Problem Set 11/7

Problem 1

Consider the parameters for a proton storage ring in which there is a single circulating bunch:

\[ N_0 = 2 \times 10^{11} \text{ protons/bunch} \]

\[ l_{\text{off}} \approx 12.4 \text{ m} = \text{effective bunch length} \]

\[ r_{\text{p}} = 0.1 \text{ m} = \text{pipe radius} \]

\[ \beta = 0.875 = \frac{v}{c} \quad \text{(ion velocity)} \]

a) What is the single bunch, multivacating parameter \( f_s \)?

b) What is the characteristic energy gain per electron?

c) Assume an ionization cross section of \( 10^{-22} \text{ m}^2 \), and a desorption coefficient \( 4 \), and a linear pump rate \( S \) of \( 0.1 \text{ m}^3 \text{s}^{-1} \text{m}^{-1} \). Will the pressure be stable in this machine against runaway desorption? (Assume the beam fills the ring; for making your estimate.)

d) Assume that \( r_{\text{b}} = 0.01 \text{ m} \). Estimate the electron oscillation frequency in the field of the ion beam.
Consider the effects of Coulomb collisions for a beam with an initial temperature $T_{10} \neq T_{110}$. Assume $\nu = \text{const}$, and that

$$\frac{dT_1}{d\xi} = -\frac{1}{2} \frac{dT_{11}}{d\xi} = -\frac{(T_1 - T_{11})}{\nu}$$

Show that for initial values $T_1 = T_{10}$ and $T_{11} = T_{110}$, that the general solution (when $\nu$ is assumed constant) is:

$$T_{11} = A - \frac{2}{3} B \exp[-\kappa t]$$
$$T_1 = A + \frac{1}{3} B \exp[-\kappa t]$$

Calculate $A, B, \kappa$ in terms of $T_{10}, T_{110}, \nu$. What are the values of $T_1$ and $T_{11}$ as $t \to \infty$?
Problem 3, 25 Points

ICE Problem 4

S.M. Lund

In class we derived rms envelope equations for a coasting beam:

\[
\frac{\Gamma_x'' + \Gamma_x(s) \Gamma_x}{\Gamma_x + \Gamma_y} - \frac{Z_0}{\Gamma_x} - \frac{E_x^2}{\Gamma_x} = 0
\]

\[
\frac{\Gamma_y'' + \Gamma_y(s) \Gamma_y}{\Gamma_x + \Gamma_y} - \frac{Z_0}{\Gamma_y} - \frac{E_x^2}{\Gamma_y} = 0
\]

Take:

\[
\Gamma_x = \Gamma_{x0} + \delta \Gamma_x
\]

\[
\Gamma_y = \Gamma_{y0} + \delta \Gamma_y
\]

Continuous Focus equilibrium

\[
Q = Q + \delta Q
\]

\[
E_x = E + \delta E_x
\]

\[
E_y = E + \delta E_y
\]

and

\[
\Gamma_x = \Gamma_m + \delta \Gamma_x
\]

\[
\Gamma_y = \Gamma_m + \delta \Gamma_y
\]

Matched Envelope

\[
\Gamma_m = \text{const.}
\]

A/ Expand to linear order as in class notes and derive equilibrium and perturbed envelope eqns

B/ Show that the linear order envelope eqns in A/ decouple when taking

\[
\delta \Gamma_{+} = \frac{\delta \Gamma_x + \delta \Gamma_y}{2}
\]

\[
\delta \Gamma_{-} = \frac{\delta \Gamma_x - \delta \Gamma_y}{2}
\]

and give
\[ \begin{align*} \frac{\partial^2 \sigma^{++}}{\partial \sigma_0^2} + \frac{\partial^2 \sigma^{--}}{\partial \sigma_0^2} + \partial^2 \sigma^{++} + \frac{3 \sigma^2}{\sigma^4} \sigma^{++} &= \sigma_0^2 \\ \frac{\partial^2 \sigma^{--}}{\partial \sigma_0^2} + \frac{\partial^2 \sigma^{--}}{\partial \sigma_0^2} + \frac{3 \sigma^2}{\sigma^4} \sigma^{--} &= \sigma_0^2 \end{align*} \]

with

\[ \begin{align*} \sigma^{++} &= -\frac{\sigma^2}{\sigma^4} \left( \frac{\partial \sigma_0}{\partial \sigma_0} + \frac{\partial \sigma_0}{\partial \sigma_0} \right) + \frac{\sigma^2}{\sigma^4} (\delta \sigma + \delta \sigma) \\ \sigma^{--} &= -\frac{\sigma^2}{\sigma^4} \left( \frac{\partial \sigma_0}{\partial \sigma_0} + \frac{\partial \sigma_0}{\partial \sigma_0} \right) + \frac{\sigma^2}{\sigma^4} (\delta \sigma - \delta \sigma) \end{align*} \]

C/ Take

\[ \eta_0 = \frac{\sigma_0}{L_0} \]

\[ \sigma = \sqrt{\frac{\eta_0^2 - \sigma^2}{L_0^2 \sigma_0^2}} = \frac{\sigma L_0}{L_0} \]

\[ \sigma = \sqrt{\frac{\eta_0^2 + \sigma^2}{\sigma_0^2}} \]

and show that the eqn for \( \sigma^{++} \) in B/ becomes:

\[ \begin{align*} L_0^2 \frac{d^2}{ds^2} (\sigma^{++}) + \frac{\sigma^2}{\sigma_0^2} (\sigma^{++}) &= -\frac{\sigma^2}{\sigma_0^2} \left( \frac{\partial \sigma_0}{\partial \sigma_0} + \frac{\partial \sigma_0}{\partial \sigma_0} \right) \\ &+ \frac{(\sigma^2 - \sigma^2)}{\sigma_0^2} \frac{\partial \sigma_0}{\partial \sigma_0} + \frac{\sigma^2}{\sigma_0^2} \left( \frac{\delta \sigma + \delta \sigma}{\sigma} \right) \end{align*} \]

Solution of the eqn in this form was extensively developed in the lectures.
Problem 4, 25 Points

TCE Problem 6

Approximate Matched Envelope Solution for a Solenoid

For a solenoidal transport channel:

We can Fourier series expand \( R(s) = R_x(s) = R(s) \) as

\[
R(s) = \sum_{n=0}^{\infty} R_n \cos \left( \frac{2\pi n s}{L_p} \right)
\]

and identify

\[
R_0 = \int_0^{L_p} ds \frac{R(s)}{L_p}
\]

\[
R_n = \frac{2}{L_p} \int_0^{L_p} ds \cos \left( \frac{2\pi n s}{L_p} \right) \frac{R(s)}{L_p}
\]

calculate \( R_0 \) and \( R_n \)

B/ The matched beam envelope equation is: (Larmor Frame)

\[
\Gamma_{xm}'' + \left( R(s) \Gamma_{xm} - \frac{Q}{\Gamma_{xm}} - \frac{\varepsilon_x^2}{\Gamma_{xm}^3} \right) \Gamma_{xm} = 0
\]

Expand the matched envelope solution similarly:

\[
\Gamma_{xm}(s) = \Gamma_0 \left[ 1 + \Delta \cos \left( \frac{2\pi n s}{L_p} \right) \right] + \sum_{n=2}^{\infty} \Delta_n \cos \left( \frac{2\pi n s}{L_p} \right)
\]

where \( \Gamma_0 = \text{const} \)

\( \Delta = \text{const.} \) \(|\Delta| \ll 1 \)

is an expansion term and \( \Delta_n \) are dimensionless expansion coefficients we take to be small relative to \( \Delta \),
TCE Problem 6

Insert the expansions

- Neglect all terms $O(\Delta^2)$ and higher
- Neglect fast oscillation terms $\cos(e^{- \frac{\Delta^2 t}{2p}})$ with $n \geq 2$

Derive any and leading-order force balance terms.

C/ Explain (no need to calculate explicitly) how the constraints in B/ can be used to calculate:

\[
\text{Max } [F_{xm}] : \text{Max beam excursion in lattice period in terms of lattice parameters, avg. beam radius, and emittance}
\]

\[
Q : \text{Beam perveance as a function of lattice parameters, avg. beam radius, and emittance}
\]

D/ The unde pressed particle phase advance can be calculated as

\[
\cos \delta_0 = \cos(2\delta) - (1 - q) \Omega \sin(2\delta)
\]

\[
\delta = \frac{1}{2} \Omega \sqrt{\frac{p}{2}}
\]

Expand this expression to leading order in $\delta$ to obtain a simpler design expression relating $\delta_0$ and lattice parameters.

Results from part C/ and D/ can be employed to design solenoidal transport channels.