Problem 1

Consider the parameters for a proton storage ring in which there is a single circulating bunch:

\[ N_0 = 2 \times 10^{11} \text{ protons/bunch} \]

\[ \delta_{\text{offset}} \approx 124 \text{ m} = \text{effective bunch length} \]

\[ r_{\text{pipe}} = 0.1 \text{ m} = \text{pipe radius} \]

\[ \beta = 0.875 = \frac{v}{c} \text{ (ion velocity)} \]

a) What is the single bunch, multivacating parameter \( \beta_s \)?

b) What is the characteristic energy gain per electron?

Assume an ionization cross section of \( 10^{-22} \text{ m}^2 \), and a desorption coefficient \( 4 \), and a linear luminescence \( S \)

of \( 0.1 \text{ m}^3 s^{-1} \text{ m}^{-1} \). Will the pressure be stable in this machine against runaway desorption? (Assume the beam fills the ring for making your estimate.)

c) Assume that \( r_{\text{b}} = 0.01 \text{ m} \). Estimate the electron oscillation frequency in the field of the ion beam.
Assume a beam is propagating in a linear focusing channel with linear space charge field.

\[ x'' = -\left( \frac{e}{\delta} \right) x + \frac{2Q}{\gamma_x^3 + \gamma_y} \frac{x}{\gamma_x} \]

Assume scattering is present so the moment equations are those present in class.

\[ \frac{1}{\gamma_x} \langle x^2 \rangle = 2 \langle xx' \rangle \]

\[ \frac{d}{d\sigma} \langle xx' \rangle = \langle xx'' \rangle + \langle x'^2 \rangle \]

\[ \frac{1}{\gamma_x} \langle x'^2 \rangle = 2 \langle xx'' \rangle + C_{sc} \]

Extra term due to scattering

Show the envelope equation is:

\[ x'' + \frac{2Q}{\gamma_x^3 + \gamma_y} \frac{x}{\gamma_x} + k(s) \gamma_x + \frac{\delta_x^2}{\gamma_x^3} = 0 \]

And emittance evolution equation is:

\[ \frac{dE_x}{ds} = 4\gamma_x^2 C_{sc} \]
Problem 3
ICE Problem 9
S.M. Lund

In class we derived rms envelope equations for a coasting beam:

\[ \frac{\rho_x'' + R_x(s) \rho_x}{\rho_x} - \frac{\rho_x^2}{\rho_x} = 0 \]

\[ \frac{\rho_y'' + R_y(s) \rho_y}{\rho_y} - \frac{\rho_y^2}{\rho_y} = 0 \]

Take 1:

\[ \rho_x = \rho_{x0} + \delta \rho_x \]
\[ \rho_y = \rho_{y0} + \delta \rho_y \]

\[ Q = 0 + \delta Q \]

\[ E_x = E + \delta E_x \]
\[ E_y = E + \delta E_y \]

and

\[ \rho_x = \rho_m + \delta \rho_x \]
\[ \rho_y = \rho_m + \delta \rho_y \]

Matched Envelope

\[ \rho_m = \text{const.} \]

A/ Expand to linear order as in class notes and derive equilibrium and perturbed envelope equations.

B/ Show that the linear order envelope equations in A/ decouple when taking

\[ \delta \rho_x = \frac{\delta \rho_x + \delta \rho_y}{2} \]

\[ \delta \rho_y = \frac{\delta \rho_x - \delta \rho_y}{2} \]

and give
\[ \frac{\partial^2 \delta S_+}{\partial t^2} + \frac{\partial^2 \delta S_+}{\partial \theta^2} + \frac{1}{\Gamma_m} \frac{\partial \delta S_+}{\partial \theta} + \frac{3}{\Gamma_m} \delta S_+ = \frac{\partial^2 \delta P}{\partial t^2} \]

\[ \frac{\partial^2 \delta S_-}{\partial t^2} + \frac{\partial^2 \delta S_-}{\partial \theta^2} + \frac{3}{\Gamma_m} \delta S_- = \frac{\partial^2 \delta P}{\partial t^2} \]

with

\[ \delta P_+ = -\Gamma_m \left( \frac{\delta S_x + \delta S_y}{2} \right) + \frac{1}{\Gamma_m} \delta Q + \frac{3}{\Gamma_m^3} \left( \delta S_x + \delta S_y \right) \]

\[ \delta P_- = -\Gamma_m \left( \frac{\delta S_x - \delta S_y}{2} \right) + \frac{3}{\Gamma_m^3} \left( \delta S_x - \delta S_y \right) \]

C: Take:

\[ \gamma_0 = \frac{\delta_0}{L_p} \]

\[ \delta = \sqrt{\delta_0^2 - \frac{\gamma_0^2}{(1m/L_p)^2}} = \frac{\gamma_0 L_p}{\Gamma_m^2} \]

\[ \delta_+ = \sqrt{\delta_0^2 + 2\delta^2} \]

and show that the eqn for \( \delta S_+ \) in B/ becomes:

\[ \frac{L_p}{\Gamma_m} \frac{d^2 e}{d \gamma^2} \left( \frac{\delta S_+}{\delta m} \right) + \frac{e_+}{\Gamma_m} \frac{d^2 (\delta S_+)}{d \gamma^2} = -\delta_0 \left( \frac{d S_x + d S_y}{\gamma_0} \right) \]

\[ + \left( \delta_0^2 - \delta^2 \right) \frac{\delta Q}{Q} + \delta^2 \left( \frac{d S_x + d S_y}{\Gamma_m} \right) \]

Solution of the eqn in this form was extensively developed in the lectures.
**Problem 6**

**TCF Problem 6**

Approximate Matched Envelope Solution for a Solenoid

A/ For a solenoidal transport channel:

\[ R(s) \approx R_y(s) = R_c(s) \]

We can Fourier-series expand \( R(s) = R_y(s) = R_c(s) \) as

\[ R(s) = \sum_{n=0}^{\infty} R_n \cos \left( \frac{2\pi n s}{L_p} \right) \]

and identify

\[ R_0 = \frac{1}{L_p} \int_{0}^{L_p} R(s) ds \]

\[ R_n = \frac{2}{L_p} \int_{0}^{L_p} \cos \left( \frac{2\pi n s}{L_p} \right) R(s) ds \quad n = 1, 2, \ldots \]

Calculate \( R_0 \) and \( R_n \)

B/ The matched beam envelope equation is: (Larmor Frame)

\[ \ddot{R}_x + R(s) \dot{R}_x - \frac{Q}{\sqrt{R_x}} \frac{\dot{R}_x}{R_x} - \frac{E_x}{R_x} = 0 \]

\[ R_s(s + L_p) = R_s(s) \quad ; \quad R_x = R_y \quad ; \quad E_x = \text{const} \]

\( Q = \text{const} \)

Expand the matched envelope solution similarly:

\[ R_x(s) = R_b \left( 1 + \Delta \cos \left( \frac{2\pi n s}{L_p} \right) \right) + \sum_{n=2}^{\infty} \Delta_n \cos \left( \frac{2\pi n s}{L_p} \right) \]

Where

\[ R_b = \text{const} \]

is the average beam radius and

\[ \Delta = \text{const} \quad |\Delta| \ll | \]

\( \Delta \) is an expansion term and \( \Delta_n \) are dimensionless expansion coefficients. We take to be small, relative to \( \Delta \).
TCE Problem 6

Insert the expansions

1. Neglect all terms \( O(\Delta^3) \) and higher
2. Neglect fast oscillation terms \( \sim \cos \left( \frac{2\pi n \Delta}{T_p} \right) \) with \( n \geq 2 \)

Derive \( \alpha \), \( \gamma \), and leading-order force balance terms.

C/ Explain (no need to calculate explicitly) how the constraints in B/ can be used to calculate.

Max \( [\xi_m] \) : Max beam excursion in lattice period, in terms of lattice parameters, avg. beam radius, and emittance

- \( Q \) : Beam phase error as a function of lattice parameters, avg. beam radius, and emittance.

D/ The undepressed particle phase advance can be calculated as

\[
\cos \delta_0 = \cos(2\gamma) - (1-\gamma) \frac{Q}{1} \sin(2\gamma)
\]

\[ \delta_0 = \sqrt{\frac{Q}{2}} \]

Expand this expression to leading order in \( \delta_0 \) to obtain a simpler design expression relating \( \delta_0 \) and lattice parameters.

Results from part C/ and D/ can be employed to design solenoidal transport channels.