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## II. Envelope Equations

Paraxial Ray Equation

Envelope equations for axially symmetric beams

Cartesian equation of motion

Envelope equations for elliptically symmetric beams

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Roadmap:

Single particle equation with Lorentz force
q(E + v x B)

$\Downarrow$

Make use of:

1. Paraxial (near-axis) approximation
   (Small r and r' )
2. Conservation of canonical angular momentum

3. Axisymmetry f(r,z)

$\Downarrow$

Paraxial Ray Equation for Single Particle

Next take statistical averages over the distribution function

$\Rightarrow$ Moment equations

Express some of the moments in terms of the rms radius and emittance

$\Rightarrow$ Envelope equations (axisymmetric case)

Some focusing systems have quadrupolar symmetry
Rederive envelope equations in cartesian coordinates
(x,y,z) rather than radial (r,z)

Start with Newton's equations with the Lorentz force:

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B})$$

In cartesian coordinates this can be written:

$$\frac{d(\gamma m\dot{x})}{dt} = \gamma m\ddot{x} + \dot{\gamma} m\dot{x} = q(E_x + \dot{y}B_z - \dot{z}B_y)$$

$$\frac{d(\gamma m\dot{y})}{dt} = \gamma m\ddot{y} + \dot{\gamma} m\dot{y} = q(E_y + \dot{z}B_x - \dot{x}B_z)$$

$$\frac{d(\gamma m\dot{z})}{dt} = \gamma m\ddot{z} + \dot{\gamma} m\dot{z} = q(E_z + \dot{x}B_y - \dot{y}B_x)$$

In cylindrical coordinates: (use $\frac{d\hat{e}_r}{dt} = \dot{\theta}\ \hat{e}_\theta$ and $\frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\ \hat{e}_r$ )
(see next page).

$$\frac{d(\gamma m\dot{r})}{dt} - \gamma m r\dot{\theta}^2 = q(E_r + r\dot{\theta}B_z - \dot{z}B_\theta) \qquad (I)$$

$$\frac{1}{r}\frac{d(\gamma m r^2\dot{\theta})}{dt} = q(E_\theta + \dot{z}B_r - \dot{r}B_z) \qquad (II)$$

$$\frac{d(\gamma m\dot{z})}{dt} = q(E_z + \dot{r}B_\theta - r\dot{\theta}B_r) \qquad (III)$$

In general $\underline{E} = -\underline{\nabla}\phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}$ and $\underline{B} = \underline{\nabla} \times \underline{A}$

When $\frac{\partial}{\partial\theta} = 0$:

$$\underline{E} = \hat{e}_r\left[\frac{-\partial\phi}{\partial r} - \frac{\partial A_r}{\partial t}\right] + \hat{e}_\theta\left[-\frac{\partial A_\theta}{\partial t}\right] + \hat{e}_z\left[\frac{-\partial\phi}{\partial z} - \frac{\partial A_z}{\partial t}\right]$$

$$\underline{B} = \hat{e}_r\left[-\frac{\partial A_\theta}{\partial z}\right] + \hat{e}_\theta\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] + \hat{e}_z\left[\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}\right]$$

To calculate the rate of change of the momentum $p$ in cylindrical coordinates we must take into account that the unit vectors change directions as the particle moves:

Note: on this page $p^*_\theta$ = $\theta$-component of mechanical momentum, not to be confused with
$p_\theta \equiv \gamma m r^2\dot{\theta} + qrA_\theta \equiv$
<u>canonical</u> <u>angular</u> momentum.

$$\underline{p} = p_r\hat{e}_r + p^*_\theta\hat{e}_\theta + p_z\hat{e}_z = \gamma m\underline{v}$$

where $p_r = \gamma m\dot{r}$

$$p^*_\theta = \gamma m r\dot{\theta}$$

$$p_z = \gamma m\dot{z}$$

So $$\frac{d\underline{p}}{dt} = \dot{p}_r\hat{e}_r + p_r\dot{\hat{e}}_r + \dot{p}^*_\theta\hat{e}_\theta + p^*_\theta\dot{\hat{e}}_\theta + \dot{p}_z\hat{e}_z$$

$$= (\dot{p}_r - p^*_\theta\dot{\theta})\hat{e}_r + (p_r\dot{\theta} + \dot{p}^*_\theta)\hat{e}_\theta + \dot{p}_z\hat{e}_z$$

where we have used:

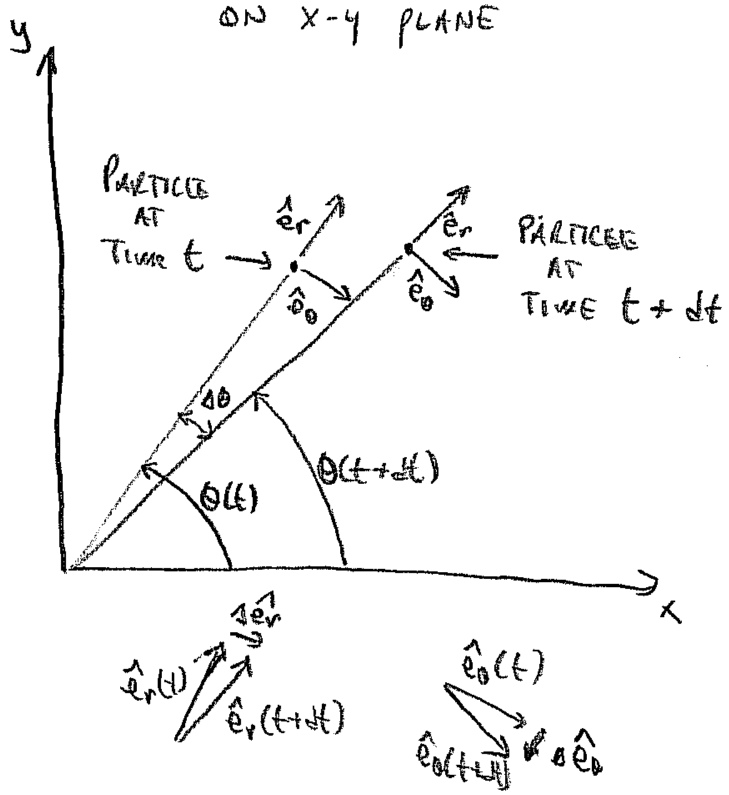
$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta\dot{\theta} \qquad \frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\dot{\theta}$$

$$\Rightarrow \boxed{\frac{d\underline{p}}{dt} = \left(\frac{d(\gamma m\dot{r})}{dt} - \gamma m r\dot{\theta}^2\right)\hat{e}_r + \left(\gamma m\dot{r}\dot{\theta} + \frac{d(\gamma m r\dot{\theta})}{dt}\right)\hat{e}_\theta + \frac{d(\gamma m\dot{z})}{dt}\hat{e}_z}$$

Note: second term $= \dfrac{1}{r}\dfrac{d}{dt}\left(\gamma m r^2\dot{\theta}\right)$

mechanical angular momentum

PROJECTION OF PARTICLE POSITION AT TIMES $t$ & $t+dt$ ON X-Y PLANE




$\Delta\hat{e}_r = \hat{e}_\theta \Delta\theta \qquad \Delta\hat{e}_\theta = -\hat{e}_r \Delta\theta$

$\dfrac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta} \qquad \dfrac{d\hat{e}_\theta}{dt} = -\hat{e}_r \dot{\theta}$

Algebraically

$$\hat{e}_r = \hat{e}_x \cos\theta + \hat{e}_y \sin\theta$$

$$\hat{e}_\theta = -\hat{e}_x \sin\theta + \hat{e}_y \cos\theta$$

$$\Rightarrow \quad \frac{d\hat{e}_r}{dt} = -\hat{e}_x\dot{\theta}\sin\theta + \hat{e}_y\dot{\theta}\cos\theta = \hat{e}_\theta\dot{\theta}$$

and $$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_x\dot{\theta}\cos\theta - \hat{e}_y\dot{\theta}\sin\theta = -\hat{e}_r\dot{\theta}$$

# Conservation of Canonical Angular Momentum

Now the RHS of eq. II multiplied by $r$ can be written:

$$qr(E_\theta + \dot{z}B_r - \dot{r}B_z) = q\left(-\frac{\partial rA_\theta}{\partial t} - \dot{z}\frac{\partial rA_\theta}{\partial z} - \dot{r}\frac{\partial rA_\theta}{\partial r}\right)$$

$$= -q\left[\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}}\right](rA_\theta)$$

$$= -q\frac{d(rA_\theta)}{dt} \qquad \textit{(IV)}$$

So eq. II and eq. IV => $\frac{d}{dt}\left(\gamma m r^2\dot{\theta} + qrA_\theta\right) = 0$

Define:

$p_\theta \equiv \gamma m r^2\dot{\theta} + qrA_\theta \equiv$ canonical angular momentum

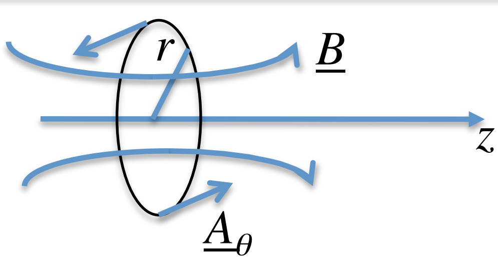
=> $\frac{dp_\theta}{dt} = 0$

Note that the flux $\psi$ enclosed by a circle of radius $r$ about the origin is given by:

$$\psi = \int \underline{B}\cdot d\underline{S} = \int \underline{\nabla}\times\underline{A}\cdot d\underline{S} = \oint \underline{A}\cdot \underline{dl} = 2\pi r A_\theta$$

So $p_\theta = \gamma m r^2\dot{\theta} + \frac{q\psi}{2\pi}$

is conserved along an orbit in axisymmetric geometries




d$\underline{S}$=element of area spanning circle; d$\underline{l}$= line element along circle

"External" electric and magnetic field with azimuthal symmetry $(\partial/\partial\theta \;=\; 0)$ (cf. Reiser section 5.3)

Consider the field $\underline{E}_{\mathrm{ext}}$ and $\underline{B}_{\mathrm{ext}}$ created by external sources (time steady, vacuum fields):

$$\nabla\times\underline{B}_{ext} = 0 \qquad \nabla\times\underline{E}_{ext} = 0 \qquad (\Rightarrow\; E_{ext},\; B_{ext} \;\sim\nabla\phi)$$

$$\nabla\cdot\underline{B}_{ext} = 0 \qquad \nabla\cdot\underline{E}_{ext} = 0 \qquad (\Rightarrow\; \nabla^2\phi = 0\;)$$

In cylindrical coordinates:

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\left(\frac{\partial^2\phi}{\partial z^2}\right)$$

Let $\phi(r,z) = \sum_{\nu=0}^{\infty} f_{2\nu}(z)r^{2\nu} = f_0(z)+f_2(z)r^2+f_4r^4+\ldots$

$$\nabla^2\phi = 0 \qquad \Rightarrow \qquad \sum_{\nu=1}^{\infty}(2\nu)^2 f_{2\nu}(z)r^{2\nu-2} \;+\; \sum_{\nu=0}^{\infty} f_{2\nu}^{//}(z)r^{2\nu} \;=0$$

Let $B_z(0,z) = B(z) = -f_0^{/}(z)$ and let $\phi(0,z) = V(z) = f_0(z)$

$$B_z(r,z) = -\frac{\partial\phi(r,z)}{\partial z} = -f_0^{/}(z)+\frac{1}{4}f_0^{///}(z)r^2-\frac{1}{64}f_0^{/////}(z)r^4+\ldots$$

$$= B(z)-\frac{r^2}{4}\frac{d^2B(z)}{dz^2}+\frac{r^4}{64}\frac{d^4B(z)}{dz^4}+\ldots$$

$$B_r(r,z) = -\frac{\partial\phi(r,z)}{\partial r} = \frac{1}{2}f_0^{//}(z)r-\frac{1}{16}f_0^{////}(z)r^3+\ldots$$

$$= -\frac{r}{2}\frac{dB(z)}{dz}+\frac{r^3}{16}\frac{d^3B(z)}{dz^3}+\ldots$$

Similarly, for the electric field define

Let $\phi(0,z) = V(z) = f_0(z)$

$$\phi(r,z) = V(z) - \frac{r^2}{4}\frac{d^2V(z)}{dz^2} + \frac{r^4}{64}\frac{d^4V(z)}{dz^4} + \ldots$$

$$E_r(r,z) = -\frac{\partial\phi(r,z)}{\partial r} = \frac{1}{2} f_0^{\prime\prime}(z) r - \frac{1}{16} f_0^{\prime\prime\prime\prime}(z) r^3 + \ldots$$

$$= \frac{r}{2} V_0^{\prime\prime}(z) + \frac{r^3}{16}\frac{d^4V(z)}{dz^4} + \ldots$$

$$E_z(r,z) = -\frac{\partial\phi(r,z)}{\partial z} = -f_0^{\prime}(z) + \frac{1}{4} f_0^{\prime\prime\prime}(z) r^2 - \frac{1}{64} f_0^{\prime\prime\prime\prime\prime}(z) r^4 + \ldots$$

$$= -V_0^{\prime}(z) + \frac{r^2}{4}\frac{d^3V(z)}{dz^3} - \frac{r^4}{64}\frac{d^5V(z)}{dz^5} + \ldots$$

RETURNING TO THE RADIAL COMPONENT OF THE MOMENTUM EQUATION IN CYLINDRICAL COORDINATES (EQ I):

$$\frac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 = q(E_r + r\dot{\theta} B_z - \dot{z} B_\theta) \qquad (I)$$

<u>for the external field use</u> (keeping only terms through linear order in $r$)

$$E_{r\,ext} = \frac{r}{2} V'' + O(r^3)$$

$$B_{z\,ext} = B_z(z) + O(r^3)$$

$$B_{\theta\,ext} = 0 \qquad \left[\text{since } \frac{\partial \phi_{mag}}{\partial \theta} = 0\right]$$

<u>for the self field use:</u>

$E_{r\,self}$ = non-zero (to be shown)

$B_{z\,self}$ = 0 in paraxial approx. ( $v_\theta B_{z\,self} \sim (\omega_c r_b/c)^2 E_{r\,self}$ )

$B_{\theta\,self}$ = non-zero (to be shown)

<u>We let:</u>

$\underline{B} = \underline{B}_{ext} + \underline{B}_{self}$

$\underline{E} = \underline{E}_{ext} + \underline{E}_{self}$

## <u>Paraxial ray equation</u>:

$$\frac{d(\gamma m\dot{r})}{dt} - \gamma m r\dot{\theta}^2 = q(E_r + r\dot{\theta}B_z - \dot{z}B_\theta)$$

$$\cong \; q(\frac{V''}{2}r + r\dot{\theta}B(z)) + q(E_r^{self} + \cancel{r\dot{\theta}B_z^{self}} - \dot{z}B_\theta^{self})$$

Inertial

Centrifugal

$E_r$ external (from field gradient)

$v_\theta B_z$ external (solenoid)

self-fields

Now use s as the independent variable: $v_z\, dt = ds$

$$v_z \frac{d(\gamma m v_z r')}{ds} - \gamma m v_z^2 r\theta'^2 = q(\frac{V''}{2}r + r v_z \theta' B(z)) + q(E_r^{self} - v_z B_\theta^{self})$$

Expanding 1st term, using $v_z \cong v$ and dividing by $\gamma m v^2$ $(=\gamma m\beta^2 c^2)$:

$$r'' - r\theta'^2 + \frac{(\gamma\beta)'}{\gamma\beta}r' = \frac{q}{\gamma m\beta^2 c^2}(\frac{V''}{2}r + r\beta c\theta' B + E_r^{self} - v_z B_\theta^{self}) \qquad \text{(PI)}$$

Define $\omega_c \equiv qB/m$. Using definition of $p_\theta$ eliminate $\theta'$ via:

$$\theta' = \frac{p_\theta - q\psi/(2\pi)}{\gamma m r^2\beta c} = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{qB}{2\gamma m\beta c} = \frac{p_\theta}{\gamma m r^2\beta c} - \frac{\omega_c}{2\gamma\beta c}$$

Adding the two $\theta'$ terms in equation (PI):

$$-r\theta'^2 - \frac{r\omega_c\theta'}{\gamma\beta c} = \frac{-p_\theta^2}{\gamma^2 m^2\beta^2 c^2 r^3} + \frac{p_\theta\omega_c}{\gamma^2 m\beta^2 c^2 r} - \frac{r\omega_c^2}{4\gamma^2\beta^2 c^2}$$

$$- \frac{p_\theta\omega_c}{\gamma^2 m\beta^2 c^2 r} + \frac{r\omega_c^2}{2\gamma^2\beta^2 c^2}$$

$$= \frac{-p_\theta^2}{\gamma^2 m^2\beta^2 c^2 r^3} + \frac{r\omega_c^2}{4\gamma^2\beta^2 c^2}$$

So eq. P1 becomes:

$$r''+\frac{(\gamma\beta)'}{\gamma\beta}r'=\frac{q}{\gamma m\beta^2c^2}(\frac{V''}{2}r)-\frac{r\omega_c^2}{4\gamma^2\beta^2c^2}+\frac{p_\theta^2}{\gamma^2m^2\beta^2c^2r^3}+\frac{q}{\gamma m\beta^2c^2}\left(E_r^{self}-v_zB_\theta^{self}\right) \qquad \text{(P2)}$$

Now

$$\frac{d\gamma mc^2}{dt}=q\underline{E}\cdot\underline{v} \quad\Rightarrow\quad \gamma' mc^2=q\frac{\underline{E}\cdot\underline{v}}{v_z}\cong qE_z \qquad \text{so}\quad \gamma''=-\frac{q}{mc^2}\left(V''+\frac{\partial^2\phi^{self}}{\partial z^2}\right)$$

How do we calculate $\frac{q}{\gamma m\beta^2c^2}\left(\frac{V''}{2}r+E_r^{self}-v_zB_\theta^{self}\right)$ ?

$$\nabla^2\phi^{self}=-\frac{\rho}{\varepsilon_0} \quad\Rightarrow\quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi^{self}}{\partial r}\right)=-\frac{\rho}{\varepsilon_0}-\frac{\partial^2\phi^{self}}{\partial z^2}$$

$$\Rightarrow\quad \frac{\partial}{\partial r}\left(r\frac{\partial\phi^{self}}{\partial r}\right)=-\frac{r\rho}{\varepsilon_0}-\frac{r\partial^2\phi^{self}}{\partial z^2}$$

$$r\frac{\partial\phi^{self}}{\partial r}=-\frac{1}{2\pi\varepsilon_0}\int_0^r 2\pi\tilde{r}\rho(\tilde{r})d\tilde{r}-\int_0^r\frac{\tilde{r}\partial^2\phi^{self}}{\partial z^2}d\tilde{r}$$

$$=-\frac{\lambda(r)}{2\pi\varepsilon_0}-\frac{r^2}{2}\frac{\partial^2\phi^{self}}{\partial z^2}$$

(Here we have included only the lowest order term for $\frac{\partial^2\phi^{self}}{\partial z^2}$).

$$\Rightarrow E_r^{self}\cong\frac{\lambda(r)}{2\pi\varepsilon_0 r}+\frac{r}{2}\frac{\partial^2\phi^{self}}{\partial z^2}$$

$$\underline{\nabla}\times\underline{B}^{self}=\mu_0\underline{J} \quad\Rightarrow\quad 2\pi rB_\theta^{self}=\mu_0\int_0^r 2\pi\tilde{r}J_z(\tilde{r})d\tilde{r}=\mu_0 v_z\lambda(r)$$

$$B_\theta^{self}=\frac{\mu_0 v_z\lambda(r)}{2\pi r}=\frac{v_z}{c^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r}$$

$$\left(\frac{V''}{2}r+E_r^{self}-v_zB_\theta^{self}\right)=\frac{r}{2}\left(V''+\frac{\partial^2\phi^{self}}{\partial z^2}\right)+\left(1-\frac{v_z^2}{c^2}\right)\frac{\lambda(r)}{2\pi\varepsilon_0 r}$$

$$=-\frac{\gamma'' mc^2}{2q}r+\frac{1}{\gamma^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r}$$

Leading to the "Paraxial Ray Equation:"

$$r'' + \frac{(\gamma\beta)'}{\gamma\beta} r' + \frac{\gamma''}{2\gamma\beta^2} r + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r + \left(\frac{p_\theta}{\gamma\beta mc}\right)^2 \frac{1}{r^3} - \frac{q}{\gamma^3 m\beta^2 c^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r} = 0$$

Inertial

Accelerative damping (of angle $r'$)

$E_r$ from converging field lines

Solenoidal focusing ($v_\theta B_z$ – part of centrifugal term)

Part of centrifugal term

Self-field $(E_r^{self} - v_z B_\theta^{self})$

which together with the conservation of canonical angular momentum,

$$p_\theta \equiv \gamma\beta mcr^2\theta' + \frac{m\omega_c r^2}{2}$$

and initial conditions, specifies the orbit a particle an axisymmetric field.

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# Moment Equations

Vlasov eqtn: $\frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} = 0$

Let $g = g(x, x', y, y')$ ; $N = \iiiint f \, dx\, dx'\, dy\, dy'$

Multiply Vlasov equation by $g$ & $\frac{1}{N}\iiiint dx\, dx'\, dy\, dy'$

$$\int dx\, dx'\, dy\, dy' \left[ g \frac{\partial f}{\partial s} + g x' \frac{\partial f}{\partial x} + g x'' \frac{\partial f}{\partial x'} + g y' \frac{\partial f}{\partial y} + g y'' \frac{\partial f}{\partial y'} \right] = 0$$

$$\Rightarrow \frac{d}{ds}\langle g \rangle + \overbrace{\underbrace{\frac{1}{N}\iiint g f \Big|_{x=-\infty}^{\infty}}_{\to 0} - \underbrace{\frac{1}{N}\iiiint \frac{\partial g}{\partial x} f x'}_{= \langle x' \frac{\partial g}{\partial x} \rangle}}^{\text{INTEGRATE BY PARTS}} + \cdots = 0$$

$$\Rightarrow \frac{d}{ds}\langle g \rangle = \langle x' \frac{\partial g}{\partial x} \rangle + \langle x'' \frac{\partial g}{\partial x'} \rangle + \langle y' \frac{\partial g}{\partial y} \rangle + \langle y'' \frac{\partial g}{\partial y'} \rangle$$

But $\frac{dg}{ds} = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial x'} x'' + \frac{\partial g}{\partial y} y' + \frac{\partial g}{\partial y'} y''$

$$\Rightarrow \frac{d}{ds}\langle g \rangle = \langle g' \rangle$$

So $\frac{d}{ds}\langle x^2 \rangle = 2\langle x x' \rangle$

$\frac{d}{ds}\langle x'^2 \rangle = 2\langle x' x'' \rangle$ etc...

$\frac{d}{ds}\langle x x' \rangle = \langle x x'' \rangle + \langle x'^2 \rangle$

# ENVELOPE EQUATION FOR AXISYMMETRIC BEAMS

LET $r_b^2 = 2\langle r^2\rangle = 2(\langle x^2\rangle + \langle y^2\rangle) = 4\langle x^2\rangle$ for an axisymmetric beam

$$2r_b r_b' = 4\langle rr'\rangle \quad \Rightarrow \quad r_b' = \frac{2\langle rr'\rangle}{r_b}$$

$$r_b'' = \frac{2\langle rr''\rangle + 2\langle r'^2\rangle}{r_b} - \frac{2\langle rr'\rangle}{r_b^2}\left(\frac{2\langle rr'\rangle}{r_b}\right)$$

$$= 2\frac{\langle rr''\rangle}{r_b} + \frac{4\langle r^2\rangle\langle r'^2\rangle - 4\langle rr'\rangle^2}{r_b^3}$$

WHAT IS $\langle rr''\rangle$?

RECALL EQUATION P1 (ON PATH TO PARAXIAL RAY EQUATION):

$$r'' - r\theta'^2 + \frac{\gamma'}{\beta^2\gamma} r' = \frac{q}{\gamma m \beta^2 c^2}\left(\frac{V''}{2} r + r\beta c \theta' B + E_r^{self} - v_z B_\theta^{self}\right),$$

P1 may be rewritten:

$$r'' - r\theta'^2 + \frac{\gamma'}{\beta^2\gamma} r' = \frac{q}{\gamma m \beta^2 c^2}\left[\frac{-mc^2}{q}\gamma'' \frac{r}{2} + \frac{\lambda(r)}{\gamma^2 2\pi\epsilon_0 r} + r\beta c \theta' B\right]$$

$$\Rightarrow \boxed{r'' + \frac{\gamma'}{\beta^2\gamma} r' + \frac{\gamma''}{2\beta^2\gamma} r - \frac{q}{\gamma^3 m v_z^2}\frac{\lambda(r)}{2\pi\epsilon_0 r} + \underbrace{\frac{-\omega_c}{\gamma\beta c}\theta' r - r\theta'^2}_{} = 0}$$

What is $\langle r r'' \rangle$?

$$\langle r r'' \rangle + \frac{-\omega_c}{\gamma\beta c}\langle \theta' r^2 \rangle - \langle r^2\theta'^2 \rangle + \ldots = 0$$

$$\langle p_\theta \rangle^2 = \gamma^2 m^2 \beta^2 c^2 \langle r^2\theta' \rangle^2 + \frac{\omega_c^2}{4} m^2 \langle r^2 \rangle^2 + \omega_c \gamma m^2 \beta c \langle r^2\theta' \rangle \langle r^2 \rangle$$

$$\Rightarrow \frac{-\omega_c}{\gamma\beta c}\langle \theta' r^2 \rangle = \frac{-\omega_c}{\gamma\beta c}\left[\frac{\langle p_\theta \rangle^2}{\omega_c \gamma m^2 \beta c \langle r^2 \rangle} - \frac{\omega_c \langle r^2 \rangle}{4\gamma\beta c} - \frac{\gamma\beta c}{\omega_c}\frac{\langle r^2\theta' \rangle^2}{\langle r^2 \rangle}\right]$$

$$\Rightarrow \langle r r'' \rangle - \frac{\langle p_\theta \rangle^2}{\gamma^2 m^2 \beta^2 c^2 \langle r^2 \rangle} + \frac{\omega_c^2 \langle r^2 \rangle}{4\gamma^2 \beta^2 c^2} - \frac{\langle r^2\theta' \rangle^2}{\langle r^2 \rangle} + \langle r^2\theta'^2 \rangle + \ldots = 0$$

$$\Rightarrow$$

$$\langle r r'' \rangle = \frac{\gamma'}{\beta^2\gamma}\langle r r'\rangle + \frac{\gamma''}{2\beta^2\gamma}\langle r^2\rangle - \frac{q}{\gamma^3 m v_z^2}\frac{\langle \lambda(r)\rangle}{2\pi\epsilon_0} + \frac{\langle p_\theta\rangle^2}{(\gamma m\beta c)^2\langle r^2\rangle} - \frac{\omega_c^2\langle r^2\rangle}{4(\gamma^2\beta c)^2} - \frac{\langle r^2\theta'\rangle^2}{\langle r^2\rangle} + \langle r^2\theta'^2\rangle$$

$$r_b'' = \frac{2\langle r r''\rangle}{r_b} + \frac{4\langle r^2\rangle\langle r'^2\rangle - 4\langle r r'\rangle}{r_b^3}$$

$$= \frac{\gamma'}{\beta^2\gamma}\frac{2\langle r r'\rangle}{r_b} + \frac{\gamma''}{2\beta^2\gamma}\frac{2\langle r^2\rangle}{r_b} - \frac{2q}{\gamma^3 m v_z^2}\frac{\langle \lambda(r)\rangle}{2\pi\epsilon_0}\frac{1}{r_b}$$

$$+ \frac{\langle p_\theta\rangle^2}{(\gamma m\beta c)^2}\frac{2}{\langle r^2\rangle r_b} - \frac{\omega_c^2}{4(\gamma\beta c)^2}\frac{2\langle r^2\rangle}{r_b} - \frac{2\langle r^2\theta'\rangle^2}{r_b\langle r^2\rangle}$$

$$+ \frac{2\langle r^2\theta'^2\rangle}{r_b} + \frac{4\langle r^2\rangle\langle r'^2\rangle - 4\langle r r'\rangle^2}{r_b^3}$$

USING $r_b^2 \equiv 2\langle r^2\rangle$ & $r_b' = \dfrac{2\langle r r'\rangle}{r_b}$

ENVELOPE EQUATION

$$\Rightarrow \boxed{r_b'' + \frac{\gamma'}{\beta^2\gamma}r_b' + \frac{\gamma''}{2\beta^2\gamma}r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b + \frac{-4\langle p_\theta\rangle^2}{(\gamma m\beta c)^2 r_b^3} - \frac{\epsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0}$$

WHERE $\epsilon_r^2 = 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle r r'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$

# ENVELOPE EQUATION -- CONTINUED

$$\boxed{r_b'' + \frac{\gamma'}{\beta^2\gamma} r_b' + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle p_\theta\rangle^2}{(\gamma m \beta c)^2 r_b^3} - \frac{\epsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0}$$

COMPARE WITH THE SINGLE PARTICLE PARAXIAL RAY EQUATION:

$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma} r}_{E_r} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{v_\theta B_z \text{ - CENTRIPETAL}} - \underbrace{\left(\frac{p_\theta}{\gamma m \beta c}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}}_{E_r - v_z B_\theta \text{ self field}} = 0$$

$$\epsilon_r^2 = 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle rr'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$$

NOTE THAT FOR AXISYMMETRIC BEAMS ($\rho = \rho(r)$ ONLY)

$$\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle = 2\langle x^2\rangle$$

$$\Rightarrow 2\langle rr'\rangle = 4\langle xx'\rangle$$

$$\& \quad \langle x'^2\rangle + \langle y'^2\rangle = 2\langle x'^2\rangle = \langle r'^2\rangle + \langle r^2\theta'^2\rangle$$

DEFINE $\varepsilon_x^2 = 16\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)$

$$\Rightarrow \boxed{\varepsilon_r^2 = \varepsilon_x^2 - 4\langle r^2\theta'\rangle^2}$$

EXAMPLES OF
SYSTEMS WITH AXIAL SYMMETRY

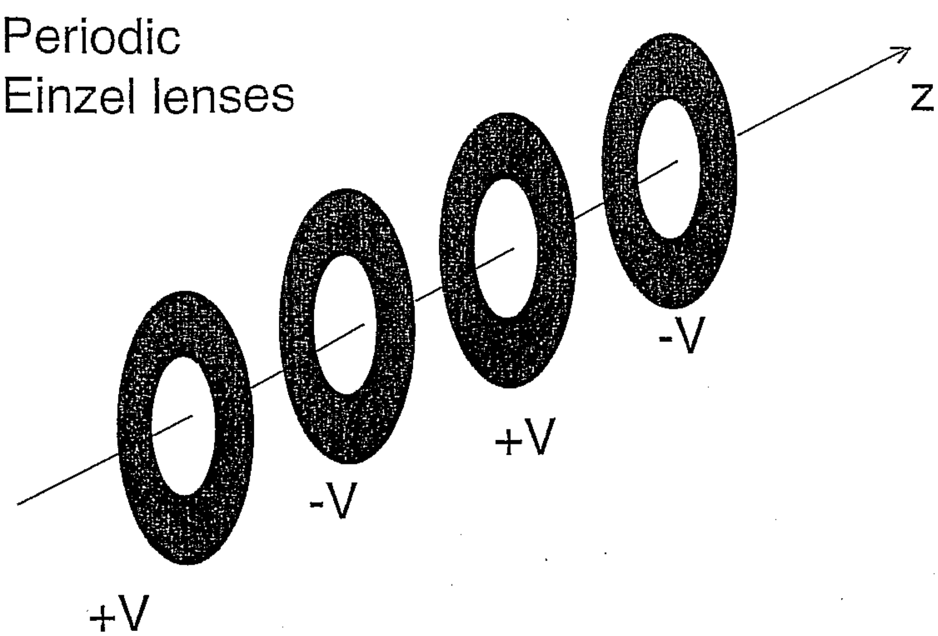
- PERIODIC SOLENOIDS
- EINZEL LENSES
- CONTINUOUS FOCUSING

EXAMPLES OF
SYSTEMS WITHOUT AXIAL SYMMETRY

- ELECTRIC OR MAGNETIC QUADRUPOLE

$\Longrightarrow$ USE CARTESIAN COORDINATES WITH
ELLIPTICAL SPACE CHARGE SYMMETRY

# EXAMPLES OF AXISYMMETRIC SYSTEMS




## PERIODIC SOLENOIDS

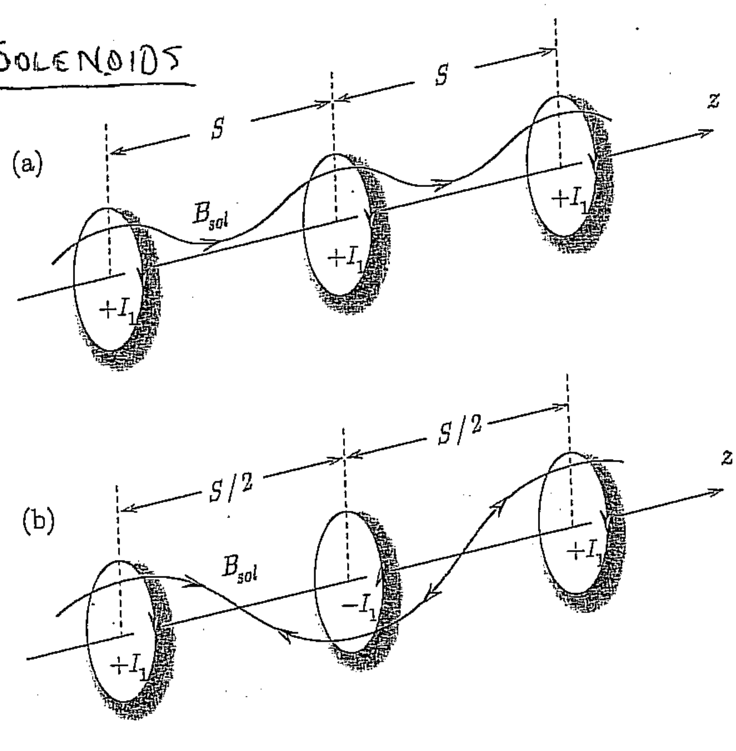



**Figure 3.2.** Schematic of magnet sets producing a periodic focusing solenoidal field with axial periodicity length $S$. In Fig. 3.2 (a), successive coils are spaced by $S$ and have the same current polarity $+I_1, +I_1, \cdots$. In Fig. 3.2 (b), successive coils are spaced by $S/2$ and have alternating current polarities $+I_1, -I_1, +I_1, \cdots$.

(FIGURE FROM DAVIDSON & QIN 2003) p. 55 "PHYSICS OF INTENSE CHARGE PARTICLE BEAMS IN HIGH ENERGY ACCELERATORS"

EXAMPLE OF NON-AXISYMMETRIC SYSTEM

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figure from
Davidson & Qin, 2003.

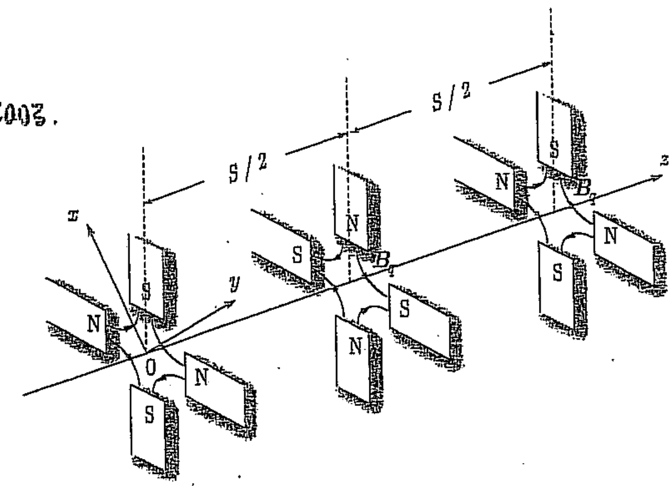


**Figure 3.1.** Schematic of magnet sets producing an alternating-gradient quadrupole field with axial periodicity length $S$.

# NON-AXISYMMERIC SYSTEM



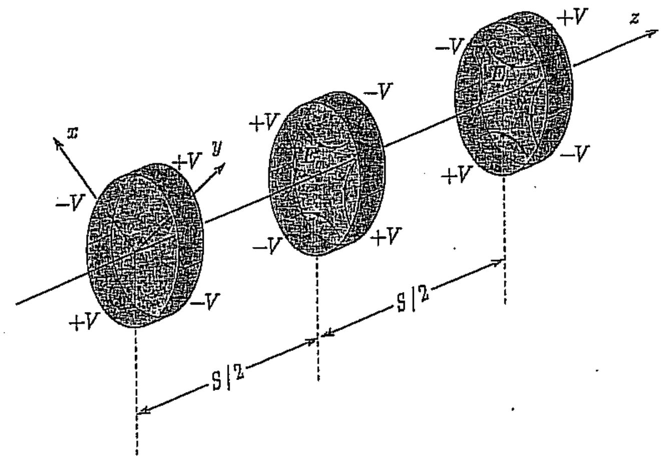


**Figure 3.3.** Schematic of conductor configuration with applied voltages producing an alternating-gradient quadrupole electric field with axial periodicity length $S$.

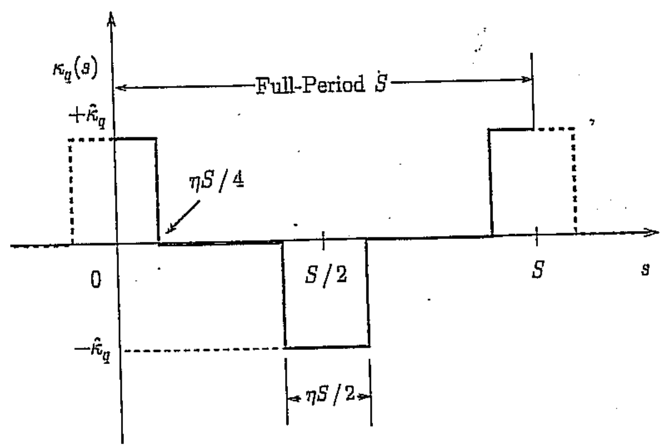


**Figure 3.7.** Alternating step-function model of a periodic quadrupole lattice with filling factor $\eta$ for the lens elements. The figure shows a plot of the quadrupole coupling coefficient $\kappa_q(s)$ versus $s$ for one full period ($S$) of the lattice. Such a configuration is often called a FODO transport lattice (acronym for focusing-off-defocusing-off).

FIGURES FROM DAVIDSON & QIN 2003

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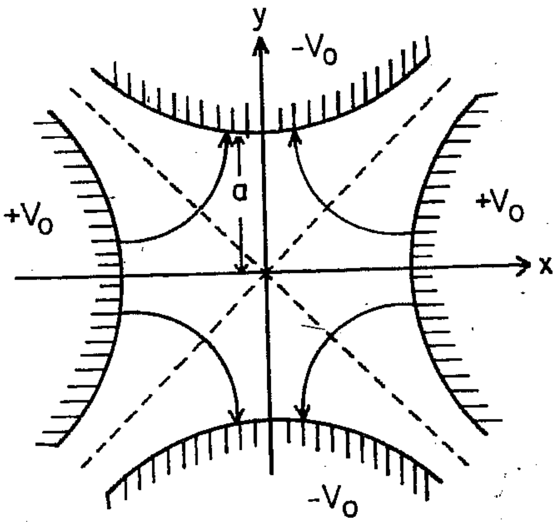
FROM
REISER, p.112

$E_x = -E'x$
$E_y = E'y$



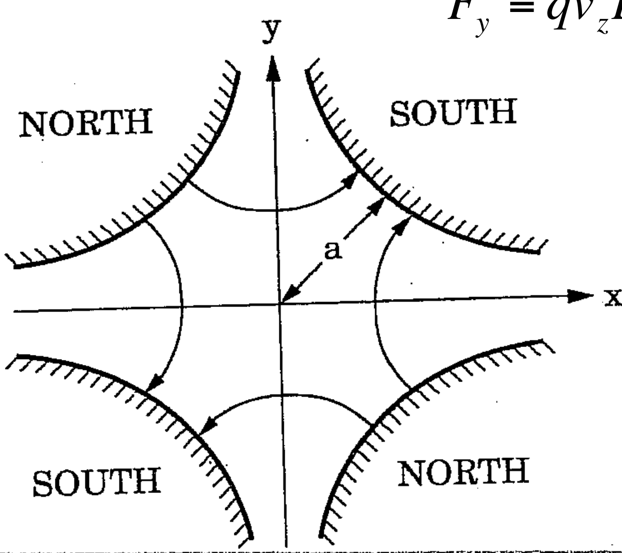

$F_x = -qE'x$
$F_y = qE'y$

ELECTROSTATIC
QUADS

**Figure 3.15.** Electrodes and force lines in an electrostatic quadrupole.

$B_x = B'y$
$B_y = B'x$

$$F_x = -qv_zB'x$$
$$F_y = qv_zB'y$$



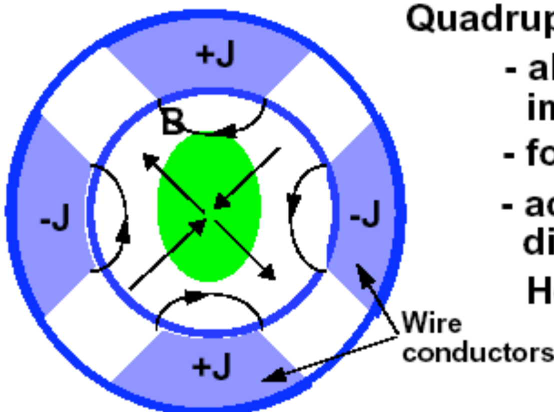

MAGNETIC
QUADS

# Heavy ion accelerators use alternating gradient quadrupoles to confine the beams

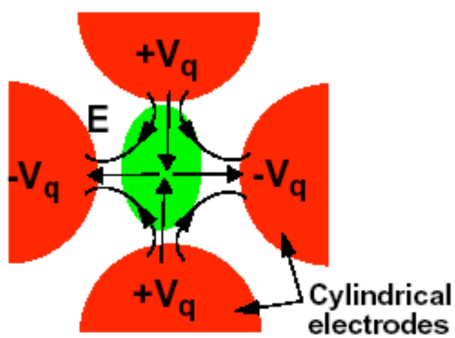
Space-charge forces and thermal forces act to expand beam

Quadrupoles (magnetic or electric):

- alternately provide inward then outward impulse
- focus in one plane and defocus in other
- act as linear lenses. (Force proportional to distance from axis).

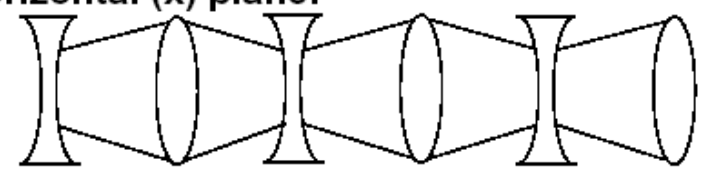



Magnetic quad

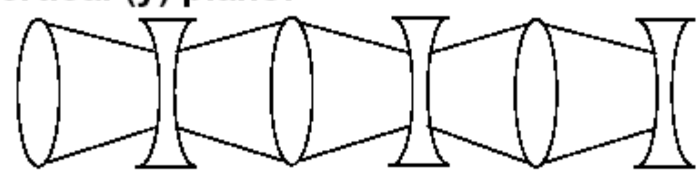



Electric quad

Horizontal (x) plane:

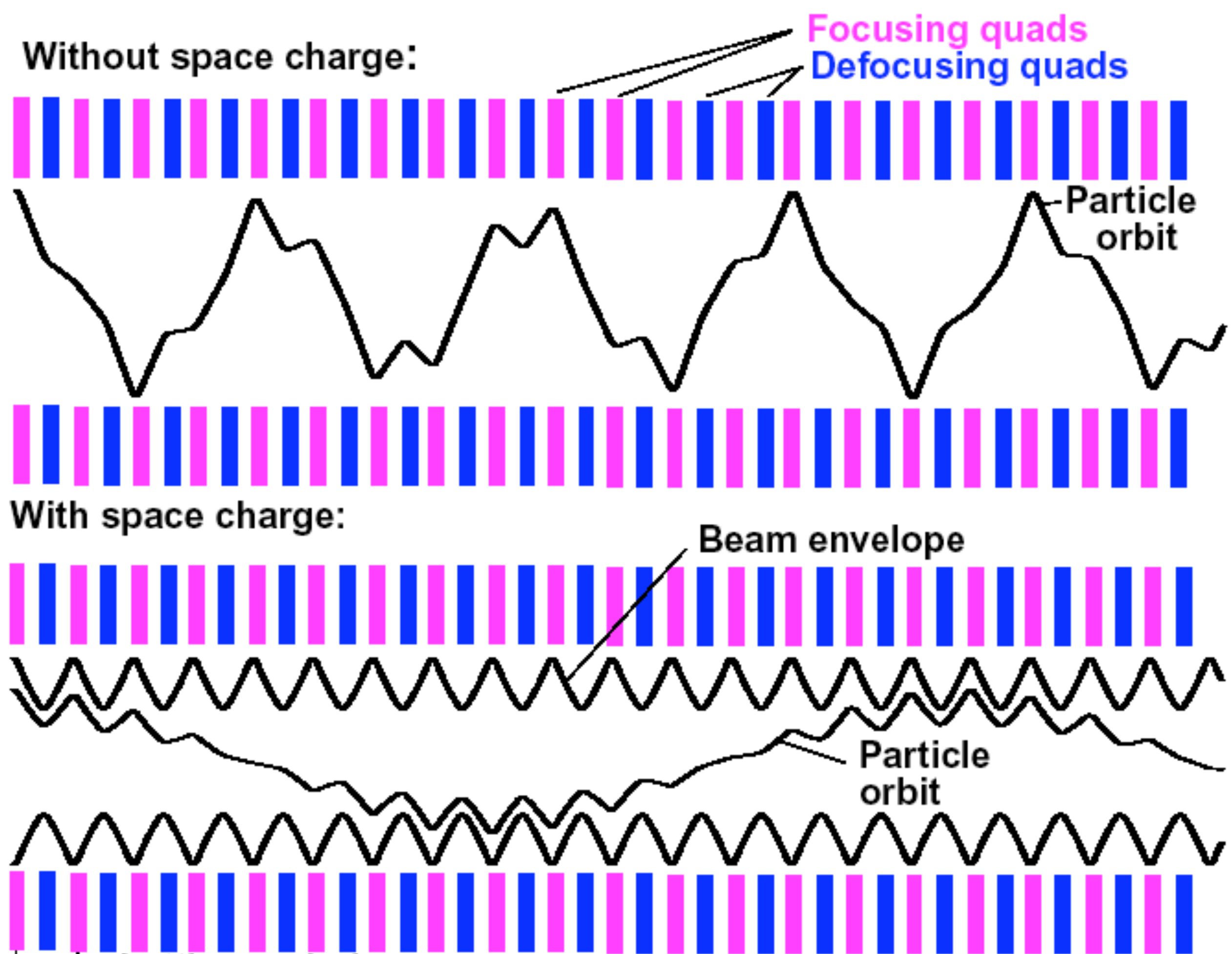


Vertical (y) plane:



Average displacement is larger in focusing lenses so the net effect is focusing.

# Space charge reduces betatron phase advance



# ENVELOPE EQUATIONS FOR NON-AXISYMMETRIC SYSTEMS 

$$r_x^2 \equiv 4\langle x^2\rangle \qquad r_y^2 \equiv 4\langle y^2\rangle$$

$$2 r_x r_x' = 8\langle x x'\rangle$$

$$r_x' = \frac{4\langle x x'\rangle}{r_x}$$

$$r_x'' = \frac{4\langle x x''\rangle}{r_x} + \frac{4\langle x'^2\rangle}{r_x} - \frac{4\langle x x'\rangle}{r_x^2} r_x'$$

$$= \frac{4\langle x x''\rangle}{r_x} + \frac{16\langle x'^2\rangle\langle x^2\rangle}{r_x^2} - \frac{16\langle x x'\rangle^2}{r_x^2}$$

DEFINE $\varepsilon_x^2 = 16\left(\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2\right)$

$$\Rightarrow \boxed{r_x'' = \frac{4\langle x x''\rangle}{r_x} + \frac{\varepsilon_x^2}{r_x^3}}$$

SO HOW DO WE CALCULATE $\langle x x''\rangle$?

RETURN TO SINGLE PARTICLE EQUATION (IN CARTESIAN COORDINATES)

$$\frac{d}{dt}(\gamma m \dot x) = \dot\gamma m \dot x + \gamma m \ddot x = q\left(E_x + \dot y B_z - \dot z B_y\right)$$

$x''$
& similarly
$y''$

QUADRUPOLE FOCUSING
SPACE-CHARGE OF ELLIPTICAL BEAMS

TO BE CONTINUED . . .

J. BARNARD

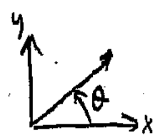
## QUADRUPOLE FOCUSING

Now, relax radial symmetry:

FOR $\nabla \cdot E = 0$ or $\nabla \times B = 0$

EXPAND FIELD IN CYLINDRICAL "MULTIPOLES":

$$E_r, B_r = \sum_{n=1}^{\infty} f_n r^{n-1} \cos(n\theta)$$




$E_x = E_r \cos\theta - E_\theta \sin\theta$

$E_y = E_r \sin\theta + E_\theta \cos\theta$

$$E_\theta, B_\theta = \sum_{n=1}^{\infty} f_n r^{n-1} \sin(n\theta)$$

$$n=1 \quad \Rightarrow \quad \text{dipole} \quad \begin{cases} E_r = f_1 \cos\theta \\ E_\theta = -f_1 \sin\theta \end{cases} \Rightarrow \quad \begin{aligned} E_x &= f_1 \\ E_y &= 0 \end{aligned}$$

$$n=2 \quad \Rightarrow \quad \text{quadrupole} \quad \begin{cases} E_r = f_2 r \cos 2\theta \\ E_\theta = -f_2 r \sin 2\theta \end{cases} \Rightarrow \quad \begin{aligned} E_x &= f_2 x \\ E_y &= -f_2 y \end{aligned}$$

NOTE: ABOVE EXPANSION IS VALID WHEN $\underline{E \text{ or } B \neq \text{function}(z)}$.
FOR MAGNETS OF FINITE AXIAL EXTENT, FOR EACH FUNDAMENTAL n-pole, A SET OF HIGHER ORDER MULTIPOLES WITH SAME AZIMUTHAL SYMMETRY ARE REQUIRED TO SATISFY $\nabla^2\phi = 0$.

FOR EXAMPLE FOR A FUNDAMENTAL QUADRUPOLE THE FIELD MAY BE EXPANDED:

$$E_r = \sum_{\nu=0}^{\infty} f_{2,\nu}(z) \, [1+\nu] \, r^{1+2\nu} \cos[2\theta]$$

$$E_\theta = \sum_{\nu=0}^{\infty} -f_{2,\nu}(z) \, r^{1+2\nu} \sin[2\theta]$$

$$E_z = \sum_{\nu=0}^{\infty} \frac{1}{2} \frac{df_{2,\nu}}{dz} r^{2+2\nu} \cos 2\theta$$

with $f_{2,\nu+1}(z) = \dfrac{-1}{4(\nu+1)(\nu+3)} \dfrac{d^2 f_{2,\nu}(z)}{dz^2}$

SEE LUND, S. M. (1996) FOR EXAMPLE. HIF note 96-1 LLNL.