Problem 6

Let the equation of motion for a test particle oscillating in a mismatched beam be modeled by:

\[ x'' + \frac{k_B}{V_{bo}^2} x' = -\frac{Q}{V_{bo}^6} x^5 + \frac{2\varepsilon Q}{V_{bo}^2} x \cos k_B s \]

Let \( x = A \sin \psi \) and \( x' = k_B A \cos \psi \)

where \( \psi = k_B s + \alpha \)

a) Calculate \( A' \) and \( \alpha' \) in terms of \( A, \psi, k_B, s \)

(b) and parameters.

Let \( \Phi'_r = 2k_B - k_B + 2\alpha' \)

b) Average over rapid variations to obtain equations for the amplitude and phase of the near resonant particles, i.e., find \( A'_r(\Phi_r, A_n) \) and \( \Phi'_r(\Phi_r, A_n) \).

c) Define \( \omega = A_r^2 \). Find \( \omega'(\Phi_r, \omega) \) and \( \Phi'_r(\Phi_r, \omega) \).

d) Find the Hamiltonian \( H \), such that

\[ \omega' = \frac{\partial H(\omega, \Phi_r)}{\partial \Phi_r} \quad \text{and} \quad \Phi'_r = -\frac{\partial H(\omega, \Phi_r)}{\partial \omega} \]

e) Verify that \( H(\omega, \Phi_r) \) is a constant.

Hint for part (c): \( \sin^6 x = \frac{1}{32} (10 - 15 \cos 2x + 6 \cos 4x - \cos 6x) \)

\( \cos x \sin^5 x = \frac{1}{5} (5 \sin 3x - 4 \sin 4x + \sin 6x) \)
PROBLEM 2

1. Suppose the space charge electric field exterior to the beam, used as

\[ E_r \sim \frac{\lambda}{4\pi \epsilon_0} \frac{V_0}{r^3} \]

instead of \[ E_r \sim \frac{\lambda}{4\pi \epsilon_0} \frac{1}{r} \]

Would you expect the radial extent of a beam halo due to mismatch to be larger or smaller than the one obeying the \( 1/r \) dependence?

HINT: Consider the location of the radius where the particle - envelope radius is strongest. (Assume halo extent is proportional to this radius).
Problem 3

TCE Problem 1

1. Envelope Radius

A. Envelope Radius of a Nonuniform Density Beam

For a uniform density elliptical beam with envelope radii $r_x$ and $r_y$,

$$ n(x, y) = \begin{cases} \frac{\lambda}{2\pi r_x r_y} & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} \leq 1 \\ 0 & \text{otherwise} \end{cases} $$

$\lambda$ = line-charge density = const.

And we showed in previous problems that,

$$ r_x = 2 \langle x^2 \rangle^{\frac{1}{2}} $$
$$ r_y = 2 \langle y^2 \rangle^{\frac{1}{2}} $$

where

$$ \langle x^2 \rangle = \frac{\int d^2 x \, x^2 \, n(x, y)}{\int d^2 x \, n(x, y)} $$

If the density profile is elliptical with

$$ n(x, y) = n(\xi) \quad \xi^2 = \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} \quad \text{for } x_e = \text{const.} \quad \text{for } y_e = \text{const.} $$

such that $n(\xi)$ is monotonically decreasing in $\xi$ with a sharp cutoff at $\xi = 1$.

![Graph of n(\xi) with beam edge]

Show that

$$ x_e > \langle x \rangle = 2 \langle x^2 \rangle^{\frac{1}{2}} $$
$$ y_e > \langle y \rangle = 2 \langle y^2 \rangle^{\frac{1}{2}} $$

where $\langle x^2 \rangle$ and $\langle y^2 \rangle$ are defined from the nonuniform density beam. You may use steps/transforms from previous problems.
B. Edge Radius Factor in a 1D Beam

For a 1D sheet beam with uniform density,

\[ n(x) = \begin{cases} \frac{N}{2x_b^2} & -x_b \leq x \leq x_b \\ 0 & \text{otherwise} \end{cases} \quad N = \text{const} \]

\[ x_b = \text{edge radius} \]

Find an edge radius coefficient \( F \) such that

\[ x_b = F \langle x^2 \rangle \]

where, in 1D phase-space,

\[ \langle x^2 \rangle = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx' f \quad f = f(x,x',s) \]

1D distribution

Compare \( F \) to the corresponding results for a centered 2D elliptical beam

\[ f_x = 2 \langle x^2 \rangle \nu \]
\[ f_y = 2 \langle y^2 \rangle \nu \]

Should you expect \( F = 2 \)? Why?
Problem 4

TCE Problem 2

21) Image Charges on a Cylindrical Pipe

Consider a perfectly conducting pipe of radius \( r_p \):

\[
\frac{\partial^2 \phi}{\partial x^2} = 0
\]

\[
\phi(r = r_p) = \text{const}
\]

A) Show that the formula for a line-charge \( \lambda \) at the origin in free-space is:

\[
\phi(r) = -\frac{\lambda}{2\pi\epsilon_0} \ln r + \text{const}
\]

B) Use the formula in part A) to show that a solution to the interior problem \( |x| < r_p \) can be found for a line charge \( \lambda \) at coordinate \( x = x_0 \) by superimposing the direct charge and an image charge at \( x = -C x_0 \). Calculate \( C \) for cylindrical geometry.

Images can be superimposed to obtain the Green's function for the 2D calculation of \( \phi \) within the cylinder.
Problem 5,
TCE Problem 8

Free Expansion of a Beam Envelope.

In the absence of applied focusing forces:

\[ \frac{f_x'' - 2Q - \varepsilon_x^2}{f_x + f_y} = 0 \]

\[ \frac{f_y'' - 2Q - \varepsilon_y^2}{f_x + f_y} = 0 \]

Initial conditions:

\[ (s = s_1): \]

\[ f_x(s = s_1) = f_{x_1}; \]

\[ f_y(s = s_1) = f_{y_1}; \]

\[ f_x'(s = s_1) = f_{x_1}'; \]

\[ f_y'(s = s_1) = f_{y_1}'; \]

Part I: \( Q = 0, \varepsilon_x \neq 0, \) free expansion without space charge.

9) Show that the envelope Hamiltonian satisfies:

\[ \frac{f_x'}{2} + \frac{\varepsilon_x^2}{2f_x} = \text{const.} \]

\[ f_y' \text{ same.} \]

b) Show that the equation in 9) can be written as an integral from the initial condition to solve for the free expansion as a function of \( s. \) You do not need to carry out the integration explicitly.

Part II: \( \varepsilon_x = 0, Q \neq 0, \) free expansion without emittance.

9) Show using \( f_\pm = (f_x \pm f_y)/2 \) that the coupled envelope equations reduce to:

\[ f_\pm'' - \frac{Q}{f_\pm} = 0, \quad f_-'' = 0 \]

and satisfy

\[ \frac{f_\pm'}{2} - Q \ln f_\pm = \text{const.} \]

\[ f_- = C_1 s + C_2 \]

\( C_1, C_2 \) constants.
d) Argue that for finite $Q \neq 0$ that the free expansion without emittance will be more rapid than the free expansion without space charge ($Q=0$, $E_0 \neq 0$) when the beam expands sufficiently.