LUND/BARNARD
USPAS 2015

FINAL EXAM

(1) CONSIDER A HEAVY ION ACCELERATOR WITH A CONSTANT AVERAGE BEAM RADIUS r_b, CONSTANT PIPE RADIUS r_p, CONSTANT UNDEPRESSED PHASE ADVANCE σ_0, AND CONSTANT MAGNETIC QUADRUPOLE GRADIENT (B' IN JOHN'S NOTATION $\equiv G$ IN STEVE'S NOTATION). (YOU MAY ASSUME THE BEAM IS NON-RELATIVISTIC; THE OCCUPANCY η OF THE QUADRUPOLES IS CONSTANT. ALSO YOU MAY USE THE THIN LENS, SMALL σ_0 APPROXIMATION TO CALCULATE THE SCALING OF σ_0.* THE CURRENT CAN BE CONSIDERED CONSTANT OVER THE PULSE LENGTH, BUT VARIES AS A FUNCTION OF POSITION s THROUGH THE ACCELERATOR. ASSUME THE BEAM IS SPACE-CHARGE DOMINATED AND IS BEING COMPRESSED AND ACCELERATED SO THAT IT IS AT THE MAXIMUM TRANSPORTABLE CURRENT. ALSO, ASSUME THE NORMALIZED LONGITUDINAL AND TRANSVERSE EMITTANCES ARE CONSTANT.

$*\sigma_0 = \eta L^2 B/(r_p[B\rho]) \equiv \eta L^2 B'/[B\rho]$ for thin lens, small σ_0 approximation

a). HOW DOES THE CURRENT I SCALE with velocity/c β?

(i.e. WHAT IS THE EXPONENT α IN THE RELATION

$$\frac{I}{I_0} = \left(\frac{\beta}{\beta_0}\right)^{\alpha_1}$$

$I_0 \equiv$ initial CURRENT

$\beta_0 \equiv$ initial velocity/c.

b). How does the emittance ϵ scale with β?

$$\frac{\epsilon}{\epsilon_0} = \left(\frac{\beta}{\beta_0}\right)^{\alpha_2}$$

(find α_2).

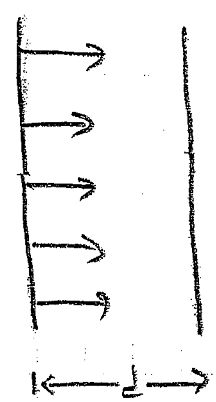
c). How does the bunch length l_b scale with β?

d). How does the half-lattice period L scale with β?

e). How does $\frac{\Delta p}{p}$ scale with β?

(2) -CALCULATE THE CURRENT DENSITY J FOR AN EXTREME RELATIVISTIC 1-D DIODE. (i.e. ignore the small region of the diode for which $\beta \cong c$ IS NOT A GOOD APPROXIMATION.) Assume space-charge limited emission. LET THE DIODE HAVE LENGTH d, AND voltage V, AND THE ION SPECIES HAVE MASS m AND CHARGE q. Sketch $\log J$ vs $\log V$.

FOR BOTH A NON-RELATIVISTIC DIODE AND AN EXTREME RELATIVISTIC DIODE. AT WHAT VALUE OF qV/mc^2 DO THE CURVES INTERSECT?



(HINT: $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ & $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$ are both RELATIVISTICALLY CORRECT EQUATIONS).

(3). LET THE SINGLE PARTICLE EQUATION OF MOTION BE:

$$\frac{d^2x}{ds^2} = -k^2 x$$

HERE k is a constant, x is the usual transverse coordinate, and s is the longitudinal coordinate.

~~Let~~ the initial value of $\langle x^2 \rangle = \langle x_0^2 \rangle$.

What are the values of $\langle x_0 x_0' \rangle$ and $\langle x_0'^2 \rangle$ for which $\frac{d}{ds}\langle x^2 \rangle$, $\frac{d}{ds}\langle x x' \rangle$, and $\frac{d}{ds}\langle x'^2 \rangle$

are all zero?

HERE $\langle \; \rangle$ denotes average over the distribution function and subscript 0 indicates initial value.

(THESE ARE THE CONDITIONS FOR A MATCHED BEAM).

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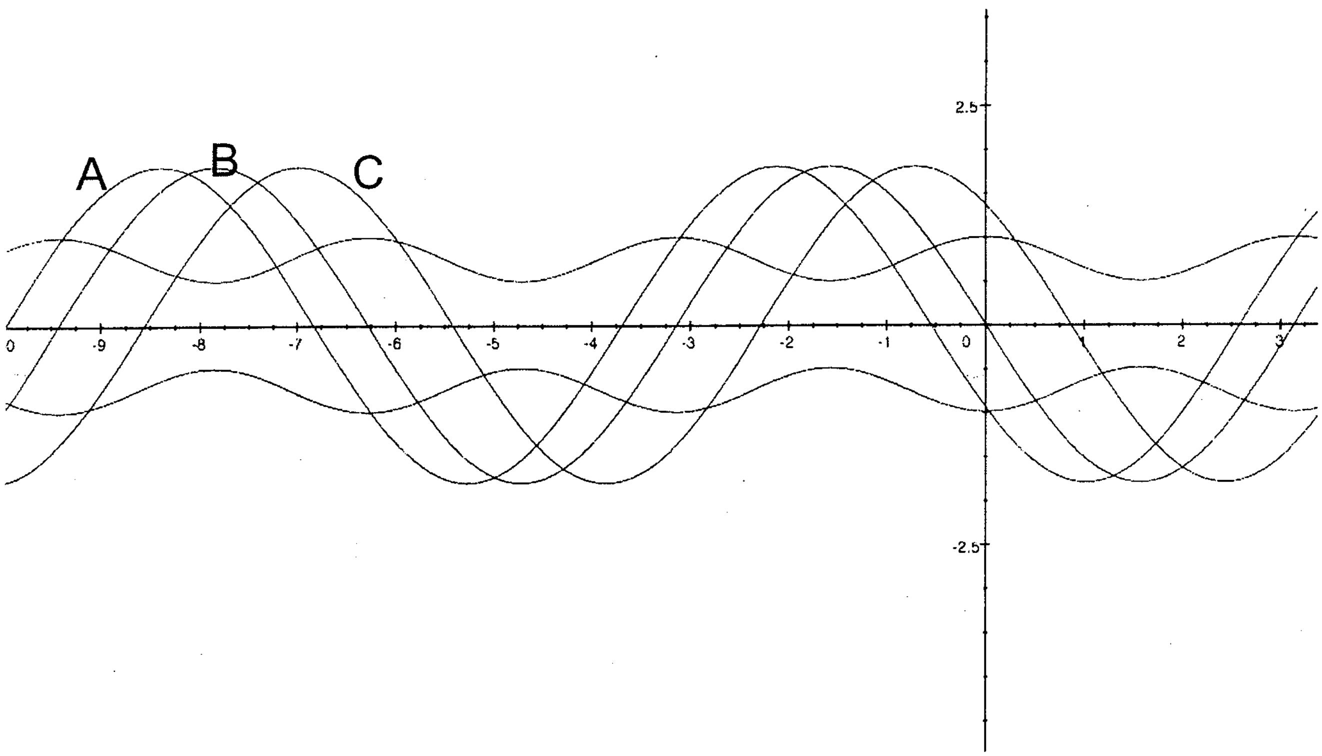
The diagram below shows a mismatched beam envelope with three particle orbits A, B, and C. All particles are propagating to the right. Which particle will have an orbit with increasingly larger amplitude? Which orbit will stay fixed in amplitude? Which particle will have an orbit of decreasing amplitude? Explain your reasoning.



TED Problem 7

S.M. Lund

(5) For a continuous focusing channel with

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$$

and a round, "matched" KV equilibrium beam with

$$f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\,\delta(H_\perp - H_b)$$

where we have:

$$H_\perp = \frac{1}{2}(x'^2 + y'^2) + \frac{k_{\beta 0}^2}{2}(x^2+y^2) + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$= \frac{1}{2}(x'^2+y'^2) + \frac{\varepsilon_x^2}{2r_b^4}(x^2+y^2)$$

and

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

$$H_b = \frac{\varepsilon_x^2}{2r_b^2}$$

Within the beam core $(0 \le r < r_b)$ the local kinetic temperature is:

$$\text{Temp} \propto \langle x'^2\rangle_{\vec{x}_\perp'} \equiv \frac{\int d^2x'\, x'^2 f_\perp(H_\perp)}{\int d^2x'\, f_\perp(H_\perp)}$$

a) Argue (symmetry)

$$\langle x'^2\rangle_{\vec{x}_\perp'} = \frac{1}{2}\langle x'^2+y'^2\rangle_{\vec{x}_\perp'} = \frac{1}{2}\langle \vec{x}_\perp'^2\rangle_{\vec{x}_\perp'}$$

b) Calculate $\langle x'^2\rangle_{\vec{x}_\perp'}$ within the beam core

Hints:
a) Use results of previous problem: $\int d^2x' f_\perp(H_\perp) = \hat{n}$
b) Use step a)
c) Steps given in class notes on angular integrals with cylindrical symmetry can be applied to easily calculate. Same methods also used in end of Appendix B.

c) What is the value of $\langle x'^2\rangle_{\vec{x}_\perp'}$ at the beam edge $(r = r_b)$? Is this value consistent with what should be expected for a sharp beam edge? Why?

(6) **Problem - Courant Snyder Invariant**

As derived in class, a coasting uniform density elliptical beam with $(\gamma_b \beta_b)' = 0$ has particle equations of motion in the beam given by:

$$x'' + \kappa_x(s)\, x - \frac{2Q\, x}{(r_x + r_y)\, r_x} = 0$$

$$y'' + \kappa_y(s)\, y - \frac{2Q\, y}{(r_x + r_y)\, r_y} = 0$$

where r_x and r_y obey the envelope equations:

$$r_x'' + \kappa_x(s)\, r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y(s)\, r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

with

$\varepsilon_x = \text{const}$

$\varepsilon_y = \text{const}$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{Perveance} = \text{const.}$$

κ_x, κ_y = x- and y-focusing forces (specified in s)

A/ Take a "Phase-Amplitude" form of the particle x-orbit with

$$x = A_x(s) \cos \psi_x(s)$$

A solution of this form is known to exist by Floquet's theorem. Taking this for granted, show that the x equation of motion is then equivalent to two equations:

$$A_x'' + \kappa_x A_x - \frac{2Q A_x}{(r_x+r_y) r_x} - A_x \psi_x'^2 = 0 \qquad -(1)$$

$$A_x \psi_x'' + 2 A_x' \psi_x' = 0 \qquad -(2)$$

B/ Show that Eq. (2) in part A/ has a solution

$$\psi_x' = \frac{C}{A_x^2} \qquad C = \text{const.}$$

and show that if we take

$$C = a^2 \varepsilon_x \qquad a = \text{const}$$

$$A_x = a\, r_x \qquad \text{(dimensionless amplitude)}$$

that the particle orbit is consistent with the x-equation for the beam envelope:

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

C/ From the results of part B/, the particle orbit in the beam can be expressed as:

$$x = a\, r_x \cos \psi_x$$

Show that the particle orbit has a single-particle invariant of the form:

$$\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 = a^2 = \text{const.}$$

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This Courant-Snyder invariant is the equation of an ellipse in x-x' phase-space.

D/ Note that ψ_x satisfies:

$$\psi_x' = \frac{C}{A_x^2} = \frac{\varepsilon_x}{r_x^2}$$

independent of a. Thus ψ_x is independent of particle amplitude and we expect the amplitudes of particle orbits of the uniform density beam to be uniformly distributed with $0 \leq a \leq 1$. Use this and the invariant in part C to show that the maximum particle orbits define an ellipse with

$$\text{Area} = \pi \varepsilon_x$$

in x-x' phase space.

Hint: The rotated ellipse:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 1$$

has area

$$\text{Area} = \frac{\pi}{\sqrt{\gamma\beta - \alpha^2}},$$

These results reinforce that the statistical emittance

$$\varepsilon_x = 4\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle x x'\rangle_\perp^2\right]^{1/2}$$

is π x the x-x' phase-space area of the maximum particle orbits in a KV beam and that all particles move on nested ellipses in x-x' phase-space.

(7) In 3D a thermal equilibrium can be constructed with $\partial/\partial z = 0$ (unbunched) that is a straightforward generalization of the cylindrical equilibrium presented in class. There is essentially one additional Gaussian integral over the longitudinal beam-frame momentum.

Assume a nonrelativistic beam ($\gamma_b = 1$). Following the procedure in class, a test charge q_T is placed at the origin of the 3D thermal equilibrium beam. With analogous approximations a 3D Poisson equation valid in the beam core can be derived:

$$\nabla^2 \delta\phi - \frac{\delta\phi}{\lambda_D^2} = -\frac{q_T}{\varepsilon_0}\delta(\vec{x})$$

where:

$$\delta(\vec{x}) = \delta(x)\delta(y)\delta(z)$$

$$\nabla^2 = \frac{\partial}{\partial\vec{x}}\cdot\frac{\partial}{\partial\vec{x}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Show that this equation has a solution regular at infinity satisfying

$$\delta\phi = \frac{q_T}{4\pi\varepsilon_0 r} e^{-r/\lambda_D} \qquad r = \sqrt{x^2+y^2+z^2}$$

Hints: You can construct the solution by
1) matching near and far solutions as in class notes — see: Transverse Equilibrium Distribution Notes.
2) In 3D spherically symmetric geometry
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \qquad r = \sqrt{x^2+y^2+z^2}$$
3) Try transforming the equation using
$\delta\phi = \tilde{\phi}/r$ and solving for $\tilde{\phi}$.

(8) Axisymmetric Envelope Equation

Take

$$X = 0 = Y \qquad \text{Zero centroid} \qquad ; \quad \varepsilon_x = \varepsilon_y = \varepsilon \quad \text{equal emittances}$$

$$r_x = r_y \equiv r_b \qquad \text{Round beam}$$

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const} \qquad \text{Cont. Focusing}$$

and a uniform density beam of circular cross-section in a cylindrical pipe of radius $r_p > r_b$.

A/ Calculate $\frac{\partial \phi}{\partial x}$ inside the beam and show that the x-particle equation of motion is:

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + k_{\beta 0}^2 x - \frac{Q}{r_b^2} x = 0$$

$$Q \equiv \frac{q\lambda}{2\pi\varepsilon_0 m \gamma_b^3 \beta_b^2 c^2} \quad ; \quad \lambda \equiv q \hat{n} \pi r_b^2 = \text{const.}$$

B/ Parallel steps in class to derive the envelope equation

$$r_b'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r_b' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

where

$$\varepsilon_x \equiv 4\left[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle x x' \rangle_\perp^2\right]^{1/2}$$

Use steps analogous to those to lecture notes in "Transverse Envelope Descriptions"

C/ For a non-uniform density axisymmetric beam with $\phi = \phi(r)$ the the particle equation of motion becomes:

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + k_{\beta 0}^2 x = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

show that the envelope equation is now:

$$r_b'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)'} r_b' + k_{\beta 0}^2 r_b + \frac{4q \left\langle x \frac{\partial\phi}{\partial x} \right\rangle_\perp}{m\gamma_b^3 \beta_b^2 c^2 r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

where

$$r_b \equiv 2\langle x^2\rangle^{1/2} \qquad \langle x^2\rangle = \frac{\int_0^{r_b} dr\, r^3 \rho(r)}{\int_0^{r_b} dr\, r\, \rho(r)}$$

$\rho(r)$ = beam charge density.

In earlier problem sets you showed that:

$$\left\langle x \frac{\partial\phi}{\partial x} \right\rangle_\perp = \frac{-\lambda}{8\pi\varepsilon_0} \qquad \lambda \equiv 2\pi \int_0^{r_b} dr\, r\, \rho(r) = \text{const.}$$

So this results in the same statistical envelope equation as in part B/ with Q defined by λ.

D/ Take: $\gamma_b\beta_b = \text{const}$ and

$$r_b(s) = \underset{\text{const.}}{r_{b0}} + \underset{\text{const.}}{\delta r_b}\, e^{iks} \qquad |\delta r_b| \ll r_{b0} \qquad k = \text{const.}$$

and require that r_{b0} satisfy the envelope equation with $\delta r_b = 0$. Then require that the form above satisfy the envelope equation to linear order in δr_b. Show that for nontrivial solutions

$$k^2 = 2k_{\beta 0}^2 + 2k_\beta^2$$

where

$$k_\beta^2 \equiv k_{\beta 0}^2 - \frac{Q}{r_{b0}^2} \equiv \text{depressed } \beta\text{-tron wavenumber}$$

—or—

$$k = k_{\beta 0}\sqrt{2 + 2(\sigma/\sigma_0)^2} \qquad \frac{\sigma}{\sigma_0} = \frac{k_\beta}{k_{\beta 0}}$$