Efficient Computation of Matched Solutions of the KV Envelope Equations for Periodic Focusing Lattices*

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Conventional root-finding methods of solving the KV envelope equations often require *a priori* knowledge of initial conditions.

**Syncopated Quadrupole Lattice**

\[ L_p = 0.5 \text{ m}, \; \eta = 0.5, \; \alpha = 0.1, \; \sigma_0 = 80^\circ, \; Q = 4 \times 10^{-4}, \; \varepsilon = 50 \text{ mm-mrad} \]

Actual initial conditions:

\[ r_{xi} = r_{yi} = 7.71 \text{ mm} \]
\[ r_{xi}' = -r_{yi}' = 0.0186 \]

Incorrect IC's leading to non-matched solutions

\[ r_{xi} = 9 \text{ mm}, \; r_{yi} = 6 \text{ mm} \]
\[ r_{xi}' = 0.016, \; r_{yi}' = -0.02 \]
New iterative numerical method converges rapidly to matched solution without prior knowledge of initial conditions.

iteration 0

Iteration 2

Iteration 1

\[
\begin{align*}
L_p &= 0.5 \text{ m} \\
\eta &= 0.5 \\
\alpha &= 0.1 \\
\sigma_0 &= 80^\circ \\
\sigma / \sigma_0 &= 0.3 \\
\varepsilon &= 50 \text{ mm-mrad}
\end{align*}
\]
Introduction: New Iterative Numerical Method to construct matched solutions to the KV envelope equations

- Based on consistency between particle orbits and the matched beam envelope
  - Uses betatron formulation
- Method works over entire parameter space
- Works for all parameterizations of matched solutions
- Valid for all linear lattices without skew coupling
- Rapidly convergent and robust, even where envelope is unstable
Outline

- Introduction (Already Done)
- Theoretical Model
- Matched Envelope Properties
- Numerical Iterative Method
- Example Applications
- Conclusions
Theoretical Model: Definition of the KV Equations and Relevant Parameters

rms/KV envelope Equations:

\[ r''_x + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0 \]
\[ r''_y + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0 \]

Periodicity:

\[ r_x(s + L_p) = r_x(s) \]
\[ \kappa_x(s + L_p) = \kappa_x(s) \]

\[ Q = \frac{qI}{2\pi\epsilon_0 mc^3\gamma_b^3\beta_b^3} = \text{const} \quad \text{perveance} \]

\[ \varepsilon_x = 4[\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2]^{1/2} \quad \text{rms edge emittance} \]

\[ \kappa_x(s), \quad \kappa_y(s) \quad \text{define applied focusing forces of the lattice} \]
Undepressed particle phase advance $\sigma_{0x}$ measures the strength of the applied focusing function $\kappa_x(s)$ of periodic lattices.

Single-particle orbit without space-charge:

$$x'' + \kappa_x(s)x = 0$$

The same applies to $y$

$$
\begin{pmatrix}
  x(s) \\
  x'(s)
\end{pmatrix}
= M_{0x}(s \mid s_i) \cdot
\begin{pmatrix}
  x(s_i) \\
  x'(s_i)
\end{pmatrix}
\quad \text{with } M_{0x} = 2 \times 2 \text{ Transfer Matrix from } s = s_i \text{ to } s.
$$

Undepressed particle phase advance:

$$\cos \sigma_{0x} = \frac{1}{2} \text{Tr} \ M_{0x}(s_i + L_p \mid s_i)$$
Undepressed Principal Orbit Equations

Transfer Matrix:

\[
\mathbf{M}_{0x}(s|s_i) = \begin{pmatrix} C_{0x}(s|s_i) & S_{0x}(s|s_i) \\ C'_{0x}(s|s_i) & S'_{0x}(s|s_i) \end{pmatrix}
\]

Cosine-like Principal Orbit Equation:

\[
C''_{0x}(s|s_i) + \kappa_x(s)C_{0x}(s|s_i) = 0
\]

Initial Conditions:

\[
\begin{align*}
C_{0x}(s_i|s_i) &= 1 & \text{Sine-like case analogous} \\
C'_{0x}(s_i|s_i) &= 0 & \text{y-plane analogous}
\end{align*}
\]

Note that stability requires:

\[
\frac{1}{2} \left| \text{Tr} \mathbf{M}_{0x}(s_i + L_p|s_i) \right| < 1 \quad \Rightarrow \quad \sigma_{0x} < 180^\circ
\]

[Courant and Snyder, Annals of Physics 3, 1 (1958)]
Depressed Principal Orbit Equations

Depression: \( Q \neq 0 \)

Maintain same basic formulation as before except:

\[
\kappa_x \rightarrow \kappa_x - \frac{2Q}{(r_x + r_y) r_x}
\]

Applied focusing \quad \text{Space-charge defocusing}

Notation: drop 0 subscript to indicate depression

Cosine-like Principal Orbit Equation:

\[
C_x'''(s|s_i) + \left[ \kappa_x(s) - \frac{2Q}{[r_x(s) + r_y(s)]]} \right] C_x(s|s_i) = 0
\]
The depressed particle phase advance provides a convenient measure of space-charge strength:

\[
\cos \sigma_x = \frac{1}{2} \left[ C_x (s_i + L_p | s_i) + S'_x (s_i + L_p | s_i) \right]
\]

\[
\sigma_x = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r^2_x}
\]

\[
\lim_{Q \to 0} \sigma_x = \sigma_{0x}
\]

Normalized space charge strength or “depressed tune”:

\[
0 \leq \frac{\sigma_x}{\sigma_{0x}} \leq 1
\]

\[
\frac{\sigma_x}{\sigma_{0x}} \to 0 \quad \text{Cold Beam (space-charge dominated)}
\]

\[
\varepsilon_x \to 0
\]

\[
\frac{\sigma_x}{\sigma_{0x}} \to 1 \quad \text{Warm Beam (kinetic dominated)}
\]

\[
Q \to 0
\]
Parameterization Classes

Examples from here on assume a symmetric system:

\[ \sigma_{0x} = \sigma_{0y} \equiv \sigma_0, \quad \sigma_x = \sigma_y \equiv \sigma, \quad \varepsilon_x = \varepsilon_y \equiv \varepsilon \]

Possible parameterizations of matched envelope solutions:

<table>
<thead>
<tr>
<th>Case</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \kappa_x (\sigma_0), Q, \varepsilon )</td>
</tr>
<tr>
<td>1</td>
<td>( \kappa_x (\sigma_0), Q, \sigma )</td>
</tr>
<tr>
<td>2</td>
<td>( \kappa_x (\sigma_0), \varepsilon, \sigma )</td>
</tr>
</tbody>
</table>

“Normal” parameterization
Typical principal orbit functions and corresponding matched envelope functions

Syncopated Quadrupole Lattice

\[ L_p = 0.5 \, \text{m}, \ \eta = 0.5, \ \alpha = 0.1, \ \sigma_0 = 80^\circ, \]
\[ \sigma/\sigma_0 = 0.3, \ \varepsilon = 50 \, \text{mm-mrad} \]
The betatron consistency condition allows us to construct matched solutions of the KV equations

Consistency Condition:

\[
\beta_x(s) = \frac{r_x^2(s)}{\varepsilon_x} \frac{S_x^2(s_i)}{S_x(s_i + L_p | s_i) / \sin \sigma} + \frac{S_x(s_i + L_p | s_i)}{\sin \sigma} \left[ C_x(s | s_i) + \frac{\cos \sigma - C_x(s_i + L_p | s_i)}{S_x(s_i + L_p | s_i)} S_x(s | s_i) \right]^2
\]

Used to formulate iterative numerical method for matched envelope solutions
The continuous limit is employed to seed the numerical method

Period Averages:

\[ \bar{\zeta} \equiv \int_{s_i}^{s_i + L_p} \frac{ds}{L_p} \zeta(s) \]

Continuous Limit Replacements:

\[ \kappa_x \rightarrow \left( \frac{\sigma_0}{L_p} \right)^2, \quad \bar{r}_x \rightarrow r_x, \quad \bar{r}_x = \text{const} \]

Continuous Limit KV Envelope Equation (x-plane):

\[ \left( \frac{\sigma_0}{L_p} \right)^2 \bar{r}_x - \frac{2Q}{\bar{r}_x + \bar{r}_y} - \frac{\varepsilon^2}{\bar{r}_x^3} = 0 \]
Form of solution of continuous limit envelope equations depends on parameters specified

Q, $\varepsilon$ parameterization:

Symmetric System: $\sigma_{0x} = \sigma_{0y} \equiv \sigma_0$, $\sigma_x = \sigma_y \equiv \sigma$, $\varepsilon_x = \varepsilon_y \equiv \varepsilon$

$$
\bar{r}_x = \bar{r}_y = \frac{1}{(\sigma_0/L_p)} \left[ \frac{Q}{2} + \frac{1}{2} \sqrt{Q^2 + 4 \left( \frac{\sigma_0}{L_p} \right)^2 \varepsilon^2} \right]^{1/2}
$$

Q, $\sigma$ parameterization:

$$
\bar{r}_x = \frac{\sqrt{2Q}L_p}{\sqrt{(\sigma_{0x}^2 - \sigma_x^2) + \frac{(\sigma_{0x}^2 - \sigma_x^2)^2}{(\sigma_{0y}^2 - \sigma_y^2)}}}
$$

$$
\bar{r}_y = \frac{\sqrt{2Q}L_p}{\sqrt{(\sigma_{0y}^2 - \sigma_y^2) + \frac{(\sigma_{0y}^2 - \sigma_y^2)^2}{(\sigma_{0x}^2 - \sigma_x^2)}}}
$$

$\varepsilon$, $\sigma$ parameterization:

$$
\bar{r}_x = \bar{r}_y = \sqrt{\frac{\varepsilon}{(\sigma/L_p)}}
$$
Numerical Iterative Method uses connection between principal orbits and envelope to generate a correction closer to actual matched solution

Notation: denote iteration order with superscript \(i (i = 0, 1, 2, \ldots)\)

For iterations \(i \geq 1\), we calculate refinements of the principal orbit functions in terms of the envelope calculated at the previous iteration from

\[
C_x^{i''} + \kappa_x C_x^i = -\frac{2Q^{i-1}}{(r_x^{i-1} + r_y^{i-1})r_x^{i-1}} C_x^i = 0
\]

Space-charge defocusing from previous iteration

Cosine-like initial conditions: \(C_x^i(s_i|s_i) = 1\)  
Sine-like case analogous

\[
\beta_x^i \text{ calculated from } C_x^i \text{ and } S_x^i \text{ using consistency condition}
\]
Unspecified parameters may be calculated with one or more of the constraint equations below

Depressed Phase Advance:

\[ \cos \sigma_x^i = \frac{1}{2} [C_x^i (s_i + L_p | s_i) + S_x^{ii} (s_i + L_p | s_i)] \]

Period Averaged Envelope Equation:

\[ \sqrt{\frac{\varepsilon_y^i}{\varepsilon_x^i}} = \frac{\kappa_x \sqrt{\beta_x^i} - 1/(\beta_x^i)^{3/2}}{\kappa_y \sqrt{\beta_y^i} - 1/(\beta_y^i)^{3/2}} \]

\[ \frac{\varepsilon_x^i}{2Q_i^i} = \frac{1}{\kappa_x \sqrt{\beta_x^i} - 1/(\beta_x^i)^{3/2}} \]

\[ \frac{\varepsilon_y^i}{2Q_i^i} = \frac{1}{\kappa_y \sqrt{\beta_y^i} - 1/(\beta_y^i)^{3/2}} \]
Seed Iteration and Cutoff

Seed iteration:

\[ C''_x + \kappa_x C_x^0 - \frac{2Q}{(\bar{r}_x + \bar{r}_y)\bar{r}_x} C_x^0 = 0 \]

Actual applied focus  Continuous focusing space-charge
\( \bar{Q}, \bar{r}_x \) calculated depending on parameterization case

Note: seed iteration is more accurate than continuous focusing limit

Cutoff: Terminate iterations when

\[ \max \left| \frac{r^{i}_x - r^{i-1}_x}{r^{i}_x} \right| \leq \text{tol} \]
Example applications - solenoid and quadrupole lattices, treating the focusing functions as piecewise constant

- **Solenoid description** carried out implicitly in Larmor frame [see Lund and Bukh, PRST-AB7, 024801 (2004)]

**Lattice Period** \( L_p \)

**Occupancy** \( \eta \in [0, 1] \)

**Syncopation Factor** \( \alpha \in [0, 1] \)

\[
\alpha = \frac{1}{2} \quad \Rightarrow \quad FODO
\]
Typical Matched Solutions

Solenoidal Lattice

Syncopated Quadrupole Lattice ($\alpha = 0.1$)

FODO Quadrupole Lattice ($\alpha = 0.5$)

\[
\begin{align*}
L_p &= 0.5 \text{ m} \\
\eta &= 0.5 \\
\sigma_0 &= 80^\circ \\
Q &= 4 \times 10^{-4} \\
\varepsilon &= 50 \text{ mm-mrad}
\end{align*}
\]
Iterative numerical method converges rapidly to matched solution for all parameterizations with specified $\sigma$

Syncopated Quadrupole Lattice

**Iteration 0**

**Iteration 1**

**Iteration 2**

$L_p = 0.5 \text{ m}$

$\eta = 0.5$

$\alpha = 0.1$

$\sigma / \sigma_0 = 0.3$

$\varepsilon = 50 \text{ mm-mrad}$
Parameter space plots illustrating the number of iterations necessary to achieve a fractional tolerance of $10^{-6}$

**Syncopated Quadrupole Lattice**

$\frac{L_p}{\lambda} = 0.5$ m, $\eta = 0.5$, $\alpha = 0.1$

<table>
<thead>
<tr>
<th>$Q$, $\sigma$ specified</th>
<th>$\varepsilon$, $\sigma$ specified</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1, $Q = 10^{-4}$</td>
<td>Case 2, $\varepsilon = 50$ mm-mrad</td>
</tr>
</tbody>
</table>
| \begin{tabular}{cccccc}
  $\sigma / \sigma_0$ & 0.0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\
  \hline
  $\sigma_0$ (degrees) & 40 & 60 & 80 & 100 & 120 & 140 & 160 \\
  \hline
  x & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
  x & 5 & 5 & 5 & 5 & 5 & 5 & x \\
  x & 6 & 6 & 6 & 6 & 6 & 6 & x \\
  x & 7 & 7 & 7 & 7 & 7 & 7 & x \\
  x & 8 & 8 & 8 & 8 & 8 & 8 & x \\

dashed line | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

**Envelope Instability Bands**
Tolerance decreases rapidly toward numerical precision

Syncopated Quadrupole Lattice

\[ L_p = 0.5 \, \text{m}, \quad \eta = 0.5, \quad \alpha = 0.1, \quad \sigma_0 = 80^\circ \]
\[ \sigma / \sigma_0 = 0.2, \quad \varepsilon = 50 \, \text{mm-mrad} \]

Tolerance decreases more slowly with:
- Increasing undepressed phase advance
- Increasing lattice complexity

![Graph showing log_{10}(tol) vs. Iteration Number with machine precision indicated.]
Problem: Simplest implementation of $Q$, $\varepsilon$ parameterization fails over approximately half of the parameter space

$L_p = 0.5$ m, $\eta = 0.5$, $\varepsilon = 50$ mm-mrad

Solenoidal Lattice

FODO Quadrupole Lattice: $\alpha = 0.5$

$x = \text{Failure point due to complex } \sigma^i \text{ in iterations:}$
Beam squeezed too hard for given $Q$; principal orbits overcompensate, grow too large, and yield complex phase advances
We attempted to implement the $Q$, $\varepsilon$ parameterization in the entire parameter space through several methods:

1) Calculate the depressed phase advances via previous iteration integral formula

$$\sigma^i = \varepsilon \int_{s_i}^{s_i + L_p} \frac{ds}{[r_x^{i-1}(s)]^2}$$

- Converges systematically to unphysical solutions

2) Vary perveance adaptively

- Raise $Q$ until method fails, lower until method works, then increase adaptively
- Found this only works for very slow increases in $Q$, leading to many iterations

3) Hybrid Method

- Assume trial $\sigma_x$, $\sigma_y$ values and find consistent values with specified $Q$ and/or $\varepsilon_x$, $\varepsilon_y$ using numerical root-finding
Fortunately, the $Q, \varepsilon$ parameterization can be extended to the entire parameter space by employing hybrid methods $(Q, \varepsilon)/(Q, \sigma)$

Hybrid
Find $\sigma_x$, $\sigma_y$ satisfying

$$\varepsilon_j(\sigma_x, \sigma_y) = \varepsilon_j|_{\text{specified}}$$

Then employ $Q$, $\sigma$ method

$$Q = 10^{-4}$$

$L_p = 0.5$ m, $\eta = 0.5$

<table>
<thead>
<tr>
<th>$\sigma_0$ (degrees)</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
<th>140</th>
<th>160</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>78</td>
<td>136</td>
<td>136</td>
<td>210</td>
<td>310</td>
<td>370</td>
<td>372</td>
</tr>
<tr>
<td>0.8</td>
<td>104</td>
<td>248</td>
<td>440</td>
<td>210</td>
<td>348</td>
<td>469</td>
<td>526</td>
</tr>
<tr>
<td>0.6</td>
<td>136</td>
<td>176</td>
<td>300</td>
<td>456</td>
<td>788</td>
<td>980</td>
<td>938</td>
</tr>
<tr>
<td>0.4</td>
<td>304</td>
<td>330</td>
<td>1440</td>
<td>1164</td>
<td>728</td>
<td>1088</td>
<td>1352</td>
</tr>
<tr>
<td>0.2</td>
<td>144</td>
<td>430</td>
<td>1416</td>
<td>947</td>
<td>700</td>
<td>1040</td>
<td>704</td>
</tr>
<tr>
<td>0.0</td>
<td>280</td>
<td>480</td>
<td>2616</td>
<td>1330</td>
<td>350</td>
<td>1184</td>
<td>2732</td>
</tr>
<tr>
<td></td>
<td>360</td>
<td>1940</td>
<td>1584</td>
<td>756</td>
<td>518</td>
<td>608</td>
<td>1392</td>
</tr>
<tr>
<td></td>
<td>136</td>
<td>670</td>
<td>252</td>
<td>3080</td>
<td>1824</td>
<td>784</td>
<td>1936</td>
</tr>
</tbody>
</table>
Conclusions

A new iterative method for generating matched envelope solutions to the KV equations has been developed

- Has a large basin of attraction
- Converges rapidly
- Works over entire parameter space, even in regions of strong instability
- Applicable to all linear lattices without skew coupling
- Straightforward to code

Downside: Direct application of $Q, \varepsilon$ parameterization fails in about half of the parameter space

- However, the $Q, \varepsilon$ method can be implemented with hybrids

Extra:

- Manuscript submitted to PRST-AB
- Programs and presentation slides (soon) available online