lattice Period}$$

$$\sigma_0 = \text{Phase advance in } x\text{- or } y\text{-directions}$$

$$\Longrightarrow \quad \nu_{0x} = \nu_{0y} \equiv \nu_0 = \mathcal{N}\frac{\sigma_0}{2\pi}$$

Model the focusing as continuous and assume an unbunched, transverse matched KV distribution with:

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const} \qquad \text{Focusing} \qquad k_{\beta 0} = \frac{\sigma_0}{L_p} = \frac{2\pi\nu_0}{\mathcal{N}L_p}$$

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon = \text{const} \qquad \text{Emittance}$$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const} \qquad \text{Perveance}$$

The matched envelope equation gives:

$$r_x = r_y = r_b = \text{const}$$

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0 \quad (r_b'' \to 0) \quad \Longrightarrow \quad r_b^2 = \frac{Q + \sqrt{4k_{\beta 0}^2\varepsilon^2 + Q^2}}{2k_{\beta 0}^2}$$

Depressed phase advance per lattice period can then be calculated from formulas in lectures on Transverse Equilibrium Distributions as:

$$\sigma = \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} L_p = \varepsilon \int_{s_i}^{s_i + L_p} \frac{ds}{r_b^2}$$

← Two forms equivalent from envelope equation

using

$$\nu = \mathcal{N} \frac{\sigma}{2\pi}$$

$$k_\beta = \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} \qquad \sigma = k_\beta L_p$$

and previous formulas gives:

$$\nu = \nu_0 \sqrt{1 - \frac{2Q}{Q + \sqrt{\frac{16\pi^2 \nu_0^2 \varepsilon^2}{\mathcal{N}^2 L_p^2} + Q^2}}}$$

Setting the phase shift to the Laslett current limit value

$$\nu|_{Q=Q_{\max}} = \nu_0 - \frac{1}{4}$$

gives a constraint for the maximum value of $Q = Q_{\rm max}$ to avoid 1/2-integer resonances:

$$\frac{2Q_{\rm max}}{Q_{\rm max} + \sqrt{\frac{16\pi^2\nu_0^2\varepsilon^2}{\mathcal{N}^2 L_p^2} + Q_{\rm max}^2}} = 1 - \left(\frac{\nu_0 - 1/4}{\nu_0}\right)^2 = \frac{1}{2\nu_0^2}(\nu_0 - 1/8)$$

This can be arraigned into a quadratic equation for $Q_{\rm max}$ and solved to show that the Laslett "current" limit expressed in terms of the maximum transportable perveance:

$$Q < Q_{\rm max} = \frac{\pi\varepsilon}{\mathcal{N}L_p}\left(\frac{\nu_0 - 1/8}{\nu_0}\right)\frac{1}{\sqrt{1 - \frac{1}{2\nu_0}\left(\frac{\nu_0 - 1/8}{\nu_0}\right)}}$$

$$\simeq \frac{\pi\varepsilon}{\mathcal{N}L_p}\left(1 + \frac{1}{8\nu_0} + \text{Order}(1/\nu_0^2)\right)$$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2} = \frac{qI}{2\pi\epsilon_0 m\gamma_b^3\beta_b c}$$

I = Beam Current

// Example: Take (typical synchrotron numbers, represents peak charge in rf bunch)

$\mathcal{N}L_p = \mathcal{C} =$ Ring Circumfrance ~ 300 m

$\varepsilon \sim 50$ mm-mrad

Neglect $1/\nu_0$ term

$$\Longrightarrow \quad Q < Q_{\rm max} \simeq \frac{\pi\varepsilon}{\mathcal{C}} \simeq 5\times 10^{-7}$$

Not a lot of charge .... //

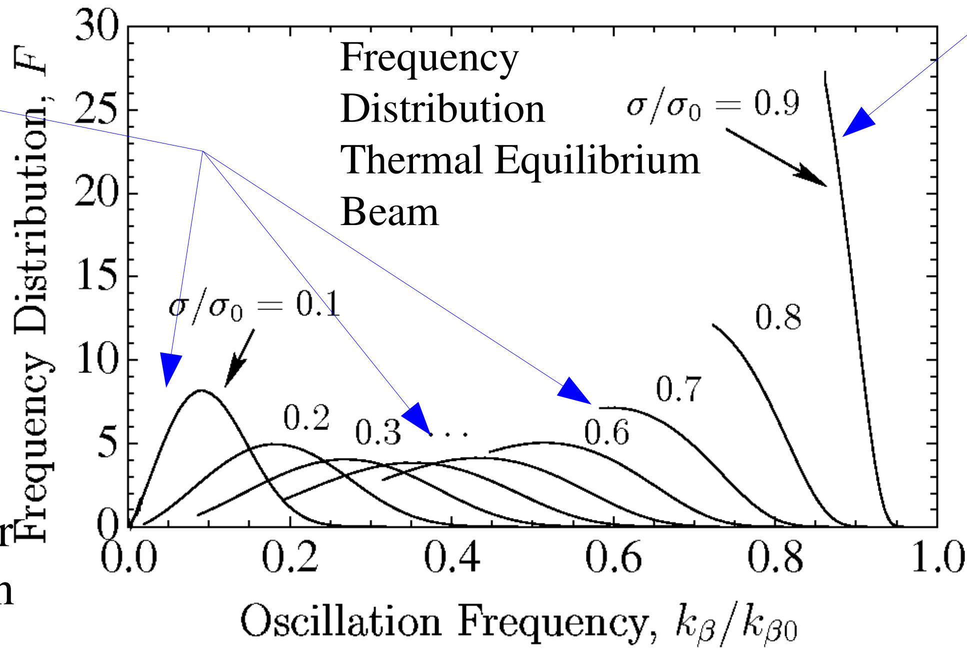
Discussion:

Laslett limit may be overly restrictive:

- KV model assumes all particles in beam have the same tune
  - Significant spectrum of particle tunes likely in real beam
    Particularly if space-charge strong: see Transverse Equilibrium Dists, S7
  - No equilibrium beam: core oscillates and space-charge may act incoherently to effectively wash-out resonances

For strong space-charge:

- Frequency spread large and KV approx bad
- Does not work in spite of beam density being near uniform density for smooth distribution



For weak space-charge:

- Frequency spread small and KV approx good
- Works in spite of beam density being far from uniform density for smooth distribution

- Simulations suggest Laslett limit poses little issues over 10s – 100s of laps in rings (Small Recirculator, LLNL) and in fast bunch compressions in rings
  - Longer simulations very difficult to resolve: see Simulation Techniques
- Future experiments can hopefully address this issue
  - University of Maryland electron ring will have strong space-charge

Discussion Continued:

- Even if internal resonances in the core of the beam are washed out due to nonlinear space-charge at high intensity, centroid resonances may still behave more as a single particle (see notes on Transverse Centroid and Envelope Descriptions of Beam Evolution) to limit beam control.
  - Steering and correction can mitigate low order centroid instabilities

More research on this topic is needed!

- Higher intensities can open new applications for energy and material processing
- Many possibilities to extend operating range of existing machines and make new use of developed technology
- Good area for graduate thesis projects!

These notes will be corrected and expanded for reference and future editions of US Particle Accelerator School and University of California at Berkeley courses:

"Beam Physics with Intense Space Charge"

"Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields"

by J.J. Barnard and S.M. Lund

Corrections and suggestions for improvements are welcome. Contact:

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References: For more information see:

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