PROBLEM SET 2

(1). PROBLEM: IN CLASS IT WAS STATED THAT IF:

$$\chi \equiv \frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}$$

AND IF $\rho(x,y) = \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right) = \hat{\rho}(\chi)\Big|_{s=0} \equiv \frac{d\eta(\chi)}{d\chi}\Big|_{s=0}$

AND IF $\Psi(x,y) = \frac{-r_x r_y}{4\varepsilon_0}\int_0^\infty \frac{\eta(\chi)\, ds}{\sqrt{r_x^2+s}\,\sqrt{r_y^2+s}}$

THEN IT FOLLOWS THAT $\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = -\frac{\rho(x,y)}{\varepsilon_0}$

and so $\Psi(x,y)$ is a solution of Poisson's equation.

a). Calculate $\frac{d\chi}{ds}$. (Note that x, y, r_x, and r_y are held fixed).

b). SHOW THAT $\nabla^2\Psi = \frac{-r_x r_y}{4\varepsilon_0}\int_0^\infty \frac{-4\eta''(\chi)\frac{d\chi}{ds} + 2\eta'\left[\frac{1}{r_x^2+s}\right] + 2\eta'\left[\frac{1}{r_y^2+s}\right]}{\sqrt{r_x^2+s}\,\sqrt{r_y^2+s}}\, ds$

HERE $\eta'(\chi) \equiv \frac{d\eta(\chi)}{d\chi}$

c). Integrate the first term in the integral ABOVE BY PARTS AND SHOW THAT ~~[illegible]~~ PART OF IT CANCELS THE TERMS PROPORTIONAL TO η'

d). EVALUATE THE INTEGRATED TERM TO SHOW THAT

$$\nabla^2\Psi = -\frac{\rho}{\varepsilon_0}$$

(2) PLOT LOG(λ) vs LOG(qV) for a heavy ion beam (mass 200 amu) between 10 keV and 1 GeV.

(λ IS THE MAXIMUM TRANSPORTABLE LINE CHARGE DENSITY, ASSUMING NEGLIGIBLE EMITTANCE. ION ENERGY = qV. ASSUME q = 1. (CHARGE STATE 1)).
ASSUME BEAM IS NON-RELATIVISTIC.

PLOT FOR EACH OF THE FOLLOWING FOCUSING METHODS:

__BIG HINT:__ THE VALUES AT 10 keV ARE GIVEN AT RIGHT.

a). SOLENOIDS $9.6 \cdot 10^{-8}$ C/m

b). ELECTRIC QUADS $1.02 \cdot 10^{-6}$ C/m

c). MAGNETIC QUADS $8.6 \cdot 10^{-8}$ C/m

d). EINZEL LENSES $8.2 \cdot 10^{-4}$ C/m

ASSUME $B_{SOLENOID} = B_{QUAD} = 2T$

$\Phi_{\substack{EINZEL\\LENSE}} = (\Phi_0 \text{ IN NOTES}) = \pm 100\,kV$

$\Phi_{QUAD} = (V_q \text{ IN NOTES}) = 100\,kV$

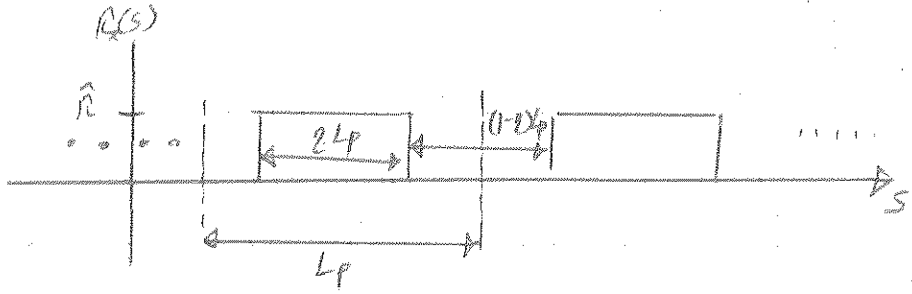
OCCUPANCY = 0.7

$\dfrac{r_{beam}}{r_{pipe}} = 0.7$ $\quad r_{beam} \leq 6\,cm$ $\quad L_{EINZEL} \geq r_{beam}$

(USE FORMULAS GIVEN ON LAST PAGE OF TODAY'S LECTURE).
NOTE $r_p \equiv r_{pipe}$ $\quad r_b \equiv r_{beam}$

Problem 3

8/ Consider a Solenoidal Periodic Lattice



L_p = lattice Period

ηL_p = length Solenoids

$(1-\eta)L_p$ = length drift.

$\hat{\kappa}$ = Solenoid strength

η = Solenoid occupancy $0 < \eta \le 1$

a) Write Larmor Frame transfer matrices $\bar{M}(s|s_i)$ for each section of the periodic lattice in terms of $\Theta = \sqrt{\hat{\kappa}}\,\eta L_p/2$, and η. Feel free to use results in S5B of the class notes.

$\bar{M}_F$: Transfer through Solenoid

$\bar{M}_D$: " " Drift

b) Write the Larmor Frame transfer matrix $\bar{M}(s_i+L_p, s_i)$ through one lattice period starting from the Solenoid.

c) Show that the Larmor Frame phase advance σ_0 of a particle through the lattice period

$$\cos\sigma_0 = \frac{1}{2}\,\mathrm{Trace}\ M(s_i+L_p|s_i)$$

can be expressed as:

$$\cos\sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta}\,\Theta\sin(2\Theta)$$

$$\Theta = \sqrt{\hat{\kappa}}\,\eta L_p/2$$

d) Will it matter where the lattice period is started in the calculation for $\bar{\sigma}_0$ in part c)? Why?

e) For $\Theta \ll 1$ (thin lens limit) show that

$$\cos \bar{\sigma}_0 \simeq 1 - \eta \frac{|\hat{\kappa}| L_p^2}{2}$$

f) If $\bar{\sigma}_0 \ll 1$, and $\eta \ll 1$, show that

$$\bar{\sigma}_0 \simeq \sqrt{\eta |\hat{\kappa}|}\, L_p$$

g) If one wanted to model a solenoidal focusing lattice by a continuous focusing channel with $\kappa(s) = k_{\beta 0}^2 = \text{const}$, how could one choose $k_{\beta 0}^2$ based on part f)?

h) <u>Qualitative only</u>:
Does our notion of phase advance apply directly (without use of Larmor frame transform) to coupled x-y motion in a solenoid? Would you expect a coupled formulation to describe phase advance in x-y coupled motion to be simple?

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9/ In class we derived the single-particle Courant-Snyder invariant:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon = \text{const.}$$

where:

$$\beta(s) = w^2(s)$$

$$\alpha(s) = -w(s)w'(s)$$

$$\gamma(s) = \frac{1}{w^2(s)} + w'(s)^2 = \frac{1+\alpha^2(s)}{\beta(s)}$$

Derive the critical values of the ellipse indicated on the figure below:

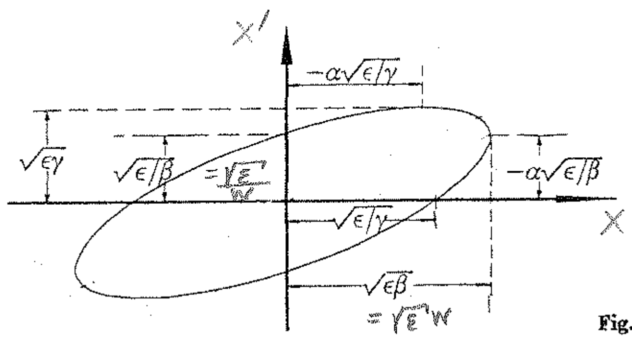

Fig. 5.22. Phase space ellipse

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From Wiedemann

Hint: to avoid messy algebra, take a differential of the constraint equation $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \text{const}$ and use this result to find turning points.

$$\Rightarrow \quad 2\gamma x\,dx + 2\alpha x\,dx' + 2\alpha x'\,dx + 2\beta x'\,dx' = 0$$

These results are important in understanding the KV distribution derived later to model beams with space-charge