I. Introduction
(related reading in parentheses)

Particle motion (Reiser 2.1)
Equation of motion (Reiser 2.1)
Dimensionless quantities (Reiser 4.2)

Plasma physics of beams (Reiser 3.2, 4.1)

Emittance and brightness (Reiser 3.1 - 3.2)
How do we describe and calculate the evolution of a collection of particles under the EM forces in an accelerator?

This array or "lattice" of focusing elements may be arranged in a linac or circular accelerator.
Consider the Lorentz force on a particle (mass \(m\), charge \(q\), momentum \(p\), velocity \(\nu = c\beta\), Lorentz factor \(\gamma\)) under the influence of an electric \((E)\) and magnetic field \((B)\):\
\[
\frac{dp}{dt} = q(E + \nu \times B) \quad \text{(SI units employed throughout)}
\]

\[
p = \gamma mv \quad \quad \gamma^2 = \frac{1}{1 - \beta^2} \quad \beta = \frac{\nu}{c}
\]

Consider the \(x\)-component of the motion (transverse to the streaming direction). \(s\) is the coordinate of the "design" (ideal) orbit (equivalent to \(z\) for a linear accelerator) and subscripts "comoving" indicate coordinates comoving with the design particle. We may transform to \(s\) as the independent variable:

\[
dt = \frac{ds}{v_z}; \quad \nu_x = \frac{dx}{dt} = v_z x' \quad \text{where prime}' = \frac{d}{ds}
\]

\[
v_z \frac{d}{ds} \left(\gamma mv_z x'\right) = q(E + \nu \times B)_x
\]

\[
\gamma mv_z^2 x'' + x' mv_z \frac{d(\gamma v_z)}{ds} = q(E + \nu \times B)_x
\]

\[
\Rightarrow x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds}\right] x' = \frac{q}{\gamma mv_z^2} (E + \nu \times B)_x
\]
Now consider an unbunched beam of uniform charge density $\rho$ and circular cross section, with radius $r_b$

Line charge density $\lambda = \pi r_b^2 \rho$

First calculate electric field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$2\pi r E_r = \pi r^2 \frac{\rho}{\varepsilon_0} \quad \text{(Gauss theorem)}$$

$$\Rightarrow E_r = \frac{\rho}{2\varepsilon_0} r = \frac{\lambda}{2\pi \varepsilon_0} r_b^2$$

$$E_x = E_r \cos \theta = \frac{\lambda}{2\pi \varepsilon_0} \frac{x}{r_b^2}$$

Similarly, calculate the magnetic field:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$2\pi B_\theta = \mu_0 \pi r^2 \rho v_z \quad \text{(Stokes theorem)}$$

$$\Rightarrow B_\theta = \mu_0 \frac{\lambda v_z}{2\pi \varepsilon_0} \frac{r}{r_b^2}$$

$$B_y = B_\theta \cos \theta = \mu_0 \frac{\lambda v_z}{2\pi} \frac{x}{r_b^2}$$

($B_z = 0$)

Let $$(\mathbf{E} + v \times \mathbf{B})_x = (E_x - v_z B_y)_\text{self} + (E_x + v_y B_z - v_z B_y)_\text{ext}$$

$$\Rightarrow x'' + \left[ \frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = -\frac{q}{\gamma mv_z^2} \frac{\lambda}{2\pi \varepsilon_0} \frac{x}{r_b^2} \left[ 1 - \frac{\mu_0 \varepsilon_0 v_z^2}{\gamma m v_z^2} \right] + \frac{q}{\gamma mv_z^2} (E_x + v_y B_z - v_z B_y)_\text{ext}$$

Now $\mu_0 \varepsilon_0 = \frac{1}{c^2}$; Assuming $\beta_x^2 + \beta_y^2 << \frac{1}{\gamma^2} \Rightarrow \gamma^2 \approx \frac{1}{1 - v_z^2 / c^2}$ (Paraxial approximation)

($\gamma^2 \approx 1/(1 - v_z^2 / c^2)$ equivalent to assuming $\beta_x^{\text{comoving}}, \beta_y^{\text{comoving}} << 1$).

$$\Rightarrow x'' + \left[ \frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = -\frac{q}{\gamma^3 mv_z^2} \frac{\lambda}{2\pi \varepsilon_0} \frac{x}{r_b^2} \left[ 1 - \frac{\mu_0 \varepsilon_0 v_z^2}{\gamma m v_z^2} \right] + \frac{q}{\gamma mv_z^2} (E_x + v_y B_z - v_z B_y)_\text{ext}$$
First consider the self-field.

\[
x'' + \left[ \frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = \frac{q}{\gamma^3 m v_z^2} \frac{\lambda}{2 \pi \epsilon_0} \frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2} (E_x + v_y B_z - v_z B_y)^\text{ext}
\]

\[
= Q \frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2} (E_x + v_y B_z - v_z B_y)^\text{ext}
\]

\[
Q = \frac{q}{\gamma^3 m v_z^2} \frac{\lambda}{2 \pi \epsilon_0} = \text{Generalized Perveance} \rightarrow \begin{cases} \frac{\lambda}{4 \pi \epsilon_0 V} & \text{for } \gamma^2 v_z^2 \ll c^2 \\ \frac{\lambda}{2 \pi \epsilon_0 V \left( \frac{q V}{m c^2} \right)^2} & \text{for } \gamma^2 v_z^2 >> c^2 \end{cases}
\]

\[
= \frac{(q/e)}{m/m_{\text{amu}}} \frac{2 I}{I_0} \frac{1}{\gamma^3 \beta^3} \quad \text{where} \quad I_0 = \frac{4 \pi \epsilon_0 m_{\text{amu}} c^3}{e} \approx 31 \text{ MA}
\]

Here \(qV=(\gamma-1)mc^2\) = ion kinetic energy, 
\(e\) is the proton charge, and \(m_{\text{amu}}\) is the atomic mass unit.
Also note in the non-relativistic limit:

\[
Q = \frac{1}{4 \pi \epsilon_0} \left( \frac{m}{2q} \right)^{1/2} \left( \frac{I}{V^{3/2}} \right) \quad \text{(non-relativistic)}
\]

(same scaling as original term "perveance" characterizing injectors)

\[
Q \equiv \frac{\phi_{\text{self}}}{V} = \int_{r_0}^{r_b} (E_r - v_z B_\theta) dr \quad \text{Potential energy of beam particle} \quad \text{Kinetic energy of beam particle}
\]
Now consider the external field. We often try to create focusing forces that are linear in $x$ (examples are: electric or magnetic quadrupoles, solenoids, Einzel lenses.) So let this focusing force be represented by $K(s)$.

$$\frac{q}{\gamma mv_z^2}(E_x + v_y B_z - v_z B_y)^{ext} = K(s)x$$

$$x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds}\right]x' = \frac{Q}{r_b^2} x + \frac{q}{\gamma mv_z^2}(E_x + v_y B_z - v_z B_y)^{ext}$$

$$= \frac{Q}{r_b^2} x + K(s)x$$

The focusing forces are often periodic:

$K(s) = K(s + L_p)$ where $L_p$ = period of focusing element

(when $dv_z/ds = 0$, and $Q$ is periodic with period $L_p$, then:

$$x'' = f(s) x$$

where $f(s)$ is periodic. (Hill's equation).

For some purposes a suitable constant can be found which captures the "average" variation (over several periods) of the particle motion (continuous focusing approximation)

Then we replace the effects of the periodic lattice with a single focusing parameter $k_{\beta_0}^2$

$$x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds}\right]x' = \frac{Q}{r_b^2} x - k_{\beta_0}^2 x$$

$k_{\beta_0}$ is defined as the "undepressed" betatron frequency

J.J. Barnard and S.M. Lund
\[
\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \left[ x'' + \frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] = Q \frac{x}{r_b^2} - k_{\beta 0}^2 x
\]

Consider a drifting beam \((dv_z/ds = 0)\). The particle equation becomes:

\[
x'' = Q \frac{x}{r_b^2} - k_{\beta 0}^2 x
\]

\[
= -k_{\beta 0}^2 \left( 1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right) x
\]

This is simple harmonic oscillator equation.

Note some frequently encountered definitions:

\[
k_{\beta 0}^2 \left( 1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right) \equiv k_{\beta}^2 = \text{depressed betatron frequency}
\]

Define also

\[
\sigma_0 \equiv k_{\beta 0} L_p = \text{undepressed phase advance (per period)}
\]

and

\[
\sigma \equiv k_{\beta} L_p = \text{depressed phase advance (per period) (includes space charge)}
\]

\[
\frac{\sigma}{\sigma_0} = \frac{k_{\beta}}{k_{\beta 0}} = \left( 1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right) = \text{tune depression}
\]

Examples:

\[
\frac{\sigma}{\sigma_0} = 0 \quad \Rightarrow \quad \text{Fully tune depressed}
\]

\[
\frac{\sigma}{\sigma_0} = 1 \quad \Rightarrow \quad \text{No space charge depression}
\]

(so two dimensionless parameters: \(Q\) characterizes space charge relative to ion kinetic energy, \(\sigma/\sigma_0\) characterizes space charge force relative to focusing force)
Space charge reduces betatron phase advance

Without space charge:

\[ x = x_i \cos[k \beta_0 (s - s_i)] + \frac{x_i'}{k \beta_0} \sin[k \beta_0 (s - s_i)] \]

Particle orbit

With space charge:

\[ \sigma/\sigma_0 \approx \frac{5}{18} \approx 0.277 \]

\[ x = x_i \cos[k \beta_0 \frac{\sigma}{\sigma_0} (s - s_i)] + \frac{x_i'}{k \beta_0} \frac{\sigma}{\sigma_0} \sin[k \beta_0 \frac{\sigma}{\sigma_0} (s - s_i)] \]

Beam envelope
Space charge reduces betatron phase advance

Without space charge:

Focusing quads
Defocusing quads
Particle orbit

With space charge:

Beam envelope
Particle orbit
BENDING BEAMS

Returning to particle equation with externally \( E, q, B \):
\[
x'' + \left[ \frac{1}{YV_2} \frac{d}{ds} (YV_2) \right] x' = \frac{q}{YMV_2} \left( \varepsilon + \nu \times B \right)_x
\]

If external force is proportional to \(-x\),
\(\Rightarrow\) focusing (harmonic oscillations)

However, if \( E + \nu \times B = \text{constant} \),
\(\Rightarrow\) bending

Example: If \( B = B_y \hat{e}_y \)
\[
v = v_0 \hat{e}_z + v_x \hat{e}_x \quad \text{where} \quad v_0 \gg v_x
\]

\[
u\left[ x'' = \frac{qB_y}{YMV_2} = \frac{B_y}{[B_y]} \right] \quad \text{[B_y]} = \text{rigidity} = \frac{YMV_2}{q} = \frac{p}{q}
\]

\[
x' = \frac{B_y}{[B_y]} z^2 + x_0'
\]

\[
x = \frac{B_y}{[B_y]} z^2 + x_0' z + x_0
\]

\(\rho = \text{radius of curvature of arc} \quad \frac{[B_y]}{B_y}
\]

(Bending can also be carried out with electric fields \( E \))
Plasma physics of beams

Physics of space charge = physics of space charge

= plasma physics of particle beams

Plasma parameter $\Lambda$:

\[ q\phi_{IP} = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r_{IP}} \]

\[ = \frac{1}{4\pi\varepsilon_0} q^2 n_0^{1/3} \]

Average potential energy $q\phi_{IP}$ of particle due to its nearest neighbor a distance $r_{IP} = n_0^{-1/3}$

$q =$ charge of particle;


If $q\phi_{IP} \ll k_B T$ \[\Rightarrow\] "Weakly coupled plasma" or simply "plasma"

Define $\lambda_D = \frac{(k_B T / m)^{1/2}}{(q^2 n_0 / (\varepsilon_0 m))^{1/2}} \equiv \frac{v_t}{\omega_p} = \left(\frac{k_B T \varepsilon_0}{q^2 n_0}\right)^{1/2}$ = Debye Length

= characteristic distance whereby charges are shielded in plasma

Define $\Lambda \equiv \frac{4\pi}{3} n_0 \lambda_D^3 = \text{Plasma Parameter}$

\[ \sim \left(\frac{k_B T}{q\phi_{IP}}\right)^{1/2} \gg 1 \quad \text{[if $q\phi_{IP} \ll k_B T$]} \]
Klimontovich Equation

\[ N(x,v,t) = \sum_{i=1}^{N_0} \delta(x - X_i(t))\delta(v - V_i(t)) \]

\( N(x,v,t) \) is the density of particles in phase space.

Note there are \( N_0 \) particles:

\[ \int N(x,v,t) \ d^3x \ d^3v = N_0 \]

\( X_i(t) \) and \( V_i(t) \) are position and velocity of the \( i^{th} \) particle.

The (non-relativistic) equations of motion are:

\[ \dot{X}_i = V_i \]

\[ m\dot{V}_i = qE \left( X_i(t), t \right) + q \left[ V_i \times B \left( X_i(t), t \right) \right] \]

Let \( u = x - X_i(t) \) \( \Rightarrow \)

\[ \frac{\partial f(u)}{\partial x} = f'(u) \quad \text{and} \quad \frac{\partial f(u)}{\partial t} = -\dot{X}(t)f'(u) = -\dot{X}(t)\frac{\partial f(u)}{\partial x} \]

So taking the derivative of \( N(x,v,t) \) with respect to \( t \):

\[ \frac{\partial N}{\partial t}(x,v,t) = -\sum_{i=1}^{N_0} \dot{X}_i(t) \cdot \nabla_x [\delta(x - X_i(t))\delta(v - V_i(t))] \]

\[ -\sum_{i=1}^{N_0} \dot{V}_i(t) \cdot \nabla_v [\delta(x - X_i(t))\delta(v - V_i(t))] \]

Maxwell's equations:

\[ \nabla \cdot E^m = \frac{\rho^m}{\varepsilon_0} = \frac{1}{\varepsilon_0} q \int N(x,v,t) d^3v \quad \nabla \cdot B^m = 0 \]

\[ \nabla \times E^m = -\frac{\partial B^m}{\partial t} \]

\[ \nabla \times B^m = \mu_0 I^m + \frac{\partial E^m}{\partial t} = \mu_0 q \int \nu N(x,v,t) d^3v + \frac{\partial E^m}{\partial t} \]

(Here superscript "m" denotes "microscopic" quantity, not averaged locally over a small volume).
\[ \frac{\partial N}{\partial t}(x,v,t) = -\sum_{i=1}^{N_0} V_i(t) \cdot \nabla_x [\delta(x - X_i(t))\delta(v - V_i(t))] \]
\[ - \sum_{i=1}^{N_0} \left( \frac{q}{m} E^m(X_i(t),t) + \frac{q}{m} [V_i \times B^m(X_i(t),t)] \right) \cdot \nabla_v [\delta(x - X_i(t))\delta(v - V_i(t))] \]

Note that \( V_i(t)\delta(v - V_i(t)) = v\delta(v - V_i(t)) \) so

\[ \frac{\partial N}{\partial t}(x,v,t) = -v \cdot \nabla_x \sum_{i=1}^{N_0} \delta(x - X_i(t))\delta(v - V_i(t)) \]
\[ - \left( \frac{q}{m} E^m(x,t) + \frac{q}{m} [v \times B^m(x,t)] \right) \cdot \nabla_v \sum_{i=1}^{N_0} \delta(x - X_i(t))\delta(v - V_i(t)) \]

\[ \frac{\partial N}{\partial t}(x,v,t) = -v \cdot \nabla_x N(x,v,t) - \left( \frac{q}{m} E^m(x,t) + \frac{q}{m} [v \times B^m(x,t)] \right) \cdot \nabla_v N(x,v,t) \]

Note that the total derivative of a quantity along an orbit in phase space:
\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \left. \frac{dx}{dt} \right|_{\text{orbit}} \cdot \nabla_x + \left. \frac{dv}{dt} \right|_{\text{orbit}} \cdot \nabla_v \]
\[ \Rightarrow \frac{d}{dt} N(x,v,t) \bigg|_{\text{orbit}} = 0 \]

Note that \( N=0 \) or infinity, nothing in between!

Average \( N \) over some box in phase space. \( \Delta x, \Delta y \) are the dimensions of the box. Assume \( n^{-1/3} << \Delta x << \lambda_D \) so that \( f(x,v,t) \) is a smoothly varying function.

Now let \( f(x,v,t) = \frac{1}{\Delta x^3 \Delta y^3} \int N(x,v,t)d^3xd^3v = \langle N(x,v,t) \rangle \)

Then \( N = f + \delta f \) \[ f \equiv \langle N \rangle \quad \langle \delta f \rangle = 0 \]
\[ E^m = E + \delta E \quad E = \langle E^m \rangle \quad \langle \delta E \rangle = 0 \]
\[ B^m = B + \delta B \quad B = \langle B^m \rangle \quad \langle \delta B \rangle = 0 \]
\[ \frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{q}{m} \left( E + v \times B \right) \cdot \nabla_v f = -\frac{q}{m} \left( \delta E + v \times \delta B \right) \cdot \nabla_v \delta f \]

LHS: Smoothly varying part
RHS: Average over "rapidly fluctuating quantities", includes "discrete particle effects" or "collisions"

If collisions are neglected (so set RHS to zero): we have the "Vlasov Equation":
\[ \frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{q}{m} \left( E + v \times B \right) \cdot \nabla_v f = 0 \]

\[ \Rightarrow \frac{d}{dt} f(x,v,t)_{\text{orbit}} = 0 \]

Phase space density on trajectories is constant. (Liouville's theorem).
The RHS represents the effects of collisions (i.e. interactions with non-smoothly varying fields). Very heuristically:

\[-\frac{q}{m} \langle \delta E + \nu \times \delta B \rangle \cdot \nabla \nu \delta f \rangle \sim \nu_c f\]

\[\nu_c \sim \sigma \nu\]

\[\sigma \sim \pi \nu_c^2\]

where $\sigma$ is the collision cross section.

For a large angle scattering the kinetic energy of a particle will be of order the potential energy at closest approach, defining a collision radius by:

\[k_B T \sim \frac{q^2}{4 \pi \varepsilon_0 r_c} \Rightarrow r_c \sim \frac{q^2}{4 \pi \varepsilon_0 k_B T}\]

\[\Rightarrow \nu_c \sim \pi \left(\frac{q^2}{4 \pi \varepsilon_0 k_B T}\right)^2 n_0 \left(\frac{k_B T}{m}\right)\]

\[\sim \frac{1}{16 \pi} \frac{\nu_{th}}{\lambda_D^3 n_0}\]

On LHS of Vlasov Equation:

\[\frac{q}{m} E \cdot \nabla v f \sim \frac{q}{m} (\lambda_D \nabla \cdot E) \nabla v f \sim \frac{q}{m} \left(\frac{q \lambda_D n_0}{\varepsilon_0}\right) f \sim \frac{\omega_p^2 \lambda_D}{\nu_{th}^2} f \]

where $\nu_t \sim \left(\frac{k_B T}{m}\right)^{1/2}$

\[\sim \omega_p f\]

\[\text{Collision term} \sim \frac{1}{16 \pi \lambda_D^3 n_0} \sim \frac{1}{16 \Lambda}\]
Accelerator beams are non-neutral plasmas for Heavy Ion Fusion.
Phase space density conservation

Liouville's theorem: \( \frac{df}{dt} = 0 \) along a trajectory in phase space.

Let \( \text{d}N = f \, \text{d}x \, \text{d}y \, \text{d}z \, \text{d}p_x \, \text{d}p_y \, \text{d}p_z \)

The continuity equation in phase space is:

\[
\frac{\partial f}{\partial t} + \nabla_6 \cdot (f \, v_6) = 0
\]

where \( v_6 = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \)

and \( \nabla_6 \cdot a_6 = \frac{\partial a_1}{\partial q_1} + \frac{\partial a_2}{\partial q_2} + \frac{\partial a_3}{\partial q_3} + \frac{\partial a_4}{\partial p_1} + \frac{\partial a_5}{\partial p_2} + \frac{\partial a_6}{\partial p_3} \)

If the system is governed by a Hamiltonian \( H(q,p,t) \)

\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}
\]

Now \( \nabla_6 \cdot v_6 = \sum_{i=1}^{3} \left( \frac{\partial}{\partial q_i} \left( \frac{dq_i}{dt} \right) + \frac{\partial}{\partial p_i} \left( \frac{dp_i}{dt} \right) \right) = \sum_{i=1}^{3} \left( \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0 \)

\[
\Rightarrow \frac{df}{dt} + \nabla_6 \cdot (f \, v_6) = \frac{df}{dt} + f \, \nabla_6 \cdot v_6 + v_6 \cdot \nabla_6 f = 0
\]

\[
\Rightarrow \frac{df}{dt} = 0 \quad \text{along a 6D trajectory}
\]
**Emittance & Brightness**

- **Liouville's Equation of Motion Equation**
  \[
  \frac{\partial N}{\partial x} = \frac{\partial N}{\partial y} = \frac{\partial N}{\partial z} = \text{const}
  \]

  - If \( x'' = f(x) \) and not equations \( y'' = f(y) \) \( z'' = f(z) \)

  - Then
    \[
    \frac{dN}{dx \, dp_x} = \text{const} \quad \frac{dN}{dy \, dp_y} = \text{const} \quad \frac{dN}{dz \, dp_z} = \text{const}
    \]

  - **1st Definition:**
    Emittance: Use trace-space of all particles in a given slice of beam.

  ![](image)

  - Instead of \( p_x \), use \( x' = \frac{v_x}{v_z} \) (for non-accelerating parallel beam, \( x' \) proportional to momentum)

  \[
  \text{Emittance} = \frac{1}{\pi} \text{area of smallest ellipse which encloses all particles.}
  \]

  - **Trace-Space Definition**
    Intuitively, product of width in \( x \) times width in \( x' \) so it is essentially (within factor of \( \pi \)) = phase space area of beam.

  - **2nd Definition** involves statistical averages of 2nd order quantities (such as RMS).
    \[
    \varepsilon_x \equiv 4 \left( \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)^{1/2}
    \]

- For an upright uniform beam (in phase space),
  \[
  \varepsilon_x = \sqrt{\frac{v_x}{v_z} \varepsilon_{x,max}} = \frac{\text{Area}}{\pi}
  \]
**Normalized Emittance**

For a beam that is accelerating, return to \( x, p_x \) an

\[ p_x = y m v_x = \frac{y m v_x}{x'} \]

Again, assuming \( v \approx v_0 \)

\[ \epsilon_{nx} = \frac{4 y^2}{\sqrt{\left( \langle x^2 \rangle \langle p_x^2 \rangle - \langle x \rangle \langle p_x \rangle \right)^2}} \]

Since emittance is the average phase space area of beam

(Averaging over empty state) the emittance in general grows

as a beam filament (engulfing empty state).

**Brightness**

The density of particles in \((x,p)\) phase space is:

\[ \frac{dN}{dx dp} = f \]

**Microscopic Density**

**Define a Quantity** \( f \) which has the phase-space density in

an average sense

\[ f = \frac{\langle \frac{dN}{dx dp} \rangle}{\langle \langle x \rangle \langle p_x \rangle \rangle} = \frac{(I dt)/q}{\pi^2 \epsilon_{nx} \epsilon_{ny} E_n} \]

Note \( f(x,p) \) is constant along a trajectory, whereas \( f \) usually is

a decreasing function of \( z \).

**Normalized Brightness** \( B_n \equiv \frac{I}{\epsilon_{nx} \epsilon_{ny}} \)

is a useful measure of \( 4D \) average phase space density,

(If \( at = \text{constant} \), \( f \) still motion is uncoupled.)

For non-accelerating beams, the unnormalized brightness \( B \) (also if \( at = \text{constant} \), \( f \) still motion uncoupled)

\[ B_n \equiv \frac{I}{\epsilon_{nx} \epsilon_{ny}} \text{measures phase space density}. \]
Emittance is constant for linear force profiles and matched beams

Linear force profile \((x'' = -k^2 x)\) \(\Rightarrow\) Phase space area preserved, ellipse stays elliptical.

Non-linear forces (e.g. \(x'' = -k^2 x + \varepsilon x^3\)) \(\Rightarrow\) position-dependent frequency
\(\Rightarrow\) phase mixing, increasing effective area \(\Rightarrow\) Emittance increases if forces non-linear