Assume that, as was done in class each acceleration gap or module is represented by a lumped circuit element with successive elements separated by a distance $d$.

Assume that the beam is a current source $I_0$, with perturbation $I_1$, and that at each gap the beam acquires a voltage increment $\Delta V$:

$$I = I_0 + I_1 e^{-i\omega t} \quad \Delta V = \Delta V_0 + \Delta V_1 e^{-i\omega t}$$

Show that the impedance for such a system is:

$$Z^*(\omega) = \frac{R^*}{1 - i\omega C^* R^* + iR^*/\omega L^*}$$

where $R^* = \frac{R}{d}$, $C^* = \frac{C}{d}$, $L^* = \frac{L}{d}$ and

$$\Delta V = \frac{-\Delta V}{d} = -I_1 Z^*(\omega)$$

Note that the voltage drop across an element is given by:

$$\Delta V = \left\{ \begin{array}{ll}
\frac{1}{L} \frac{dI}{dt} & \text{for inductor with inductance } L \\
\frac{Q}{C} & \text{for capacitor with capacitance } C \\
IR & \text{for resistor with resistance } R
\end{array} \right.$$
Problem 2: When longitudinal emittance is included in the non-relativistic longitudinal envelope equation, describing the length $L_0$ of a pulse with parabolic line charge density undergoing bunch compression,

$$\frac{1}{2} \frac{L}{L_0} = \frac{16 E_z^2}{L^3} + \frac{12 \varphi \eta Q_e}{4 \eta E_0 m v^2 L^2}$$

where $Q_e$ is the total charge in the bunch,

$$E_z = \text{longitudinal emittance} = 25 [<z^2> - <z'^2>].$$

Show that the velocity tilt $\Delta v$ required to compress the beam to " stagnation" (i.e., to the point where $L = 0$) is given by:

$$\Delta v^2 = \frac{16 E_z^2}{L_0^2} \left[ C^2 - 1 \right] + \frac{2499 Q_e}{4 \eta E_0 m v_0^2} \left[ C - 1 \right]$$

where $L_0 = L$ at $t = 0$,

$L_f = L$ at the stagnation point,

$$C = \frac{L_0}{L_f} = \text{compression ratio} > 1$$

and

$$\Delta v = -L_0'$$
Problem 1

Consider the driven harmonic oscillator equation for \( u(\varphi) \):

\[
\frac{d^2 u(\varphi)}{d\varphi^2} + \omega^2 u(\varphi) = A \cos(2\varphi) + B \sin(2\varphi)
\]

\( \omega \) = constant driving frequency.
\( A, B \) = constant amplitudes.

The general solution for \( u(\varphi) \) can be expanded as

\[
u(\varphi) = u_h(\varphi) + u_p(\varphi)
\]

where \( u_h \) is the general solution to the homogeneous equation:

\[
\frac{d^2 u_h}{d\varphi^2} + \omega_0^2 u_h = 0
\]

\[\Rightarrow u_h = C_1 \cos(\omega_0 \varphi) + C_2 \sin(\omega_0 \varphi)\]

\( C_1, C_2 \) = constants

and \( u_p \) is any particular solution to

\[
\frac{d^2 u_p}{d\varphi^2} + \omega_0^2 u_p = A \cos(2\varphi) + B \sin(2\varphi)
\]

(a) For \( \omega \neq \omega_0 \) show that a solution \( u_p \) exists proportional to the driving term and find the constant of proportionality.
b) Use the results of part a) to construct the solution \( U(\psi) \) for \( U(\psi) \) satisfying the initial conditions at \( \psi = 0 \):

\[
U(\psi = 0) = U_0 \\
\left. \frac{dU}{d\psi} \right|_{\psi = 0} = U_0 \\
\frac{dU}{d\psi} = \dot{U}
\]

c) Set \( U = \psi_0 + \Delta U \) and find the leading order form of the solution valid for \( |\Delta U| \ll 1 \) and \( |\dot{\psi} \psi_0| \ll 1 \). What does this limit imply on the amplitude of the particle oscillation as \( \Delta \rightarrow 0 \)?

d) What do these results imply for a general periodic forcing function:

\[
\ddot{U}(\psi) + \omega^2 U(\psi) = f(\psi) \quad \text{forcing function} \\
f(\psi + 2\pi) = f(\psi)
\]

How does this fit in with the analysis of machine tunes carried out in the class notes?