FINAL EXAM

1. Consider a heavy ion accelerator with a constant average beam radius $R_1$, constant pipe radius $R_p$, constant undulation (have advance $\phi_0$ and constant magnetic quadrupole gradient $B'$ in John's notation $B_2^g$ in Steen's notation). You may assume the beam is non-relativistic, the occupancy $N$ of the quadrupoles is constant. Also, you may use the thin lens, small $\phi_0$ approximation to calculate the scaling or $\phi_0$. The current can be considered constant over the pipe length, but varies as a function of position $s$ through the accelerator.

Assume the beam is space-charge dominated and is being compressed and accelerated so that it is at the maximum transmissible current. Also, assume the normalized longitudinal and transverse emittances are constant.

2. $\phi_0 = \frac{NL^2B}{(r_0B_p)} = \frac{NL^2B_2}{B_p}$ for thin lens, small $\phi_0$ approximation.

a). How does the current $I$ scale with velocity/c $\beta$? (I.e., what is the exponent $\alpha$ in the relation $I = \left(\frac{\beta}{\beta_0}\right)^\alpha$, $I_0 = $ initial current, $\beta_0 = $ initial velocity/c.)

b). How does the emittance $\epsilon$ scale with $\beta$?  

$$\frac{\epsilon}{\epsilon_0} = \left(\frac{\beta}{\beta_0}\right)^{\alpha_2}$$  (Find $\alpha_2$).

c). How does the bunch length $L$ scale with $\beta$?

d). How does the half-lattice period $L$ scale with $\beta$?

e). How does $\Delta p$ scale with $\beta$?
2. Calculate the current density $J$ for an extreme relativistic 1-D diode. (i.e. ignore the small region of the diode for which $\gamma \approx 1$ is not a good approximation.) Let the diode have length $d$, and voltage $V$, and the ion species have mass $m$ and charge $q$. Sketch $\log J$ vs $\log V$ for both a non-relativistic diode and an extreme relativistic diode. At what value of $qV/mc^2$ do the curves intersect?

(Hint: $N \cdot E = \frac{p}{\varepsilon_0}$ and $\frac{dp}{dt} + \nabla \cdot J = 0$ are both relativistically correct equations.)
Let the single particle equation of motion be:
\[
\frac{d^2 x}{ds^2} = -k^2 x
\]
Here \( k \) is a constant, \( x \) is the usual transverse coordinate, and \( s \) is the longitudinal coordinate.
Let the initial value of \( \langle x^2 \rangle = \langle x_0^2 \rangle \).
What are the values of \( \langle x_0 \cdot x_0' \rangle \) and \( \langle x_0^2 \rangle \) for which \( \frac{d}{ds} \langle x^2 \rangle \), \( \frac{d}{ds} \langle xx' \rangle \), and \( \frac{d}{ds} \langle x^2 \rangle \) are all zero?
Here \(< >\) denotes average over the distribution function and subscript 0 indicates initial value.
(These are the conditions for a matched beam.)
The diagram below shows a mismatched beam envelope with three particle orbits A, B, and C. All particles are propagating to the right. Which particle will have an orbit with increasingly larger amplitude? Which orbit will stay fixed in amplitude? Which particle will have an orbit of decreasing amplitude? Explain your reasoning.
For a continuous focusing channel with
\[ \rho_x = \rho_y = \frac{e \rho_0}{\omega_p} = \text{const} \]
and a round, "matched" KV equilibrium beam with
\[ A_+ (H_\perp) = \frac{n_1^2}{2 \pi} S (H_\perp - H_b) \]

where we have:
\[ H_\perp = \frac{1}{\varepsilon} (x^2 + y^2) + \frac{e \rho_0}{2} (x^2 + y^2) \frac{\rho_0}{\varepsilon} \varepsilon \]
\[ = \frac{1}{\varepsilon} (x^2 + y^2) + \frac{e \rho_0}{2} \frac{\rho_0}{\varepsilon} \varepsilon \]
and
\[ \frac{e \rho_0}{2} \frac{\rho_0}{\varepsilon} \varepsilon - \frac{Q}{\Gamma_b} - \frac{E_x^2}{2 \Gamma_b^2} = 0 \]
\[ H_b = \frac{E_x^2}{2 \Gamma_b^2} \]

Within the beam core \( (0 \leq r \leq \Gamma_b) \) the local kinetic temperature is:
\[ \text{Temp} \propto \frac{\langle x^2 y^2 \rangle}{\langle x^2 \rangle} = \frac{\int d^2 \mathbf{r} \cdot x^2 y^2 A_+ (H_\perp)}{\int d^2 \mathbf{r} \cdot A_+ (H_\perp)} \]

a) Argue (symmetry)
\[ \langle x^2 \rangle_x = \frac{1}{2} \langle x^2 + y^2 \rangle_x = \frac{1}{2} \frac{\langle x^2 \rangle_y}{\langle x^2 \rangle_x} \]

b) Calculate \( \langle x^2 \rangle_x \) within the beam core
Hint: Use results of previous problem. \( \int d^2 \mathbf{r} \cdot A_+ (H_\perp) = A \)

Steps given in class notes on angular integrals with cylindrical symmetry can be applied to easily calculate.
Same methods also used in end of Appendix B.

c) What is the value of \( \langle x^2 \rangle_x \) at the beam edge \( (r = \Gamma_b) \)?
Is this value consistent with what should be expected for a sharp beam edge? Why?
Problem - Courant Snyder Invariant

As derived in class, a coasting uniform density elliptical beam with \((\mathbf{A}_b, \mathbf{B}_b)' = \mathbf{0}\) has particle equations of motion in the beam given by:

\[
x'' + \frac{R_x(s)}{\Gamma_x} x' - \frac{Z_0 x}{\Gamma_x} = 0
\]

\[
y'' + \frac{R_y(s)}{\Gamma_y} y' - \frac{Z_0 y}{\Gamma_y} = 0
\]

where \(\Gamma_x\) and \(\Gamma_y\) obey the envelope equations:

\[
\Gamma_x'' + \frac{R_x(s)}{\Gamma_x} \Gamma_x - \frac{Z_0}{\Gamma_x} - \frac{E_x^2}{\Gamma_x^3} = 0
\]

\[
\Gamma_y'' + \frac{R_y(s)}{\Gamma_y} \Gamma_y - \frac{Z_0}{\Gamma_y} - \frac{E_y^2}{\Gamma_y^3} = 0
\]

with

\[E_x = \text{const}\]
\[E_y = \text{const}\]
\[Q = \frac{g\lambda}{21780 m \mathbf{B}_b^2}\]

\[R_x, R_y = x-\text{ and } y-\text{ focusing forces (specified in } s)\]

1. Take a "Phase-Amplitude" form of the particle

   - Orbit with
     \[x = A(s) \cos \psi(s)\]

   A solution of this form is known to exist by Floquet's theorem. Taking this for granted, show that the \(x\) equation of motion is then equivalent to two equations:
\[ A_x'' + \frac{v}{\ell_x} A_x - \frac{zQ}{\ell_x^2} A_x - A_x \psi_x'' = 0 \]  
\[-(1)\]

\[ A_x \psi_x'' + 2A_x \psi_x' = 0 \]  
\[-(2)\]

B/ Show that Eq. (2) in part A/ has a solution

\[ \psi_x' = \frac{C}{A_x^2} \quad C = \text{const.} \]

and show that if we take

\[ C = \frac{Q^2}{\ell_x} \quad Q = \text{const} \]

\[ A_x = \frac{Q}{\ell_x} \quad (\text{dimensionless amplitude}) \]

that the particle orbit is consistent with the x-equation for the beam envelope:

\[ \frac{\ell_x''}{\ell_x + \ell_y} + \frac{\ell_x'^2}{\ell_x^3} - \frac{2Q}{\ell_x} - \frac{\ell_x^2}{\ell_x^3} = 0 \]

C/ From the results of part B/, the particle orbit in the beam can be expressed as:

\[ x = a \ell_x \cos \psi_x \]

Show that the particle orbit has a single-particle invariant of the form:

\[ \left( \frac{x}{\ell_x} \right)^2 + \left( \frac{\ell_x x' - \ell_x' x}{\ell_x^2} \right)^2 = a^2 = \text{const.} \]
TED Problem 8

This Courant-Snyder invariant is the equation of an ellipse in x-x' phase-space.

D1. Note that $\Psi_x$ satisfies:

$$\Psi_x' = \frac{c}{A_x^2} = \frac{E_x}{f_x^2}$$

independent of $a$. Thus $\Psi_x$ is independent of particle amplitude and we expect the amplitudes of particle orbits of the uniform density beam to be uniformly distributed with $0 \leq a \leq 1$. Use this and the invariant in part C to show that the maximum particle orbits define an ellipse with

$$A_{\text{max}} = \pi E_x$$

in x-x' phase space.

Hint: The rotated ellipse:

$$ax^2 + 2\alpha x x' + \beta x'^2 = 1$$

has area

$$A_{\text{area}} = \pi \frac{1}{\sqrt{8\beta - a^2}}$$

These results reinforce that the statistical emittance

$$E_x = \frac{\pi}{4} \left[ \langle x^2 \rangle + \langle x'^2 \rangle - \langle xx' \rangle^2 \right]^{1/2}$$

is $\pi x$ the x-x' phase-space area of the maximum particle orbits in a kV beam and that all particles move on nested ellipses in x-x' phase-space.
Thermal Equilibrium

In 3D a thermal equilibrium can be constructed with \( \partial^2 / \partial t^2 = 0 \) (unbunched) that is a straightforward generalization of the cylindrical equilibrium presented in class. There is essentially one additional Gaussian integral over the longitudinal beam-frame momentum.

Assume a nonrelativistic beam \((\gamma_0 = 1)\). Following the procedure in class, a test charge \( q \) is placed at the origin of the 3D thermal equilibrium beam. With analogous approximations a 3D Poisson equation valid in the beam core can be derived:

\[
\nabla^2 \phi - \frac{\phi}{\lambda_0^2} = -\frac{q}{\varepsilon_0} \delta(x)
\]

where:

\[
\delta(x) = \delta(x) \delta(y) \delta(z)
\]

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

Show that this equation has a solution regular at infinity satisfying

\[
\phi = \frac{q}{4\pi \varepsilon_0} \frac{-1}{\gamma} e^{-r / \lambda_0}
\]

\[
\gamma = \sqrt{x^2 + y^2 + z^2}
\]

Hints:
1) Matching near and far solutions as in class notes
2) In 3D spherically symmetric geometry

\[
\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right)
\]

\[
\gamma = \sqrt{x^2 + y^2 + z^2}
\]

3) Try transforming the equation using \( \phi = \phi(r) / \gamma \) and solving for \( \phi \).
Axisymmetric Envelope Equation

Take

\[ X = 0 = Y \quad \text{Zero centroid} \quad \therefore E_x = E_y = E \]

\[ \Gamma_x = \Gamma_y = \Gamma_b \quad \text{Round beam} \quad \text{equal emittances} \]

\[ k_x = k_y = k \rho_0 = \text{const} \quad \text{Cont. Focusing} \]

and a uniform density beam of circular cross-section in a cylindrical pipe of radius \( \Gamma_p > \Gamma_b \).

A/ Calculate \( \frac{\partial \Phi}{\partial t} \) inside the beam and show that the \( x \)-
particle equation of motion is:

\[ X'' + \left( \frac{\rho_0 \Gamma_b}{\sqrt{\rho_0}} \right) X' + \frac{k \rho_0}{\Gamma_b} X = -Q X = 0 \]

\[ Q = \frac{M}{\Gamma_b \rho_0 m \lambda \sqrt{\lambda^2 - \lambda_0^2}} \quad \lambda = 2 \pi \Gamma_b \rho_0 \text{ const.} \]

B/ Parallel steps in class to derive the envelope equation:

\[ \Gamma''_b + \left( \frac{\rho_0 \Gamma_b}{\sqrt{\rho_0}} \right) \Gamma'_b + \frac{k \rho_0 \Gamma_b}{\Gamma_b} = -Q - \frac{E_x^2}{\Gamma_b} = 0 \]

where

\[ E_x = 4 \left[ \langle x^2 \rangle \langle x' x' \rangle - \langle x x' \rangle^2 \right] \frac{1}{\sqrt{2}} \]

Use steps analogous to those to lecture note on "Transverse Envelope Descriptions".

C/ For a non-uniform density axisymmetric beam with \( \rho = \rho(r) \) the
the particle equation of motion becomes:

\[ X'' + \left( \frac{\rho_0 \Gamma_b}{\sqrt{\rho_0}} \right) X' + \frac{k \rho_0}{\Gamma_b} X = -g \frac{\partial \rho}{\partial \Gamma} \frac{\rho \partial^2 x}{m \sqrt{\rho_0} \Gamma_b^3 \rho_0} \frac{\partial \rho}{\partial x} \]

Show that the envelope equation is now:
\[
\frac{\phi''}{\phi_0} + (\phi_0 \beta_0)' \frac{\phi'}{\phi_0} + \kappa_0 \phi + \frac{\hbar c}{m \phi_0 \beta_0^2} c^2 \frac{\phi'}{\beta_0} = -\frac{e_x}{\beta_0^3} = 0
\]

where
\[
\phi \equiv 2 \langle x^2 \rangle^{1/2}, \quad \langle x^2 \rangle = \frac{\int_0^\infty r^3 p(r) dr}{\int_0^\infty r^2 p(r) dr}
\]
\[p(r) = \text{beam charge density}.
\]

In earlier problem sets you showed that:
\[
\langle x^2 \phi \rangle = -\frac{\lambda}{8 \pi \epsilon_0} \quad \lambda = 2\pi \int_0^\infty dr r p(r)
\]
\[= \text{const.}
\]

So this results in the same statistical envelope equation as in part B/ with \(Q\) defined by \(\lambda\).

D/ Take: \(\beta_0 \beta_0 = \text{const.}\) and
\[
\phi_0(s) = \phi_0 + \delta \phi_0 c \quad |\delta \phi_0| \ll \phi_0
\]
\[\text{const.} \quad \text{const.}
\]
\[\delta \phi = \text{const.}
\]
and require that \(\phi_0\) satisfy the envelope equation with \(\delta \phi_0 = 0\). Then require that the form above satisfy the envelope equation to linear order in \(\delta \phi_0\). Show that for nontrivial solutions
\[
\delta x^2 = 2 \phi_0 + 2 \kappa \phi
\]
where
\[
\kappa \phi \equiv \frac{\phi^2}{\phi_0} - \frac{Q}{\phi_0} = \text{depressed } \beta\text{-tron wavenumber}
\]
\[\kappa = \frac{\delta \phi}{\phi_0} \]
\[\delta x = \kappa \phi_0 \sqrt{2 + 2(\delta \phi_0)^2}
\]