# Transverse Equilibrium Distributions*

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## Transverse Equilibrium Distribution Functions: Outline





## Transverse Equilibrium Dist. Functions: Detailed Outline

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## S0: Review: Equations of Motion and Approximations

### Overview results from Transverse Particle Dynamics to frame formulation

Transverse particle equations of motion in terms of applied field components $\mathbf{E}^a$, $\mathbf{B}^a$ were derived as:

$$\mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}_\perp^a + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}_\perp^a + \frac{qB_z^a}{m\gamma_b\beta_b c}\mathbf{x}_\perp'\times\hat{\mathbf{z}} - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi$$

Here, $\phi$ is the beam self-field potential given by the solution to the Poisson equation with beam charge density $\rho$

$$\nabla^2\phi = \frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial\mathbf{x}}\phi = -\frac{\rho}{\epsilon_0}$$
$$+\ \text{Boundary Conditions on } \phi$$

Equations derived under assumptions:
- No bends (fixed *x-y-z* coordinate system with no local bends)
- Paraxial equations ( $x'^2, y'^2 \ll 1$ )
- No dispersive effects ($\beta_b$ same all particles), acceleration allowed ( $\beta_b \neq \text{const}$ )
- Electrostatic and leading-order (in $\beta_b$ ) self-magnetic interactions



These equations can be reduced when the applied focusing fields are linear to:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi$$
$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi$$

where
$$\kappa_x(s) = x\text{-focusing function of lattice}$$
$$\kappa_y(s) = y\text{-focusing function of lattice}$$

These equations can be applied to:

Continuous Focusing:
$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Good qualitative guide but not physically realizable

Solenoidal Focusing: (implicitly expressed within a rotating frame)

$$\kappa_x(s) = \kappa_y(s) = k_L^2(s) = \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2$$

$$[B\rho] = \frac{m\gamma_b\beta_b c}{q} \qquad \omega_c(s) = \frac{qB_{z0}(s)}{m}$$

Quadrupole Focusing:

$$\kappa_x(s) = -\kappa_y(s) = \begin{cases} \frac{G(s)}{\beta_b c[B\rho]}, & \text{Electric} \\ \frac{G(s)}{c[B\rho]}, & \text{Magnetic} \end{cases}$$

$G$ is the field gradient which for linear applied fields is:

$$G(s) = \begin{cases} -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2}, & \text{Electric} \\ \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \frac{B_p}{r_p}, & \text{Magnetic} \end{cases}$$

If "normalized" variables are employed to compensate for acceleration induced damping of particle oscillations, the equations can then be analyzed using a coasting beam formulation with $\gamma_b\beta_b = \text{const}$

$$\boxed{\begin{aligned} x'' + \kappa_x(s)x &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi \\ y'' + \kappa_y(s)y &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi \end{aligned}}$$

See Transverse Particle Dynamics notes for details of interpretation

Using adjusted focusing strength

$$\kappa_x \to \kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}$$



## S1: Transverse Vlasov-Poisson Model: for a 2D coasting, single species beam with electrostatic self-fields propagating in a linear focusing lattice:

$\mathbf{x}_\perp,\ \mathbf{x}'_\perp$ transverse particle coordinate, angle

$q,\ \ m$ charge, mass

$\gamma_b,\ \ \beta_b$ axial relativistic factors

$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle distribution

$H_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle Hamiltonian

Vlasov Equation (see J.J. Barnard, Introductory Lectures):

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Particle Equations of Motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Hamiltonian (see S.M. Lund, lectures on Transverse Particle Dynamics):

$$H_\perp = \frac{1}{2}\mathbf{x}'^{\,2}_\perp + \frac{1}{2}\kappa_x(s)x^2 + \frac{1}{2}\kappa_y(s)y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Poisson Equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0} = -\frac{q}{\epsilon_0}\int d^2\mathbf{x}'_\perp\ f_\perp$$

Charge Density:

$$\rho = q\int d^2\mathbf{x}'_\perp\ f_\perp$$

+ boundary conditions on $\phi$



## Comments on Normalization

Normalization choices of distribution function $f_\perp$

$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)d^2x_\perp d^2x'_\perp =$ Number of particles per unit axial length within $d^2x_\perp\ d^2x'_\perp$ of $\mathbf{x}_\perp,\ \mathbf{x}'_\perp$ at lattice position $s$

Transverse distribution $f_\perp$ is actually projection of 3D distribution $f$

$f(x, y, z, x', y', p_z, s)dxdydzdx'dy'dp_z$
$=$ Number of particles within $dxdydzdx'dy'dp_z$ of $\mathbf{x},\ \mathbf{x}'_\perp, p_z$ at lattice position $s$

Project:

$$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \int_{-\infty}^{\infty} dp_z\ f(x, y, z, x', y', p_z, s)$$

- Vlasov equation is more typically derived in 3D variables $\mathbf{x},\ \mathbf{p}$ in texts:
- "Particles" in 2D transverse model are really charged rods uniform in z
- Later work will motivate how this 2D geometry can get the right answers in many contexts to physical 3D systems
  - Analysis much easier in lower dimensions!



## Projections of Distribution

Integrate over coordinate to "project" distribution

- Certain projections have are needed to solve for beam self fields and have well developed interpretations

Number Density:

$$n(\mathbf{x}_\perp, s) = \int d^2\mathbf{x}'_\perp\ f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) \qquad [[n]] = \frac{\text{number}}{\text{meter}^3}$$

Charge Density:

$$\rho(\mathbf{x}_\perp, s) = qn(\mathbf{x}_\perp, s) = q\int d^2\mathbf{x}'_\perp\ f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) \qquad [[\rho]] = \frac{\text{Coulombs}}{\text{meter}^3}$$

Line-Charge:

- Constant of motion if particles not lost/created (see problem sets)
  - Particles must go somewhere so total weight/number conserved

$$\begin{aligned} \lambda &= q\int d^2x_\perp \int d^2x'_\perp\ f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) \qquad [[\lambda]] = \frac{\text{Coulombs}}{\text{meter}} \\ &= q\int d^2x\ n(\mathbf{x}_\perp, s) \quad = \quad \text{const} \end{aligned}$$

## Averages over the distribution

Take projections of distribution with quantities of interest to average over the distribution

- Phase-space 6D (4D here): Hard to see what is going on in high dimensions so take averages on projection to more easily interpret beam evolution

Phase-Space Average:

- Averaged quantity depends only on $s$

$$\boxed{\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp} \quad = \quad \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\lambda/q}}$$

Example: Statistical edge measure of beam $x$-edge $\qquad r_x(s) \equiv 2\langle x^2 \rangle_\perp^{1/2}$

Restricted (angle) Average:

- Averaged quantity depends on

$$\boxed{\langle \cdots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \; \cdots f_\perp}{\int d^2x'_\perp \; f_\perp} \quad = \quad \frac{\int d^2x'_\perp \; \cdots f_\perp}{n}}$$

Example: $x$-plane flow $\qquad X'(\mathbf{x}_\perp, s) \equiv \langle \mathbf{x}'_\perp \rangle_{\mathbf{x}'_\perp}$



## Expression of Vlasov Equation

Hamiltonian expression of the Vlasov equation:

$$\frac{d}{ds} f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

$$= \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

<u>Reminder</u>: Hamiltonian form eqations of motion

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}$$

$$\frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Using the equations of motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} = \mathbf{x}'_\perp$$

$$\frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} = -\left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right)$$

$$\boxed{\frac{\partial f_\perp}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right) \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0}$$

In formal dynamics, a "Poisson Bracket" notation is often employed:

$$\frac{d}{ds} f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

$$\equiv \frac{\partial f_\perp}{\partial s} + \underbrace{\{H_\perp, f_\perp\}}_{\text{Poisson Bracket}} = 0$$



## Comments on Vlasov-Poisson Model

- Collisionless Vlasov-Poisson model good for intense beams with many particles
  - Collisions negligible, see: S.M. Lund, Transverse Particle Dynamics, S13
- Vlasov-Poisson model can be solved as an initial value problem

1) $f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s = s_i) =$ Initial "condition" (function) specified

2) Vlasov-Poisson model solved for subsequent evolution in $s$ for $f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ for $s \geq s_i$

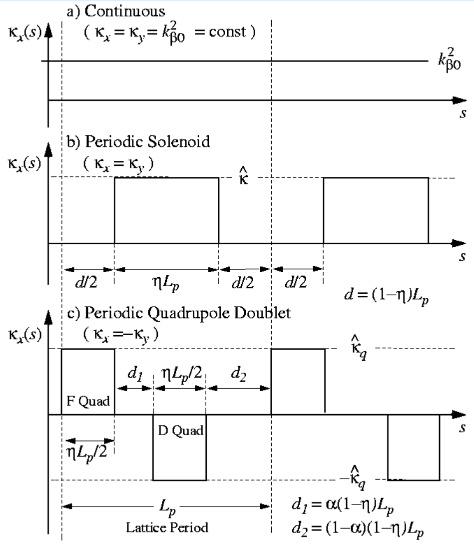
- The Vlasov distribution function $f_\perp \geq 0$ can be thought of as a probability distribution evolving in $\mathbf{x}_\perp - \mathbf{x}'_\perp$ phase-space.
  - Particles/probability neither created nor destroyed
  - Flows along characteristic particle trajectories in $\mathbf{x}_\perp - \mathbf{x}'_\perp$ phase-space
  - Vlasov equation a higher-dimensional continuity equation describing incompressible flow in $\mathbf{x}_\perp - \mathbf{x}'_\perp$ phase-space
- The coupling to the self-field via the Poisson equation makes the Vlasov-Poisson model *highly* nonlinear

$$\boxed{\rho = q\int d^2x'_\perp \; f_\perp \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0}}$$



- Vlasov-Poisson system is written without acceleration, but the transforms developed to identify the normalized emittance in the lectures on Transverse Particle Dynamics can be exploited to generalize all result presented to (weakly) accelerating beams (interpret in tilde variables)
- For solenoidal focusing the system can be interpreted in the rotating Larmor Frame, see: lectures on Transverse Particle Dynamics
- System as expressed applies to 2D (unbunched) beam as expressed
  - Considerable difficulty in analysis for 3D version for transverse/longitudinal physics

## Review: Focusing lattices, continuous and periodic (simple piecewise constant):




Lattice Period $L_p$

Occupancy $\eta$
$\eta \in [0, 1]$

Solenoid description carried out implicitly in Larmor frame
[see: S.M. Lund, lectures on Transverse Particle Dynamics]

Syncopation Factor $\alpha$

$$\alpha \in [0, \frac{1}{2}]$$

$$\alpha = \frac{1}{2} \implies FODO$$



## Example Hamiltonians:

See S.M. Lund Lectures on Transverse Particle Dynamics for more details

Continuous focusing: $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$\boxed{H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi}$$

Solenoidal focusing: (in Larmor frame variables) $\kappa_x = \kappa_y = \kappa(s)$

$$\boxed{H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi} \qquad = \left(\frac{B_{z0}}{2[B\rho]}\right)^2$$

Quadrupole focusing: $\kappa_x = -\kappa_y = \kappa(s)$

$$\kappa = \begin{cases} G/[B\rho] & \text{Magnetic} \\ G/(\beta_b c[B\rho] & \text{Electric} \end{cases}$$

$$\boxed{H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa x^2 - \frac{1}{2}\kappa y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi}$$

$$[B\rho] = \frac{m\gamma_b\beta_b c}{q} = \text{Rigidity}$$



## Review: Undepressed particle phase advance $\sigma_0$ is typically employed to characterize the applied focusing strength of periodic lattices:

see: S.M. Lund lectures on Transverse Particle Dynamics

$x$-orbit without space-charge satisfies Hill's equation

$$x''(s) + \kappa_x(s)x(s) = 0$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \mathbf{M}_x(s \mid s_i) \cdot \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

$\mathbf{M}_x =$ 2 x 2 Transfer Matrix from $s = s_i$ to $s$

Undepressed phase advance

$$\boxed{\cos\sigma_{0x} = \frac{1}{2}\text{Tr }\mathbf{M}_x(s_i + L_p|s_i)}$$

- Subscript $0x$ used stresses $x$-plane value and zero ($Q = 0$) space-charge effects

Single particle (and centroid) stability requires:

$$\frac{1}{2}|\text{Tr }\mathbf{M}_x(s_i + L_p|s_i)| \leq 1 \longrightarrow \boxed{\sigma_{0x} < 180^\circ}$$

[Courant and Snyder, Annals of Phys. **3**, 1 (1958)]

Analogous equations hold in the $y$-plane



The undepressed phase advance can also be equivalently calculated from:

$$w_{0x}'' + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} = 0 \qquad w_{0x}(s + L_p) = w_{0x}(s)$$

$$w_{0x} > 0$$

$$\sigma_{0x} = \int_{s_i}^{s_i+L_p} \frac{ds}{w_{0x}^2}$$

- Subscript $0x$ stresses $x$-plane value and zero ($Q = 0$) space-charge effects
  - Need to generalize notation since we will add space-charge effects
  - Will find space-charge tends to cancel out part of applied focusing
  - Focusing can also be different in $x$- and $y$-planes

## S2: Vlasov Equilibria: Plasma physics-like approach is to resolve the system into an equilibrium + perturbation and analyze stability

Equilibrium solution to the Vlasov equation is constructed from single-particle constants of motion $C_i$

$$\boxed{f_\perp = f_\perp(\{C_i\}) \geq 0 \qquad \Longrightarrow \qquad \text{Equilibrium}}$$

$$\frac{d}{ds} f_\perp(\{C_i\}) = \sum_i \frac{\partial f_\perp}{\partial C_i} \cancelto{0}{\frac{dC_i}{ds}} = 0$$

And the distribution satisfies Vlasov's equation without $f_\perp$ changing *form*

Comments:

- Equilibrium $f_\perp$ is an exact solution to Vlasov's equation that *does not change* in 4D phase-space *functional form* as $s$ advances
  - Distribution values can still evolve in 4D $x,\ y,\ x',\ y'$ phase-space as $s$ advances
  - Equilibrium distribution periodic in lattice period in periodic lattice
  - Projections of the distribution can evolve in $s$ in non-continuous lattices
- Equilibrium is "time independent" ( $\partial/\partial s = 0$ ) in continuous focusing



Comments Continued:

- Requirement of non-negative $f_\perp(\{C_i\})$ follows from the distribution representing (probability of) particle counts in the continuum model
- Particle constants of the motion $\{C_i\}$ are in the presence of (possibly $s$-varying) applied and space-charge forces
  - Highly non-trivial!
  - Only one exact solution known for $s$-varying focusing using Courant-Snyder invariants: the KV distribution to be analyzed in these lectures



/// Example: Continuous focusing with $f_\perp = f_\perp(H_\perp)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

no explicit $s$ dependence
$k_{\beta 0}^2 = \text{const}$

Vlasov's equation expressed in Hamiltonian form is:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}_\perp'}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0$$
$$= \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0$$

Reminder:
Hamiltonian form equations of motion

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'} \qquad \frac{d}{ds}\mathbf{x}_\perp' = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Take $f_\perp = f_\perp(H_\perp)$ and apply the chain rule:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial H_\perp}\frac{\partial H_\perp}{\partial s} + \frac{\partial f_\perp}{\partial H_\perp}\cancelto{0}{\left(\frac{\partial H_\perp}{\partial \mathbf{x}_\perp'}\cdot\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial H_\perp}{\partial \mathbf{x}_\perp'}\right)} = 0$$
$$= \frac{\partial f_\perp}{\partial H_\perp}\frac{\partial H_\perp}{\partial s} = 0$$
$$\Longrightarrow \qquad \frac{\partial H_\perp}{\partial s} = 0 \qquad \text{For nontrivial solution with} \quad \frac{\partial f_\perp(H_\perp)}{\partial H_\perp} \neq 0$$



Apply chain rule for the total change of $H_\perp$ along particle orbit in the distribution:

$$\frac{d}{ds}H_\perp = \frac{\partial H_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{d\mathbf{x}_\perp}{ds} + \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'}\cdot\frac{d\mathbf{x}_\perp'}{ds}$$

Apply Hamilton's equations of motion

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'} \qquad \frac{d}{ds}\mathbf{x}_\perp' = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

to obtain:

$$\frac{d}{ds}H_\perp = \frac{\partial H_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial H_\perp}{\partial \mathbf{x}_\perp'} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'}\cdot\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} = \cancelto{0}{\frac{\partial H_\perp}{\partial s}} = 0$$

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$$\Longrightarrow H_\perp = \text{const}$$

Showing that $f_\perp = f_\perp(H_\perp)$ implies that $H_\perp = \text{const}$ following particle orbits and that Vlasov's equation is satisfied to produce a stationary equilibrium

Discussion:

- Also, for physical solutions must require: $f_\perp(H_\perp) \geq 0$
  - To be appropriate for single species with positive density
- Huge variety of equilibrium function choices $f_\perp(H_\perp)$ can be made to generate many radically different equilibria
  - Infinite variety in function space

## Discussion Continued:

- Does *NOT* apply to systems with $s$-varying focusing with $\kappa_{x,y} \neq \text{const}$
  - However, $\kappa_x \rightarrow k_{\beta 0}^2$ can provide a guide in many reasonable cases
- In this special case the equilbrium distribution does not change form but is also *stationary* ($\partial/\partial s = 0$ ) with no evolution in local phase-space density when viewing $f_\perp(x, y, x', y')$

///



## Typical single particle constants of motion:

Transverse Hamiltonian for continuous focusing:

$$\boxed{\begin{aligned} H_\perp &= \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi = \text{const} \\ k_{\beta 0}^2 &= \text{const} \end{aligned}}$$

- Not valid for periodic focusing systems!

Angular momentum for systems invariant under azimuthal rotation:

$$\boxed{P_\theta = xy' - yx' = \text{const}}$$

- Subtle point: This form is really a Canonical Angular Momentum and applies to solenoidal magnetic focusing when the variables are expressed in the rotating Larmor frame (i.e., in the “tilde” variables)
  - see: S.M. Lund, lectures on Transverse Particle Dynamics

Axial kinetic energy for systems with no acceleration:

$$\boxed{\mathcal{E} = (\gamma_b - 1)mc^2 = \text{const}}$$

- Trivial for a coasting beam with $\gamma_b\beta_b = \text{const}$

More on other classes of constraints later ...



## Plasma physics approach to beam physics:

Resolve:

$$\boxed{f(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \underbrace{f_\perp(\{C_i\})}_{\text{equilibrium}} + \underbrace{\delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)}_{\text{perturbation}} \quad f_\perp \gg |\delta f_\perp|}$$

and carry out equilibrium + stability analysis

Comments:

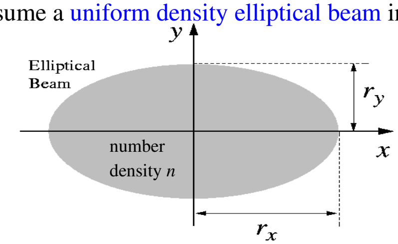
- Attraction is to parallel the impressive successes of plasma physics
  - Gain insight into preferred state of nature
- Beams are born off a source and may not be close to an equilibrium condition
  - Appropriate single particle constants of the motion unknown for periodic focusing lattices other than the (unphysically idealistic) KV distribution
- Intense beam self-fields and finite radial extent vastly complicate equilibrium description and analysis of perturbations
  - Unknown if smooth Vlasov equilibria exist (exact sense) in periodic focusing though recent perturbation theory/simulations suggest self-similar classes of distributions have near equilibrium form
  - Higher model detail vastly complicates picture!
- If system can be tuned to more closely resemble a relaxed, equilibrium, one might expect less deleterious effects based on plasma physics analogies



## S3: The KV Equilibrium Distribution

[Kapchinskij and Vladimirskij, Proc. Int. Conf. On High Energy Accel., p. 274 (1959); and Review: Lund, Kikuchi, and Davidson, PRSTAB **12**, 114801 (2009)]

Assume a uniform density elliptical beam in a periodic focusing lattice




Line-Charge:

$$\lambda = qn(s)\pi r_x(s) r_y(s)$$
$$= \text{const} \quad \text{(charge conservation)}$$

Beam Edge:

$$\frac{x^2}{r_x^2(s)} + \frac{y^2}{r_y^2(s)} = 1 \quad \text{(ellipse)}$$

Free-space self-field solution within the beam (see: Appendix A) is:

$$\phi = -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{(r_x + r_y)r_x} + \frac{y^2}{(r_x + r_y)r_y}\right] + \text{const}$$

$$\boxed{\begin{aligned} -\frac{\partial\phi}{\partial x} &= \frac{\lambda}{\pi\epsilon_0}\frac{x}{(r_x + r_y)r_x} \\ -\frac{\partial\phi}{\partial y} &= \frac{\lambda}{\pi\epsilon_0}\frac{y}{(r_x + r_y)r_y} \end{aligned}}$$

valid only within the beam!

- Nonlinear outside beam

The particle equations of motion:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial x}$$
$$y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial y}$$

become within the beam:

$$\boxed{\begin{aligned} x''(s) + \left\{\kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)}\right\} x(s) = 0 \\ y''(s) + \left\{\kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)}\right\} y(s) = 0 \end{aligned}}$$

Here, $Q$ is the dimensionless perveance defined by:

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2} = \text{const}}$$

- Same measure of space-charge intensity used by J.J. Barnard in Intro. Lectures
- Properties/interpretations of the perveance will be extensively developed in in this and subsequent lectures
  - Will appear in same form in many different space-charge problems



If we regard the envelope radii $r_x, \ r_y$ as specified functions of $s$, then these equations of motion are Hill's equations familiar from elementary accelerator physics:

$$\boxed{\begin{aligned} x''(s) + \kappa_x^{\text{eff}}(s)x(s) = 0 \\ y''(s) + \kappa_y^{\text{eff}}(s)y(s) = 0 \\ \kappa_x^{\text{eff}}(s) = \kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)} \\ \kappa_y^{\text{eff}}(s) = \kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)} \end{aligned}}$$

Suggests Procedure:

- Calculate Courant-Snyder invariants under assumptions made
- Construct a distribution function of Courant-Snyder invariants that generates the uniform density elliptical beam projection assumed
  - Nontrivial step: guess and show that it works: KV construction

Resulting distribution will be an equilibrium that does not evolve in functional form, but phase-space projections will evolve in $s$ when focusing functions vary in $s$



## Review (1): The Courant-Snyder invariant of Hill's equation

[Courant and Snyder, Annl. Phys. **3**, 1 (1958)]

Hill's equation describes a zero space-charge particle orbit in linear applied focusing fields:

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

As a consequence of Floquet's theorem, the solution can be cast in phase-amplitude form:

$$x(s) = A_i w(s)\cos\psi(s) \qquad \psi'(s) \equiv \frac{1}{w^2(s)}$$

where $w(s)$ is the periodic amplitude function satisfying

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$
$$w(s+L_p) = w(s) \qquad w(s) > 0$$

$\psi(s)$ is a phase function given by

$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

$A_i$ and $\psi_i$ are constants set by initial conditions at $s = s_i$



## Review (2): The Courant-Snyder invariant of Hill's equation

From this formulation, it follows that

$$x(s) = A_i w(s)\cos\psi(s) \qquad \psi'(s) \equiv \frac{1}{w^2(s)}$$
$$x'(s) = A_i w'(s)\cos\psi(s) - \frac{A_i}{w(s)}\sin\psi(s)$$

or

$$\frac{x}{w} = A_i\cos\psi$$
$$wx' - w'x = A_i\sin\psi$$

square and add equations to obtain the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

- Simplifies interpretation of dynamics
- Extensively used in accelerator physics

## Phase-amplitude description of particles evolving within a uniform density beam:

Phase-amplitude form of $x$-orbit equations:

$$x(s) = A_{xi} w_x(s) \cos\psi_x(s)$$

$$x'(s) = A_{xi} w_x'(s) \cos\psi_x(s) - \frac{A_{xi}}{w_x(s)} \sin\psi_x(s)$$

initial conditions yield: $(s = s_i)$

$$A_{xi} = \text{const}$$
$$\psi_{xi} = \psi_x(s=s_i) = \text{const}$$

where

$$w_x''(s) + \kappa_x(s) w_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)} w_x(s) - \frac{1}{w_x^3(s)} = 0$$

$$w_x(s+L_p) = w_x(s) \qquad w_x(s) > 0$$

$$\psi_x(s) = \psi_{xi} + \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})}$$

identifies the Courant-Snyder invariant

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_{xi}^2 = \text{const}$$

Analogous equations hold for the $y$-plane



## The KV envelope equations:

Define *maximum* Courant-Snyder invariants:

$$\varepsilon_x \equiv \text{Max}(A_{xi}^2)$$
$$\varepsilon_y \equiv \text{Max}(A_{yi}^2)$$

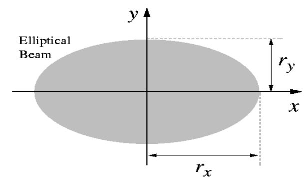
$$x = A_{xi} w_x \cos\psi_x \quad \xrightarrow{\cos\psi_x = 1} \quad r_x = A_{x,\max} w_x$$

Values must correspond to the beam-edge radii:

$$r_x(s) = \sqrt{\varepsilon_x} w_x(s)$$
$$r_y(s) = \sqrt{\varepsilon_y} w_y(s)$$

Edge Ellipse:

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} = 1$$




The equations for $w_x$ and $w_y$ can then be rescaled to obtain the familiar KV envelope equations for the matched beam envelope

$$r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0$$

$$r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0$$

$$r_x(s+L_p) = r_x(s) \qquad r_x(s) > 0$$
$$r_y(s+L_p) = r_y(s) \qquad r_y(s) > 0$$



Use variable rescalings to denote $x$- and $y$-plane Courant-Snyder invariants as:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_{xi}^2 = \text{const}$$

$$\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 \equiv C_x = \text{const}$$

$$\left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 \equiv C_y = \text{const}$$

Kapchinskij and Vladimirskij constructed a delta-function distribution of a linear combination of these Courant-Snyder invariants that generates the correct uniform density elliptical beam needed for consistency with the assumptions:

$$f_\perp = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[C_x + C_y - 1\right]$$

- Delta function means the sum of the $x$- and $y$-invariants is a constant
- Other forms cannot generate the needed uniform density elliptical beam projection (see: S9)
- Density inversion theorem covered later can be used to derive result



## The KV equilibrium is constructed from the Courant-Snyder invariants:

KV equilibrium distribution write out full arguments in $x, x'$:

$$f_\perp(\mathbf{x}_\perp, \mathbf{x}_\perp', s) = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right]$$

$\delta(x) =$ Dirac delta function

This distribution generates (see: proof in Appendix B) the correct uniform density elliptical beam:

$$n = \int d^2x_\perp' \; f_\perp = \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

Obtaining this form consistent with the assumptions, thereby demonstrating full self-consistency of the KV equilibrium distribution.

- Full 4-D form of the distribution does not evolve in $s$
- Projections of the distribution can (and generally do!) evolve in $s$

/// Comment on notation of integrals:

- 2nd forms useful for systems with azimuthal spatial or annular symmetry

Spatial

$$\int d^2x_\perp \cdots \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ \cdots$$

$$= \int_0^\infty dr\, r \int_{-\pi}^{\pi} d\theta \ \cdots$$

Cylindrical Coordinates:
$x = r\cos\theta$
$y = r\sin\theta$

Angular

$$\int d^2x'_\perp \cdots \equiv \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \ \cdots$$

$$= \int_0^\infty d\tilde{r}'\, \tilde{r}' \int_{-\pi}^{\pi} d\tilde{\theta}' \ \cdots$$

Angular
Cylindrical Coordinates:
$x' = \tilde{r}'\cos\tilde{\theta}'$
$y' = \tilde{r}'\sin\tilde{\theta}'$



Use care when interpreting dimensions of symbols in cylindrical form of angular integrals:

$$\tilde{r}' \neq \frac{d}{ds} r = \frac{d}{ds}\sqrt{x^2+y^2} \qquad [[\tilde{r}']] = \text{Angle} \qquad \tilde{r}' \in [0,\infty)$$

$$\tilde{\theta}' \neq \frac{d}{ds}\theta = \frac{d}{ds}\text{ArcTan}[y,x] \qquad [[\tilde{\theta}']] = \text{rad} \qquad \tilde{\theta}' \in [-\pi,\pi]$$

$$x' = \tilde{r}'\cos\tilde{\theta}' \qquad [[x']] = \text{Angle} \qquad x' \in (-\infty,\infty)$$

$$y' = \tilde{r}'\sin\tilde{\theta}' \qquad [[y']] = \text{Angle} \qquad y' \in (-\infty,\infty)$$

- Tilde is used in angular cylindrical variables to stress that cylindrical variables are chosen in form to span the correct ranges in $x'$ and $y'$ but are not d/ds of the usual cylindrical polar coordinates.



Comment on notation of integrals (continued):

Axisymmetry simplifications

Spatial: for some function $f(\mathbf{x}_\perp^2) = f(r^2)$

$$\int d^2x_\perp\, f(\mathbf{x}_\perp^2) = 2\pi \int_0^\infty dr\, r f(r^2)$$

$$= \pi \int_0^\infty dr^2\, f(r^2)$$

$$= \pi \int_0^\infty dw\, f(w)$$

Cylindrical Coordinates:
$x = r\cos\theta$
$y = r\sin\theta$

$w = r^2$

Angular: for some function $g(\mathbf{x}_\perp'^2) = g(\tilde{r}'^2)$

$$\int d^2x'_\perp\, g(\mathbf{x}_\perp'^2) = 2\pi \int_0^\infty d\tilde{r}'\, \tilde{r}' g(\tilde{r}'^2)$$

$$= \pi \int_0^\infty d\tilde{r}'^2\, g(\tilde{r}'^2)$$

$$= \pi \int_0^\infty du\, g(u)$$

Angular
Cylindrical Coordinates:
$x' = \tilde{r}'\cos\tilde{\theta}'$
$y' = \tilde{r}'\sin\tilde{\theta}'$

$u = \tilde{r}'^2$

///



Moments of the KV distribution can be calculated directly from the distribution to further aid interpretation: [see: Appendix B for methods to simply calculate]

Full 4D average:
$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \ \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \ f_\perp}$$

Restricted angle average:
$$\langle \cdots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \ \cdots f_\perp}{\int d^2x'_\perp \ f_\perp}$$

Envelope edge radius:
$$r_x = 2\langle x^2\rangle_\perp^{1/2}$$

Envelope edge angle:
$$r'_x = 2\langle xx'\rangle_\perp / \langle x^2\rangle_\perp^{1/2}$$

rms edge emittance (maximum Courant-Snyder invariant):

$$\varepsilon_x = 4[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2} = \text{const}$$

Coherent flows (within the beam, zero otherwise):

$$\langle x'\rangle_{\mathbf{x}'_\perp} = r'_x \frac{x}{r_x}$$

Angular spread ($x$-temperature, within the beam, zero otherwise):

$$T_x \equiv \langle (x' - \langle x'\rangle_{\mathbf{x}'_\perp})^2\rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)$$

## Summary of 1st and 2nd order moments of the KV distribution:

| Moment | Value |
|---|---|
| $\int d^2x'_\perp \; x' f_\perp$ | $r'_x \frac{x}{r_x} n$ |
| $\int d^2x'_\perp \; y' f_\perp$ | $r'_y \frac{y}{r_y} n$ |
| $\int d^2x'_\perp \; x'^2 f_\perp$ | $\left[r'^2_x \frac{x^2}{r_x^2} + \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] n$ |
| $\int d^2x'_\perp \; y'^2 f_\perp$ | $\left[r'^2_y \frac{y^2}{r_y^2} + \frac{\varepsilon_y^2}{2r_y^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] n$ |
| $\int d^2x'_\perp \; xx' f_\perp$ | $\frac{r'_x}{r_x} x^2 n$ |
| $\int d^2x'_\perp \; yy' f_\perp$ | $\frac{r'_y}{r_y} y^2 n$ |
| $\int d^2x'_\perp \; (xy' - yx') f_\perp$ | $0$ |
| $\langle x^2 \rangle_\perp$ | $\frac{r_x^2}{4}$ |
| $\langle y^2 \rangle_\perp$ | $\frac{r_y^2}{4}$ |
| $\langle x'^2 \rangle_\perp$ | $\frac{r'^2_x}{4} + \frac{\varepsilon_x^2}{4r_x^2}$ |
| $\langle y'^2 \rangle_\perp$ | $\frac{r'^2_y}{4} + \frac{\varepsilon_y^2}{4r_y^2}$ |
| $\langle xx' \rangle_\perp$ | $\frac{r_x r'_x}{4}$ |
| $\langle yy' \rangle_\perp$ | $\frac{r_y r'_y}{4}$ |
| $\langle xy' - yx' \rangle_\perp$ | $0$ |
| $16[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]$ | $\varepsilon_x^2$ |
| $16[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2]$ | $\varepsilon_y^2$ |

All 1st and 2nd order moments not listed vanish, i.e.,

$$\int d^2x'_\perp \; xy f_\perp = 0$$

$$\langle xy \rangle_\perp = 0$$

see reviews by:

(limit of results presented)
Lund and Bukh, PRSTAB **7**, 024801 (2004), Appendix A

S.M. Lund, T. Kikuchi, and R.C. Davidson, PRSTAB **12**, 114801 (2009)



## Canonical transformation illustrates KV distribution structure:

[Davidson, Physics of Nonneutral Plasmas, Addison-Wesley (1990), and Appendix B]

Phase-space transformation:

$$\boxed{\begin{aligned} X &= \frac{\sqrt{\varepsilon_x}}{r_x} x \\ X' &= \frac{r_x x' - r'_x x}{\sqrt{\varepsilon_x}} \end{aligned}}$$

$$dx\, dy = \frac{r_x r_y}{\sqrt{\varepsilon_x \varepsilon_y}} dX\, dY$$

$$dx'\, dy' = \frac{\sqrt{\varepsilon_x \varepsilon_y}}{r_x r_y} dX'\, dY'$$

$$dx\, dy\, dx'\, dy' = dX\, dY\, dX'\, dY'$$

Courant-Snyder invariants in the presence of beam space-charge are then simply:

$$\boxed{X^2 + X'^2 = \text{const}}$$

and the KV distribution takes the simple, *symmetrical* form:

$$\boxed{f_\perp(x, y, x', y', s) = f_\perp(X, Y, X', Y') = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[\frac{X^2 + X'^2}{\varepsilon_x} + \frac{Y^2 + Y'^2}{\varepsilon_y} - 1\right]}$$

from which the density and other projections can be (see: Appendix B) calculated more easily:

$$n = \int d^2x'_\perp \; f_\perp = \frac{\lambda}{q\pi r_x r_y} \int_0^\infty dU^2 \; \delta\left[U^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right]$$

$$= \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$



## KV Envelope equation

The envelope equation reflects low-order force balances

$$r''_x + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r''_y + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

| Terms: | Applied Focusing | Space-Charge Defocusing | Thermal Defocusing |
|---|---|---|---|
| | Lattice | Perveance | Emittance |

Matched Solution:

$$r_x(s + L_p) = r_x(s)$$
$$r_y(s + L_p) = r_y(s)$$

$$\kappa_x(s + L_p) = \kappa_x(s)$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

Comments:

- Envelope equation is a projection of the 4D invariant distribution
  - Envelope evolution equivalently given by moments of the 4D equilibrium distribution
- Most important basic design equation for transport lattices with high space-charge intensity
  - Simplest consistent model incorporating applied focusing, space-charge defocusing, and thermal defocusing forces
  - Starting point of almost all practical machine design!



Comments Continued:

- Beam envelope matching where the beam envelope has the periodicity of the lattice

  $$r_x(s + L_p) = r_x(s)$$
  $$r_y(s + L_p) = r_y(s)$$

  will be covered in much more detail in S.M. Lund lectures on Centroid and Envelope Description of Beams. Envelope matching requires specific choices of initial conditions

  $$r_x(s_i), \; r_y(s_i) \qquad r'_x(s_i), \; r'_y(s_i)$$

  for periodic evolution.
- Instabilities of envelope equations are well understood and real (to be covered: see S.M. Lund lectures on Centroid and Envelope Description of Beams)
  - Must be avoided for reliable machine operation

## The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

<u>Matching Condition</u>

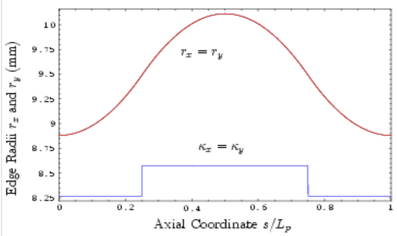
$$\boxed{\begin{aligned} r_x(s+L_p) &= r_x(s) \\ r_y(s+L_p) &= r_y(s) \end{aligned}}$$

<u>Example Parameters</u>

$$L_p = 0.5 \text{ m}, \quad \sigma_0 = 80^\circ, \quad \eta = 0.5$$
$$\varepsilon_x = \varepsilon_y = 50 \text{ mm-mrad}$$
$$\sigma/\sigma_0 = 0.2$$

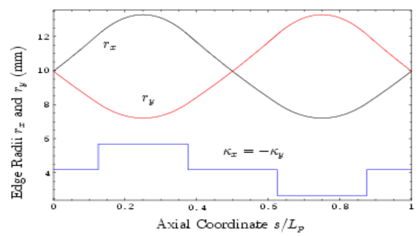
<u>Solenoidal Focusing</u>

$(Q = 6.6986 \times 10^{-4})$




<u>FODO Quadrupole Focusing</u>

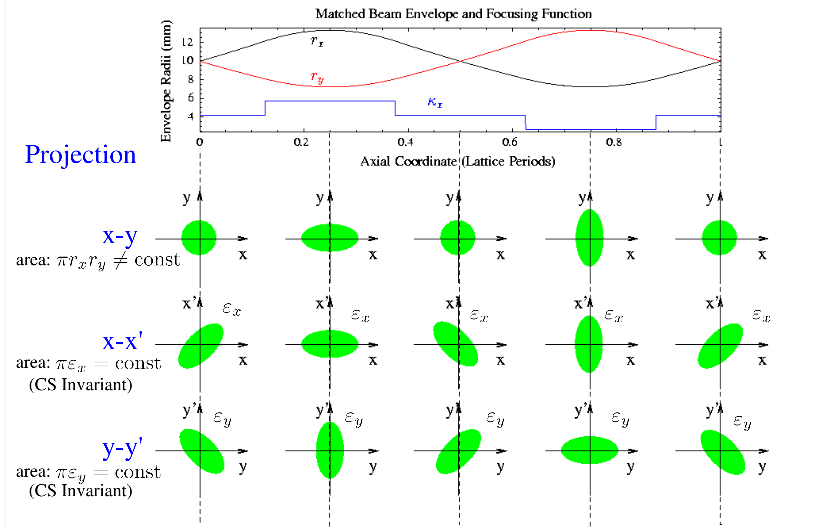
$(Q = 6.5614 \times 10^{-4})$




The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport



## 2D phase-space projections of a matched KV equilibrium beam in a periodic FODO quadrupole transport lattice






## KV model shows that particle orbits in the presence of space-charge can be strongly modified – space charge slows the orbit response:

Matched envelope:

$$r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0$$
$$r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0$$
$$r_x(s+L_p) = r_x(s) \qquad r_x(s) > 0$$
$$r_y(s+L_p) = r_y(s) \qquad r_y(s) > 0$$

Equation of motion for x-plane "depressed" orbit in the presence of space-charge:

$$x''(s) + \kappa_x(s) x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)} x(s) = 0$$

All particles have the *same value* of depressed phase advance (similar Eqns in $y$):

$$\sigma_x \equiv \psi_x(s_i + L_p) - \psi_x(s_i) = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)}$$



## Contrast: Review, the undepressed particle phase advance calculated in the lectures on Transverse Particle Dynamics

The undepressed phase advance is defined as the phase advance of a particle in the absence of space-charge ($Q = 0$):

- Denote by $\sigma_{0x}$ to distinguished from the "depressed" phase advance $\sigma_x$ in the presence of space-charge

$$w_{0x}'' + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} = 0 \qquad w_{0x}(s+L_p) = w_{0x}(s)$$
$$\sigma_{0x} = \int_{s_i}^{s_i+L_p} \frac{ds}{w_{0x}^2} \qquad w_{0x} > 0$$

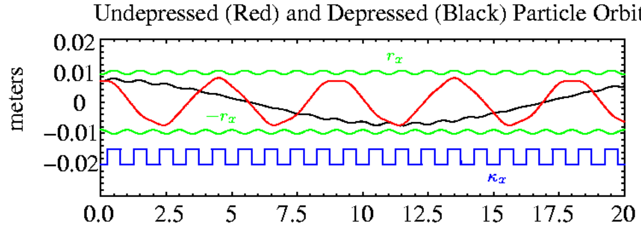
This can be equivalently calculated from the matched envelope with $Q = 0$:

$$r_{0x}'' + \kappa_x r_{0x} - \frac{\varepsilon_x^2}{r_{0x}^3} = 0 \qquad r_{0x}(s+L_p) = r_{0x}(s)$$
$$\sigma_{0x} = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r_{0x}^2} \qquad r_{0x} > 0$$

- Value of $\varepsilon_x$ is arbitrary (answer for $\sigma_{0x}$ is independent)

## Depressed particle $x$-plane orbits within a matched KV beam in a periodic FODO quadrupole channel for the matched beams previously shown

Solenoidal Focusing (Larmor frame orbit):



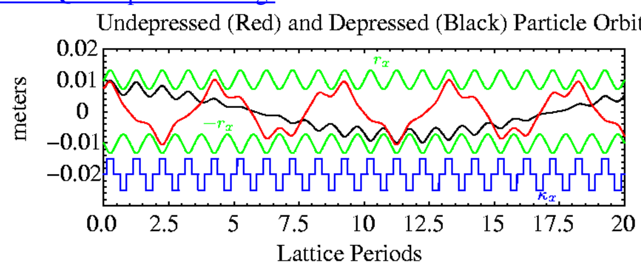

$x$-plane orbit: $y = 0 = y'$

Both Problems Tuned for:

$\sigma_0 = 80^\circ$

$\frac{\sigma}{\sigma_0} = 0.2$

FODO Quadrupole Focusing: Lattice Periods




$x$-plane orbit: $y = 0 = y'$



## Clarification Comment on previous plots:

For the shown undepressed orbit (no beam space-charge), the particle is integrated from the same initial condition as the depressed orbit (in presence of space-charge). In this context the matched envelope which is shown including space-charge has no meaning.

- A beam rms "edge" envelope without space-charge $r_{0x}$ could also be shown taking

$$r_{0x}(s) = \sqrt{\varepsilon_x} w_{0x}(s) = \sqrt{\varepsilon_x \beta_{0x}(s)}$$

- This envelope will be different than the depressed beam.

The undepressed particle orbit can be calculated using phase-amplitude methods or by simply integrating the ODE describing the particle moving in linear applied fields:

$$x'' + \kappa_x(s) x = 0$$

$$x(s = s_i) = x_i$$
$$x'(s = s_i) = x_i' \qquad \text{Same initial condition as depressed}$$



## Depressed particle phase advance provides a convenient measure of space-charge strength

For simplicity take (plane symmetry in average focusing and emittance)

$$\sigma_{0x} = \sigma_{0y} \equiv \sigma_0 \qquad \varepsilon_x = \varepsilon_y \equiv \varepsilon$$

Depressed phase advance of particles moving within a matched beam envelope:

$$\boxed{\sigma = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)} = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_y^2(s)}}$$

Limits:

1) $\lim_{Q \to 0} \sigma = \sigma_0$ Envelope just rescaled amplitude: $r_x^2 = \varepsilon w_{0x}^2$

2) $\lim_{\varepsilon \to 0} \sigma = 0$ Matched envelope exists with $\varepsilon = 0$
Then $\varepsilon = 0$ multiplying phase advance integral

Normalized space charge strength

$$\boxed{0 \leq \sigma/\sigma_0 \leq 1}$$

$\sigma/\sigma_0 \to 0$ Cold Beam (space-charge dominated) $\varepsilon \to 0$

$\sigma/\sigma_0 \to 1$ Warm Beam (kinetic dominated) $Q \to 0$

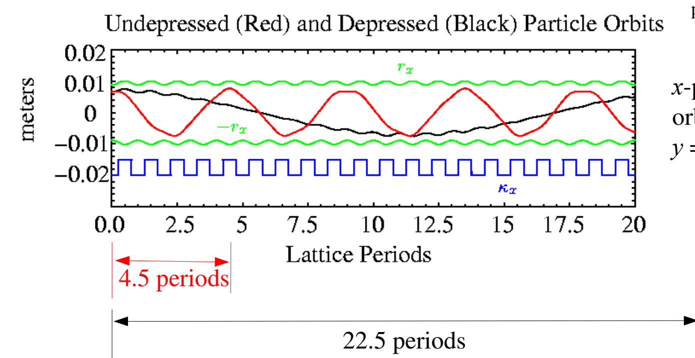


For example matched envelope presented earlier:

| | | | repeat periods |
|---|---|---|---|
| Undepressed phase advance: | $\sigma_0 = 80^\circ$ | | 4.5 |
| Depressed phase advance: | $\sigma = 16^\circ$ | $\to \sigma/\sigma_0 = 0.2$ | 22.5 |

Periods for 360 degree phase advance

Solenoidal Focusing (Larmor frame orbit):




$x$-plane orbit $y = 0 = y'$

Comment:
All particles in the distribution will, of course, always move in response to both applied and self-fields. You cannot turn off space-charge for an undepressed orbit. It is a convenient conceptual construction to help understand focusing properties.

## The rms equivalent beam model helps interpret general beam evolution in terms of an "equivalent" local KV distribution

Real beams distributions in the lab will not be KV form. But the KV model can be applied to interpret arbitrary distributions via the concept of *rms equivalence*.

For the same focusing lattice, replace any beam charge $\rho(x, y)$ density by a uniform density KV beam of the same species ($q, m$) and energy ($\beta_b$) in each axial slice ($s$) using averages calculated from the actual "real" beam distribution with:

$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp} \qquad f_\perp = \text{ real distribution}$$

rms equivalent beam (identical 1st and 2nd order moments):

| Quantity | KV Equiv. | Calculated from Distribution |
|---|---|---|
| Perveance | $Q$ | $= q^2 \int d^2x_\perp \int d^2x'_\perp \; f_\perp \; / [2\pi\epsilon_0 \gamma_b^3 \beta_b^2 c^2]$ |
| $x$-Env Rad | $r_x$ | $= 2\langle x^2 \rangle_\perp^{1/2}$ |
| $y$-Env Rad | $r_y$ | $= 2\langle y^2 \rangle_\perp^{1/2}$ |
| $x$-Env Angle | $r'_x$ | $= 2\langle xx' \rangle_\perp / \langle x^2 \rangle_\perp^{1/2}$ |
| $y$-Env Angle | $r'_y$ | $= 2\langle yy' \rangle_\perp / \langle y^2 \rangle_\perp^{1/2}$ |
| $x$-Emittance | $\varepsilon_x$ | $= 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp]^{1/2}$ |
| $y$-Emittance | $\varepsilon_y$ | $= 4[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle yy' \rangle_\perp]^{1/2}$ |



## Comments on rms equivalent beam concept:

- The emittances will generally evolve in $s$
  - Means that the equivalence must be recalculated in every slice as the emittances evolve
  - For reasons to be analyzed later (see S.M. Lund lectures on Kinetic Stability of Beams), this evolution is often small
- Concept is highly useful
  - KV equilibrium properties well understood and are approximately correct to model lowest order "real" beam properties
  - See, Reiser, *Theory and Design of Charged Particle Beams* (1994, 2008) for a detailed and instructive discussion of rms equivalence



## Sacherer expanded the concept of rms equivalency by showing that the equivalency works exactly for beams with elliptic symmetry space-charge

[Sacherer, IEEE Trans. Nucl. Sci. 18, 1101 (1971), J.J. Barnard, Intro. Lectures]

For any beam with elliptic symmetry charge density in each transverse slice:

$$\boxed{\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)}$$

Based on:

$$\langle x \frac{\partial \phi}{\partial x} \rangle_\perp = -\frac{\lambda}{4\pi\epsilon_0} \frac{r_x}{r_x + r_y}$$

see J.J. Barnard intro. lectures

the KV envelope equations

$$\boxed{\begin{aligned} r''_x(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2(s)}{r_x^3(s)} &= 0 \\ r''_y(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2(s)}{r_y^3(s)} &= 0 \end{aligned}}$$

remain valid when (averages taken with the full distribution):

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \qquad \lambda = q \int d^2x_\perp \; \rho \; = \; \text{const}$$

$$r_x = 2\langle x^2 \rangle_\perp^{1/2} \qquad \varepsilon_x = 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp^2]^{1/2}$$

$$r_y = 2\langle y^2 \rangle_\perp^{1/2} \qquad \varepsilon_y = 4[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle yy' \rangle_\perp^2]^{1/2}$$

The emittances may evolve in $s$ under this model

(see SM Lund lectures on Transverse Kinetic Stability)



## Interpretation of the dimensionless perveance $Q$

The dimensionless perveance:

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const}} \qquad \begin{aligned} \lambda &= q\hat{n}\pi r_x r_y = \text{line-charge} = \text{const} \\ \hat{n} &= \text{beam density} \end{aligned}$$

- Scales with size of beam ( $\lambda$ ), but typically has small characteristic values even for beams with high space charge intensity ( ~ $10^{-4}$ to $10^{-8}$ common)
- Even small values of $Q$ can matter depending on the relative strength of other effects from applied focusing forces, thermal defocusing, etc.

Can be expressed equivalently in several ways:

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \frac{qI_b}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^3 c^3} = \frac{2}{(\gamma_b \beta_b)^3} \frac{I_b}{I_A}$$

$$= \frac{q^2 \pi r_x r_y \hat{n}}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^3 c^3} = \frac{\hat{\omega}_p^2 r_x r_y}{2\gamma_b^3 \beta_b^2 c^2}$$

$$\begin{aligned} I_b &= \lambda \beta_b c = \text{beam current} \\ I_A &= 4\pi\epsilon_0 m c^3 / q = \text{Alfven current} \\ \hat{\omega}_p &= \sqrt{q^2 \hat{n} / (m\epsilon_0)} = \text{plasma freq.} \end{aligned}$$

- Forms based on $\lambda, \; I_b$ generalize to *nonuniform density beams*

To better understand the perveance $Q$, consider a round, uniform density beam with

$$r_x = r_y = r_b$$

then the solution for the potential within the beam reduces:

$$\phi = -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{(r_x+r_y)r_x} + \frac{y^2}{(r_x+r_y)r_y}\right] + \text{const}$$

$$= -\frac{\lambda}{4\pi\epsilon_0}\frac{r^2}{r_b^2} + \text{const}$$

$$\Longrightarrow \quad \Delta\phi = \phi(r=0) - \phi(r=r_b) = \frac{\lambda}{4\pi\epsilon_0} \qquad \text{for potential drop across the beam}$$

If the beam is also nonrelativistic, then the axial kinetic energy $\mathcal{E}_b$ is

$$\mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \frac{1}{2}m\beta_b^2c^2$$

and the perveance can be alternatively expressed as

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} \simeq \frac{q\Delta\phi}{\mathcal{E}_b}}$$

- Perveance can be interpreted as space-charge potential energy difference across beam relative to the axial kinetic energy



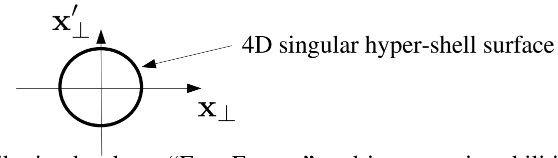
## Further comments on the KV equilibrium: Distribution Structure

KV equilibrium distribution:

$$\boxed{f_\perp \sim \delta[\text{Courant-Snyder invariants}]}$$

Forms a highly singular hyper-shell in 4D phase-space

Schematic:



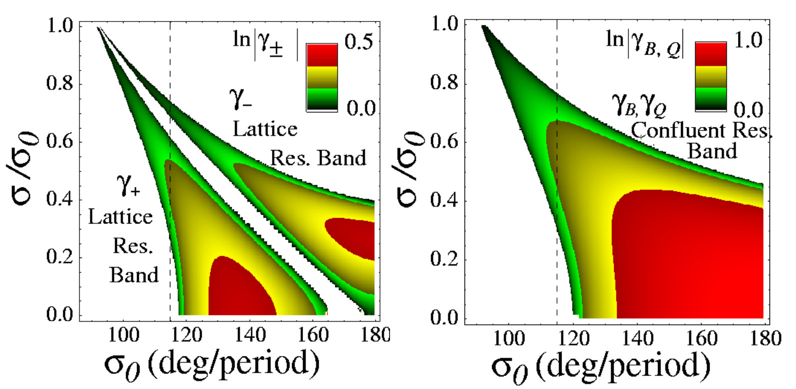

- Singular distribution has large "Free-Energy" to drive many instabilities
  - Low order envelope modes are physical and highly important (see: lectures by S.M. Lund on Centroid and Envelope Descriptions of Beams)
- Perturbative analysis shows strong collective instabilities
  - Hofmann, Laslett, Smith, and Haber, Part. Accel. **13**, 145 (1983)
  - Higher order instabilities (collective modes) have unphysical aspects due to (delta-function) structure of distribution and must be applied with care (see: lectures by S.M. Lund on Kinetic Stability of Beams)
  - Instabilities can cause problems if the KV distribution is employed as an initial beam state in self-consistent simulations



## Preview: lectures on Centroid and Envelope Descriptions of Beams: Instability bands of the KV envelope equation are well understood in periodic focusing channels and must be avoided in machine operation

Envelope Mode Instability Growth Rates

Solenoid ( $\eta$ = 0.25) Quadrupole FODO ( $\eta$ = 0.70)



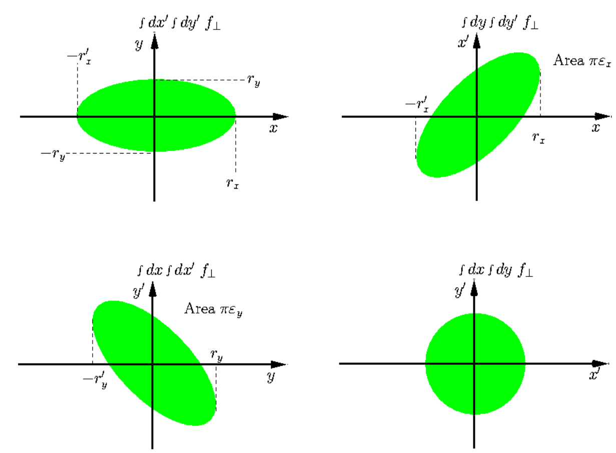

[S.M. Lund and B. Bukh, PRSTAB **7** 024801 (2004)]



## Further comments on the KV equilibrium: 2D Projections

All 2D projections of the KV distribution are uniformly filled ellipses

- Not very different from what is often observed in experimental measurements and self-consistent simulations of stable beams with strong space-charge
- Falloff of distribution at "edges" can be rapid, but smooth, for strong space-charge



## Further comments on the KV equilibrium: Angular Spreads: Coherent and Incoherent

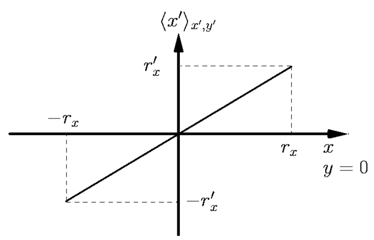
Angular spreads within the beam:

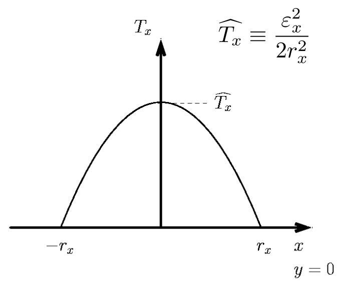
**Coherent (flow):**

$$\langle x' \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \; x'_\perp \; f_\perp}{\int d^2x'_\perp \; f_\perp} = r'_x \frac{x}{r_x}$$

**Incoherent (temperature):**

$$\langle (x' - r'_x x / r_x)^2 \rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)$$




- Coherent flow required for periodic focusing to conserve charge
- Temperature must be zero at the beam edge since the distribution edge is sharp
- Parabolic temperature profile is consistent with linear grad P pressure forces in a fluid model interpretation of the (kinetic) KV distribution



## Further comments on the KV equilibrium:

The KV distribution is the *only* exact equilibrium distribution formed from Courant-Snyder invariants of linear forces valid for periodic focusing channels:

- Low order properties of the distribution are physically appealing
- Illustrates relevant Courant-Snyder invariants in simple form
  - Later arguments demonstrate that these invariants should be a reasonable approximation for beams with strong space charge
- KV distribution does not have a 3D generalization [see F. Sacherer, Ph.d. thesis, 1968]

Strong Vlasov instabilities associated with the KV model render the distribution inappropriate for use in evaluating machines at high levels of detail:

- Instabilities are not all physical and render interpretation of results difficult
  - Difficult to separate physical from nonphysical effects in simulations

Possible Research Problem (unsolved in 40+ years!):
Can an *exact* Vlasov equilibrium be constructed for a *smooth* (non-singular), nonuniform density distribution in a linear, periodic focusing channel?

- Not clear what invariants can be used or if any can exist
  - Nonexistence proof would also be significant
- Recent perturbation theory and simulation work suggest prospects
  - Self-similar classes of distributions
- Lack of a smooth equilibrium does not imply that real machines cannot work!



Because of a lack of theory for a smooth, self-consistent distribution that would be more physically appealing than the KV distribution we will examine smooth distributions in the idealized continuous focusing limit (after an analysis of the continuous limit of the KV theory):

- Allows more classic "plasma physics" like analysis
- Illuminates physics of intense space charge
- Lack of continuous focusing in the laboratory will prevent over generalization of results obtained

A 1D analog to the KV distribution called the "Neuffer Distribution" is useful in longitudinal physics

- Based on linear forces with a "g-factor" model
- Distribution not singular in 1D and is fully stable in continuous focusing
- See: J.J. Barnard, lectures on Longitudinal Physics



## Appendix A: Self-Fields of a Uniform Density Elliptical Beam in Free-Space

1) Direct Proof:

The solution to the 2D Poisson equation:

$$\boxed{\begin{aligned} &\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \begin{cases} -\frac{\lambda}{\pi\epsilon_0 r_x r_y}, & \text{if } \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1 \\ 0, & \text{if } \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{cases} \\ &\lim_{r\to\infty} \frac{\partial \phi}{\partial r} \sim \frac{\lambda}{2\pi\epsilon_0 r} \end{aligned}}$$

has been formally constructed as:

- Solutions date from early Newtonian gravitational field solutions of stars with ellipsoidal density
- See Landau and Lifshitz, *Classical Theory of Fields* for a simple presentation

$$\boxed{\begin{aligned} &\phi = -\frac{\lambda}{4\pi\epsilon_0}\left\{\int_0^\xi \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}} + \int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}\right)\right\} + \text{const} \\ &\xi = 0 \ \text{ when } \ x^2/r_x^2 + y^2/r_y^2 < 1 \\ &\xi \text{ root of: } \ \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} = 1, \ \text{ when } \ \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{aligned}}$$

A1

We will A) demonstrate that this solution works and then B) simplify the result.
A) Verify by direct substitution:

$$\frac{\partial \phi}{\partial x} = -\frac{\lambda}{4\pi\epsilon_0}\left\{\int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{2x}{r_x^2+s}\right) - \frac{1}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left[1-\frac{x^2}{r_x^2+\xi}-\frac{y^2}{r_y^2+\xi}\right]\frac{\partial \xi}{\partial x}\right\}$$

But:

$$\text{if } \xi = 0 \implies 1 = \frac{x^2}{r_x^2+\xi}+\frac{y^2}{r_y^2+\xi}$$

$$\text{if } \xi = 0 \implies \frac{d\xi}{dx} = 0$$

$\Longrightarrow$ In either case the 2[nd] term above vanishes

Giving:

$$\frac{\partial \phi}{\partial x} = -\frac{\lambda}{2\pi\epsilon_0}\int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{x}{r_x^2+s}\right)$$

$$\frac{\partial \phi}{\partial y} = -\frac{\lambda}{2\pi\epsilon_0}\int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{y}{r_y^2+s}\right)$$

Differentiate again and apply the chain rule:

A2



$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{\int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{1}{r_x^2+s}+\frac{1}{r_y^2+s}\right) - \frac{1}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left[\frac{x\partial\xi/\partial x}{r_x^2+\xi}+\frac{y\partial\xi/\partial y}{r_y^2+\xi}\right]\right\}$$

Must show that the right hand side reduces to the required elliptical form for a uniform density beam for:

Case 1: Exterior $\quad \dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi} = 1$

Case 2: Interior $\quad \xi = 1$

Case 1: Exterior $\quad \dfrac{x^2}{r_x^2}+\dfrac{y^2}{r_y^2} > 1$

Differentiate: $\quad \dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi} = 1$

$$\Longrightarrow \quad \frac{\partial \xi}{\partial x} = \frac{2x}{(r_x^2+\xi)}\frac{1}{\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]}$$

+ analogous eqn in $y$

A3



Using these results:

$$\frac{x\partial\xi/\partial x}{r_x^2+\xi}+\frac{y\partial\xi/\partial y}{r_y^2+\xi} = 2\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]\frac{1}{\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]} = 2$$

Also, need to calculate integrals like: $\quad w^2 = r_x^2+\tilde{\xi}$

$$I_x(\xi) \equiv \int_\xi^\infty \frac{d\tilde{\xi}}{[(r_x^2+\tilde{\xi})(r_y^2+\tilde{\xi})]^{1/2}}\frac{1}{r_x^2+\tilde{\xi}} = \int_{\sqrt{r_x^2+\xi}}^\infty \frac{dw}{(r_x^2-r_y^2+w^2)^{3/2}}$$

+ analogous integrals in $y$

This integral can be done using tables or symbolic programs like Mathematica:

$$I_x(\xi) = \left.\frac{2w}{(r_x^2-r_y^2)\sqrt{r_x^2-r_y^2+w^2}}\right|_{w=\sqrt{r_x^2+\xi}}^{w\to\infty} = \frac{2}{r_x^2-r_y^2}+\frac{2\sqrt{r_y^2+\xi}}{(r_x^2-r_y^2)\sqrt{r_x^2+\xi}}$$

Applying this integral and the analogous $I_y(\xi)$

$$\int_0^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left[\frac{1}{r_x^2+s}+\frac{1}{r_y^2+s}\right] = I_x(\xi)+I_y(\xi)$$

$$= \frac{2}{r_x^2-r_y^2}\left(\frac{\sqrt{r_x^2+\xi}}{\sqrt{r_y^2+\xi}}-\frac{\sqrt{r_y^2+\xi}}{\sqrt{r_x^2+\xi}}\right) = \frac{2}{\sqrt{(r_x^2+\xi)(r_y^2+\xi)}}$$

A4



Applying both of these results, we obtain:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{\frac{2}{\sqrt{(r_x^2+\xi)(r_y^2+\xi)}}-\frac{2}{\sqrt{(r_x^2+\xi)(r_y^2+\xi)}}\right\}$$

$$= 0$$

Thereby verifying the exterior case !

Case 2: Interior $\quad \dfrac{x^2}{r_x^2}+\dfrac{y^2}{r_y^2} < 1$

$$\xi = 0 \quad \Longrightarrow \quad \frac{x\partial\xi/\partial x}{r_x^2+\xi}+\frac{y\partial\xi/\partial y}{r_y^2+\xi} = 0$$

The integrals defined and calculated above give in this case:

$$I_x(\xi=0) = \frac{2}{(r_x+r_y)r_x} \qquad I_y(\xi=0) = \frac{2}{(r_x+r_y)r_y}$$

Applying both of these results, we obtain:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{\frac{2}{r_xr_y}-0\right\} = -\frac{\lambda}{\epsilon_0\pi r_xr_y} = -\frac{q\hat{n}}{\epsilon_0}$$

Thereby verifying the interior case !

A5

Verify that the correct large-$r$ limit of the potential is obtained outside the beam:

$$-\frac{\partial \phi}{\partial x} = \frac{\lambda}{2\pi\epsilon_0} x I_x(\xi) \qquad \lim_{r\to\infty} I_x(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

$$r \text{ large} \Longrightarrow \xi \text{ large}$$

$$-\frac{\partial \phi}{\partial y} = \frac{\lambda}{2\pi\epsilon_0} y I_y(\xi) \qquad \lim_{r\to\infty} I_y(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

Thus:

$$\lim_{r\to\infty} -\frac{\partial \phi}{\partial x} = -\frac{\lambda}{2\pi\epsilon_0}\frac{x}{r^2}$$

$$\lim_{r\to\infty} -\frac{\partial \phi}{\partial y} = -\frac{\lambda}{2\pi\epsilon_0}\frac{y}{r^2}$$

$$\Longrightarrow \quad \lim_{r\to\infty} -\frac{\partial \phi}{\partial r} = \frac{\lambda}{2\pi\epsilon_0 r}$$

Thereby verifying the exterior limit!

Together, these results fully verify that the integral solution satisfies the Poisson equation describing a uniform density elliptical beam in free space

A6



Finally, it is useful to apply the steps in the verification to derive a simplified formula for the potential within the beam where:

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1, \quad \xi = 0$$

This gives:

$$\begin{aligned}\phi &= -\frac{\lambda}{4\pi\epsilon_0}\left\{x^2 I_x(\xi=0) + y^2 I_y(\xi=0)\right\} \ + \text{const} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{2x^2}{r_x(r_x+r_y)} + \frac{2y^2}{r_y(r_x+r_y)}\right\} \ + \text{const}\end{aligned}$$

$$\boxed{\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{\frac{x^2}{r_x(r_x+r_y)} + \frac{y^2}{r_y(r_x+r_y)}\right\} \ + \text{const}}$$

- This formula agrees with the simple case of an axisymmetric beam with $r_x = r_y = r_b$
  - Discussed further in a simple homework problem

A7



2) Indirect Proof:

- More efficient method
- Steps useful for other constructions including moment calculations
  - See: J.J. Barnard, Introductory Lectures

Density has elliptical symmetry:

$$n(x,y) = n\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right) \qquad \text{function } n(\text{argument}) \text{ arbitrary}$$

The solution to the 2D Poisson equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{qn}{\epsilon_0}$$

in free-space is then given by

$$\phi = -\frac{q r_x r_y}{4\epsilon_0}\int_0^\infty d\xi \, \frac{\eta(\chi)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}} \qquad \chi \equiv \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi}$$

where $\eta(\chi)$ is a function defined such that

$$n(x,y) = \left.\frac{d\eta(\chi)}{d\chi}\right|_{\xi=0}$$

- Can show that a choice of $\eta$ realizable for any elliptical symmetry $n$

A8



Prove that the solution is valid by direct substitution

$$\chi = \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} \quad \Longrightarrow \quad \begin{aligned}\frac{\partial \chi}{\partial x} &= \frac{2x}{r_x^2+\xi} & \frac{\partial^2 \chi}{\partial x^2} &= \frac{2}{r_x^2+\xi} \\ \frac{\partial \chi}{\partial y} &= \frac{2y}{r_y^2+\xi} & \frac{\partial^2 \chi}{\partial y^2} &= \frac{2}{r_y^2+\xi}\end{aligned}$$

Substitute in Poisson's equation, use the chain rule, and apply results above:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2}\right)\phi = -\frac{q r_x r_y}{4\epsilon_0}\int_0^\infty d\xi \, \frac{\left(\frac{d^2\eta}{d\chi^2}\right)\left(\frac{4x^2}{(r_x^2+\xi)^2} + \frac{4y^2}{(r_y^2+\xi)^2}\right) + \left(\frac{d\eta}{d\chi}\right)\left(\frac{2}{r_x^2+\xi} + \frac{2}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

Note:

$$d\chi = -\left[\frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)^2}\right] d\xi$$

Using this result the first integral becomes:

$$\int_0^\infty d\xi \, \frac{\left(\frac{d^2\eta}{d\chi^2}\right)\left(\frac{4x^2}{(r_x^2+\xi)^2} + \frac{4y^2}{(r_y^2+\xi)^2}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}} = -4\int_0^\infty d\xi \, \frac{\frac{d\eta^2}{d\chi^2}\frac{d\chi}{d\xi}}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

A9

Apply partial integration:

$$-4\int_0^\infty d\xi\, \frac{\frac{d\eta^2}{d\chi^2}\frac{d\chi}{d\xi}}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}} = -4\int_0^\infty d\xi\, \frac{\frac{d}{d\xi}\left(\frac{d\eta}{d\chi}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$= -4\int_0^\infty d\xi\, \frac{d}{d\xi}\left[\frac{\frac{d\eta}{d\chi}}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}\right] + 4\int_0^\infty d\xi\, \frac{d\eta}{d\chi}\frac{d}{d\xi}\frac{1}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$= -4\left.\frac{\frac{d\eta}{d\chi}}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}\right|_{\xi=0}^{\xi\to\infty} - 2\int_0^\infty d\xi\, \frac{\frac{d\eta}{d\chi}\left(\frac{1}{r_x^2+\xi}+\frac{1}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

in first term, upper limit vanishes since denominator ~ $\xi \to \infty$

$$= \frac{4}{r_x r_y}\left.\frac{d\eta}{d\chi}\right|_{\xi=0} - \boxed{2\int_0^\infty d\xi\, \frac{\frac{d\eta}{d\chi}\left(\frac{1}{r_x^2+\xi}+\frac{1}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}}$$

Term cancels 2nd integral

Giving:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -q\frac{r_x r_y}{4\epsilon_0}\frac{4}{r_x r_y}\left.\frac{d\eta(\chi)}{d\chi}\right|_{\xi=0} = -\frac{q}{\epsilon_0}n(x,y)$$

$$d\eta(\chi)/d\chi|_{\xi=0} = n(x,y)$$

Which verifies the ansatz.

A10



For a uniform density ellipse, we take:

$$\boxed{\eta(\chi) = \frac{\lambda}{q\pi r_x r_y}\begin{cases}\chi, & \text{if } \chi<1\\ 1, & \text{if } \chi>1\end{cases}} \longrightarrow \quad \frac{d\eta(\chi)}{d\chi} = \begin{cases}\frac{\lambda}{q\pi r_x r_y}, & \text{if } \chi<1\\ 0, & \text{if } \chi>1\end{cases}$$

Then

$$\left.\frac{d\eta(\chi)}{d\chi}\right|_{\xi=0} = \begin{cases}\frac{\lambda}{q\pi r_x r_y}, & \text{if } \chi|_{\xi=0}<1\\ 0, & \text{if } \chi|_{\xi=0}>1\end{cases} = \begin{cases}\frac{\lambda}{q\pi r_x r_y}, & \text{if } x^2/r_x^2+y^2/r_y^2<1\\ 0, & \text{if } x^2/r_x^2+y^2/r_y^2>1\end{cases}$$

Therefore, for this choice of

$$\left.\frac{d\eta(\chi)}{d\chi}\right|_{\xi=0} = n(x,y) \quad \text{for a uniform density elliptical beam with radii } r_x,\ r_y \text{ and density } \lambda/(q\pi r_x r_y)$$

Apply these results to calculate

$$\phi = -\frac{q r_x r_y}{4\epsilon_0}\int_0^\infty d\xi\, \frac{\eta(\chi)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$\chi = \frac{x^2}{r_x^2+\xi}+\frac{y^2}{r_y^2+\xi} \quad \Longrightarrow \quad \text{if } \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}<1, \text{ then}$$

$$\chi<1 \quad \text{for all} \quad 0\leq\xi<\infty$$

A11



Then:

$$\phi = -\frac{q r_x r_y}{4\epsilon_0}\int_0^\infty d\xi\, \frac{\lambda}{q\pi r_x r_y}\left[\frac{x^2}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}}+\frac{y^2}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}}\right]$$

Using Mathematica or integral tables

$$\int_0^\infty d\xi\, \frac{1}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}} = \frac{2}{r_x(r_x+r_y)}$$

$$\int_0^\infty d\xi\, \frac{1}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}} = \frac{2}{r_y(r_x+r_y)}$$

Showing that:

$$\boxed{\phi = -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{r_x(r_x+r_y)}+\frac{y^2}{r_y(r_x+r_y)}\right] \quad + \quad \text{const}}$$

since an overall constant can always be added to the potential (the integral had a reference choice $\phi(x=y=0)=0$ built in.

A12



The steps introduced in this proof can also be simply extended to show that

- For steps, see JJ Barnard, Introductory Lectures

$$\langle x\frac{\partial\phi}{\partial x}\rangle_\perp = -\frac{\lambda}{4\pi\epsilon_0}\frac{r_x}{r_x+r_y} \qquad \lambda\equiv q\int d^2x_\perp\, n \qquad r_x\equiv\langle x^2\rangle_\perp^{1/2}$$

$$\langle y\frac{\partial\phi}{\partial y}\rangle_\perp = -\frac{\lambda}{4\pi\epsilon_0}\frac{r_y}{r_x+r_y} \qquad r_y\equiv\langle y^2\rangle_\perp^{1/2}$$

for *any* elliptic symmetry density profile

$$n(x,y) = \text{func}\left(\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}\right)$$

In the introductory lectures, these results were applied to show that the KV envelope equations with evolving emittances can be applied to elliptic symmetry beams.

- Result first shown by Sacherer, IEEE Trans. Nuc. Sci. 18, 1105 (1971)

A13

## Appendix B: Canonical Transformation of the KV Distribution

The single-particle equations of motion:

$$x''(s) + \left\{\kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)}\right\} x(s) = 0$$

$$y''(s) + \left\{\kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)}\right\} y(s) = 0$$

can be derived from the Hamiltonian:

$$\boxed{\begin{aligned} H_\perp(x,y,x',y';s) = &\frac{1}{2}x'^2 + \left[\kappa_x(s) + \frac{2Q}{r_x(s)[r_x(s)+r_y(s)]}\right]\frac{x^2}{2} \\ &+ \frac{1}{2}y'^2 + \left[\kappa_y(s) + \frac{2Q}{r_y(s)[r_x(s)+r_y(s)]}\right]\frac{y^2}{2} \end{aligned}}$$

using:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

B1



Perform a canonical transform to new variables X,Y, X',Y' using the generating function

$$\boxed{F_2(x,y,X',Y') = \frac{x}{w_x}\left[X' + \frac{1}{2}xw'_x\right] + \frac{y}{w_y}\left[Y' + \frac{1}{2}yw'_y\right]}$$

Then we have from Canonical Transform theory (see: Goldstein, Classical Mechanics, 2nd Edition, 1980)

$$X = \frac{\partial F_2}{\partial X'} = \frac{x}{w_x} \qquad x' = \frac{\partial F_2}{\partial x} = \frac{1}{w_x}(X' + xw'_x)$$

$$Y = \frac{\partial F_2}{\partial Y'} = \frac{y}{w_y} \qquad y' = \frac{\partial F_2}{\partial y} = \frac{1}{w_y}(Y' + yw'_y)$$

which give

**Transform**

$$X = x/w_x \qquad X' = w_x x' - x w'_x$$
$$Y = y/w_y \qquad Y' = w_y y' - y w'_y$$

**Inverse Transform**

$$x = w_x X \qquad x' = X'/w_x + w'_x X$$
$$y = w_y Y \qquad y' = Y'/w_y + w'_y Y$$

B2



The structure of the canonical transform results in transformed equations of motion in proper canonical form:

$$\tilde{H}_\perp = H_\perp + \frac{\partial F_2}{\partial s} \qquad \tilde{H}_\perp = \tilde{H}_\perp(X,Y,X',Y';s)$$

$$\tilde{H} = \frac{1}{2w_x^2}X'^2 + \frac{1}{2w_y^2}Y'^2 + \frac{1}{2w_x^2}X^2 + \frac{1}{2w_y^2}Y^2$$

$$\frac{d}{ds}X = \frac{\partial \tilde{H}_\perp}{\partial X'} = \frac{X'}{w_x^2} \qquad \frac{d}{ds}X' = -\frac{\partial \tilde{H}_\perp}{\partial X} = -\frac{X}{w_x^2}$$

$$\frac{d}{ds}Y = \frac{\partial \tilde{H}_\perp}{\partial Y'} = \frac{Y'}{w_y^2} \qquad \frac{d}{ds}Y' = -\frac{\partial \tilde{H}_\perp}{\partial Y} = -\frac{Y}{w_y^2}$$

- Caution: X' merely denotes the conjugate variable to X : $\frac{d}{ds}X \neq X'$
- X and X' both have dimensions sqrt(meters)
- Equations of motion can be verified directly from transform equations (see problem sets)
- Transformed Hamiltonian $\tilde{H}_\perp$ is explicitly s dependent due to *w_x* and *w_y* lattice functions

B3



Following Davidson (Physics of Nonneutral Plasmas), the equations of motion:

$$\frac{d}{ds}X = \frac{X'}{w_x^2} \qquad \frac{d}{ds}X' = -\frac{X}{w_x^2}$$

$$\frac{d}{ds}Y = \frac{Y'}{w_y^2} \qquad \frac{d}{ds}Y' = -\frac{Y}{w_y^2}$$

have a psudo-harmonic oscillator solution:

- Straightforward to verify by direct substitution

$$X(s) = X_i \cos\psi_x(s) + X'_i \sin\psi_x(s)$$
$$X'(s) = -X_i \sin\psi_x(s) + X'_i \cos\psi_x(s)$$

$$\psi_x(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})} \qquad \begin{aligned} X_i &= \text{const} \\ X'_i &= \text{const} \end{aligned} \quad \text{set by initial conditions}$$

$$\Longrightarrow \quad \begin{aligned} X^2 + X'^2 = &X_i^2[\underbrace{\sin^2\psi_x + \cos^2\psi_x}_{1}] + X_i'^2[\underbrace{\sin^2\psi_x + \cos^2\psi_x}_{1}] \\ &+ X_i X'_i[\underbrace{\sin\psi_x\cos\psi_x - \sin\psi_x\cos\psi_x}_{0}] \\ = &X_i^2 + X_i'^2 = \text{const} \end{aligned}$$

\+ Same form solution in *y*-plane

B4

Using the transforms:

$$X = x/w_x \qquad X' = w_x x' - x w_x'$$
$$Y = y/w_y \qquad Y' = w_y y' - y w_y'$$

in this expression verifies the simple, symmetrical form of the Courant-Snyder invariants in the transformed variables:

$$X^2 + X'^2 = \left(\frac{x}{w_x}\right)^2 + (w_x x' - x w_x')^2 = X_i^2 + X_i'^2 = \text{const}$$
$$Y^2 + Y'^2 = \left(\frac{y}{w_y}\right)^2 + (w_y y' - y w_y')^2 = Y_i^2 + Y_i'^2 = \text{const}$$

B6



The canonical transforms render the KV distribution much simpler to express. First examine how phase-space areas transform:

$$dxdy = w_x w_y dXdY \qquad dx'dy' = \frac{dX'dY'}{w_x w_y} \qquad \Longrightarrow \qquad dxdydx'dy' = dXdYdX'dY'$$

- The property $dx\,dy\,dx'\,dy' = dX\,dY\,dX'\,dY'$ is a consequence of proper canonical transforms preserving phase-space area

Because phase space area is conserved, the distribution in transformed phase-space variables is identical to the original distribution. Therefore, for the KV distribution

$$f_\perp = \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right]$$
$$= \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\frac{X^2+X'^2}{\varepsilon_x} + \frac{Y^2+Y'^2}{\varepsilon_y} - 1\right] \qquad r_x = \sqrt{\varepsilon_x}w_x$$

- Transformed form simpler and more symmetrical
- Exploited to simplify calculation of distribution moments and projections

B7



## Density Calculation:

As a first example application of the canonical transform, prove that the density projection of the KV distribution is a uniform density ellipse. Doing so will prove the consistency of the KV equilibrium:

- If density projection is as assumed then the Courant-Snyder invariants are valid
- Steps used can be applied to calculate other moments/projections
- Steps can be applied to continuous focusing without using the transformations

$$n(x,y) = \int dx'dy'\, f_\perp = \int \frac{dX'dY'}{w_x w_y} f_\perp$$

$$r_x = \sqrt{\varepsilon_x}w_x \qquad U_x = X'/\sqrt{\varepsilon_x} \qquad dU_x dU_y = \frac{dX'dY'}{\sqrt{\varepsilon_x\varepsilon_y}}$$
$$r_y = \sqrt{\varepsilon_y}w_y \qquad U_y = Y'/\sqrt{\varepsilon_y}$$

$$n = \frac{\lambda}{q\pi^2 r_x r_y}\int dU_x dU_y\, \delta\left[U_x^2 + U_y^2 - \left(1 - \frac{X^2}{\varepsilon_x} - \frac{Y^2}{\varepsilon_y}\right)\right]$$

B8



Exploit the cylindrical symmetry

$$U_\perp^2 = U_x^2 + U_y^2 \qquad dU_x dU_y = d\psi U_\perp dU_\perp = d\psi\frac{dU_\perp^2}{2}$$

$$n(x,y) = \frac{\lambda}{q\pi^2 r_x r_y}\int_{-\pi}^{\pi} d\psi \int_0^\infty \frac{dU_\perp^2}{2}\,\delta\left[U_\perp^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right]$$

giving

$$n(x,y) = \frac{\lambda}{q\pi r_x r_y}\int_0^\infty dU_\perp^2\,\delta\left[U_\perp^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right]$$
$$= \begin{cases} \frac{\lambda}{q\pi r_x r_y} = \hat{n}, & \text{if } x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & \text{if } x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

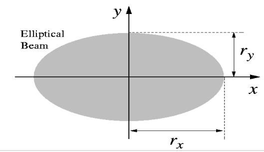
Shows that the singular KV distribution yields the required uniform density elliptical projection required for self-consistency!

Note:

$$\hat{n} = \frac{\lambda}{q\pi r_x r_y}$$

Line Charge: $\lambda = \text{const}$

Area Ellipse = $\pi r_x r_y$




B9

// Aside

An interesting footnote to this Appendix is that an infinity of canonical generating functions can be applied to transform the KV distribution in standard quadratic form

$$f_\perp \sim \delta[X^2 + X'^2 + Y^2 + Y'^2 - \text{const}]$$

to other sets of variables. These distributions have underlying KV form.

- Not logical to label transformed KV distributions as “new” but this has been done in the literature
  - Could generate an infinity of KV like equilibria in this manner
- Identifying specific transforms with physical relevance can be useful even if the canonical structure of the distribution is still KV
  - Helps identify basic design criteria with envelope consistency equations etc.
  - Example of this is a self-consistent KV distribution formulated for quadrupole skew coupling

//

B10



## S4: Continuous Focusing limit of the KV Equilibrium Distribution

Continuous focusing, axisymmetric beam

$$\boxed{\begin{aligned} &\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const} \\ &\varepsilon_x = \varepsilon_y \equiv \varepsilon \\ &r_x = r_y \equiv r_b \end{aligned}}$$

Undepressed betatron wavenumber

KV envelope equation

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

reduces to:

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

with matched ( $r_b' = 0$ ) solution to the quadratic in $r_b^2$ envelope equation

$$\boxed{r_b = \left( \frac{Q + \sqrt{4k_{\beta 0}^2 \varepsilon^2 + Q^2}}{2k_{\beta 0}^2} \right)^{1/2} = \text{const}}$$



Similarly, the particle equations of motion within the beam are:

$$x'' + \left\{ \kappa_x - \frac{2Q}{[r_x + r_y] r_x} \right\} x = 0$$

$$y'' + \left\{ \kappa_y - \frac{2Q}{[r_x + r_y] r_y} \right\} y = 0$$

reduce to

$$\mathbf{x}_\perp'' + k_\beta^2 \mathbf{x}_\perp = 0 \qquad \boxed{k_\beta \equiv \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} = \text{const}}$$

Depressed betatron wavenumber

with solution

$$\boxed{\mathbf{x}_\perp(s) = \mathbf{x}_{\perp i} \cos[k_\beta(s - s_i)] + \frac{\mathbf{x}_{\perp i}'}{k_\beta} \sin[k_\beta(s - s_i)]}$$

Space-charge tune depression (rate of phase advance same everywhere, $L_p$ arb.)

$$\frac{k_\beta}{k_{\beta 0}} = \frac{\sigma}{\sigma_0} = \left( 1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right)^{1/2} \qquad 0 \leq \frac{\sigma}{\sigma_0} \leq 1$$

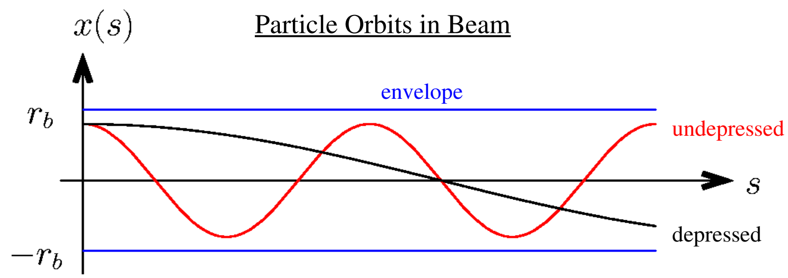
$\varepsilon \to 0$ envelope equation $\Rightarrow r_b = \sqrt{Q}/k_{\beta 0}$

$Q \to 0$ envelope equation $\Rightarrow r_b = \sqrt{\varepsilon / k_{\beta 0}}$



## Continuous Focusing KV Equilibrium – Undepressed and depressed particle orbits in the *x*-plane

$$k_\beta = \frac{\sigma}{\sigma_0} k_{\beta 0} \qquad \frac{\sigma}{\sigma_0} = 0.2 \qquad y = 0 = y'$$




Much simpler in details than the periodic focusing case, but qualitatively similar in that space-charge “depresses” the rate of particle phase advance

## Continuous Focusing KV Beam – Equilibrium Distribution Form

Using

$$\lambda = q\pi\hat{n}r_b^2 \qquad \hat{n} = \text{const} \quad \text{density within the beam}$$

for the beam line charge and

$$\delta(\text{const}\cdot x) = \frac{\delta(x)}{\text{const}}$$

the full elliptic beam KV distribution can be expressed as :

- See next slide for steps involved in the form reduction

$$\boxed{\begin{aligned} f_\perp &= \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right] \\ &= \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b}) \end{aligned}}$$

where

$$\begin{aligned} H_\perp &= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \\ &= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 \qquad \text{-- Hamiltonian (on-axis } \phi = 0 \text{ ref taken)} \\ H_{\perp b} &\equiv \frac{\varepsilon^2}{2r_b^2} = \text{const} \qquad \text{-- Hamiltonian at beam edge, } r = r_b \end{aligned}$$



/// Aside: Steps of derivation

Using:

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon \qquad \lambda = q\pi\hat{n}r_b^2 = \text{const}$$

$$r_x = r_y \equiv r_b = \text{const}$$

$$\begin{aligned} f_\perp &= \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right] \\ &= \frac{\hat{n}r_b^2}{\pi\varepsilon^2}\delta\left(\frac{x^2}{r_b^2} + \frac{y^2}{r_b^2} + \frac{r_b^2x'^2}{\varepsilon^2} + \frac{r_b^2y'^2}{\varepsilon^2} - 1\right) \end{aligned}$$

Using:

$$\delta(\text{const}\cdot x) = \frac{\delta(x)}{\text{const}}$$

$$f_\perp = \frac{\hat{n}}{2\pi}\delta\left(\frac{1}{2}\mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 - \frac{\varepsilon^2}{2r_b^2}\right)$$

The solution for the potential for the uniform density beam *inside* the beam is:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} = -\frac{\lambda}{\pi\epsilon_0 r_b^2} \quad\longrightarrow\quad \phi = -\frac{\lambda}{4\pi\epsilon_0 r_b^2}\mathbf{x}_\perp^2 + \text{const}$$



The Hamiltonian becomes:

$$\begin{aligned} H_\perp &= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \\ &= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 - \frac{q\lambda}{4\pi m\gamma_b^3\beta_b^2c^2r_b^2}\mathbf{x}_\perp^2 + \text{const} \\ &= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 - \frac{Q}{2r_b^2}\mathbf{x}_\perp^2 + \text{const} \end{aligned} \qquad \begin{aligned} Q &\equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} \\ &= \text{const} \end{aligned}$$

From the equilibrium envelope equation:

$$k_{\beta 0}^2 = \frac{Q}{r_b^2} + \frac{\varepsilon^2}{r_b^4}$$

Substituting this result, the term $\propto Q$ can be eliminated in $H_\perp$ to obtain:

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 + \text{const}$$

The value of $H_\perp$ for an edge particle (turning point with zero angle) is:

$$H_{\perp b} \equiv H_\perp(r = r_b, \mathbf{x}_\perp' = 0) = \frac{\varepsilon^2}{2r_b^2} + \text{const}$$

Giving (constants are same in Hamiltonian and edge value and subtract out):

$$f_\perp = \frac{\hat{n}}{2\pi}\delta\left(\frac{1}{2}\mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 - \frac{\varepsilon^2}{2r_b^2}\right) = \frac{\hat{n}}{2\pi}\delta\left(H_\perp - H_{\perp b}\right)$$

///



### Equilibrium distribution

$$\boxed{f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})} \qquad H_{\perp b} = \frac{\varepsilon^2}{2r_b^2} = \text{const}$$

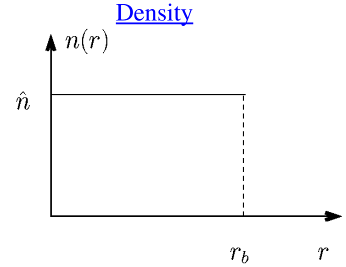
$\hat{n} = \text{const}$ because $r_b$ =const

From the equilibrium $f_\perp(H_\perp)$ can explicitly calculate (see homework problems)

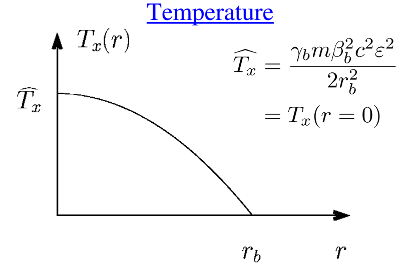
Density: $$n = \int d^2x_\perp'\, f_\perp = \begin{cases} \hat{n}, & 0 \le r < r_b \\ 0, & r_b < r \end{cases}$$

Temperature: $$T_x = \gamma_b m\beta_b^2c^2\frac{\int d^2x_\perp'\, x'^2 f_\perp}{\int d^2x_\perp'\, f_\perp} = \begin{cases} \widehat{T}_x(1 - r^2/r_b^2), & 0 \le r < r_b \\ 0, & r_b < r \end{cases}$$

Density




Temperature



## Continuous Focusing KV Beam – Comments

For continuous focusing, $H_\perp$ is a single particle constant of the motion (see problem sets), so it is not surprising that the KV equilibrium form reduces to a delta function form of $f_\perp(H_\perp)$

- Because of the delta-function distribution form, all particles in the continuous focusing KV beam have the same transverse energy with $H_\perp = H_{\perp b} = \text{const}$

Several textbook treatments of the KV distribution derive continuous focusing versions and then just write down (if at all) the periodic focusing version based on Courant-Snyder invariants. This can create a false impression that the KV distribution is a Hamiltonian-type invariant in the general form.

- For non-continuous focusing channels there is no simple relation between Courant-Snyder type invariants and $H_\perp$



## Case of a mismatched KV beam in a continuous focusing channel

If we take $f_\perp = f_\perp(H_\perp)$ in a continuous focusing channel, the resulting beam equilibrium is stationary $(\partial/\partial s = 0)$ in all statistical measures with

$$r_b = 2\langle x^2\rangle_\perp^{1/2} = \text{const}$$
$$\varepsilon_x = \varepsilon_y = \varepsilon = 4\sqrt{\langle x^2\rangle_\perp \langle x'^2\rangle_\perp} = \text{const}$$

and the beam satisfies the stationary envelope equation

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

This matched beam will be in local radial force balance with no oscillations (see S 5)

The KV case of the *matched* equilibrium distribution has been derived as

$$f_\perp = f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})$$



More generally, the KV distribution can be mismatched to the focusing lattice. In this case one *cannot* write the distribution as

$$f_\perp = f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})$$

but rather, is expressible in terms of the more general form of the KV distribution

$$f_\perp = \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right]$$

which can be written in several forms using:

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon = \text{const}$$
$$r_x = r_y \equiv r_b \neq \text{const}$$
$$r_x' = r_y' \equiv r_b' \neq \text{const}$$

with $r_b = r_b(s)$ satisfying the envelope equation:

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

$$\begin{aligned} f_\perp &= \frac{\lambda}{q\pi^2\varepsilon^2}\delta\left[\frac{r_b^2}{\varepsilon^2}(x'^2+y'^2) + \left(\frac{1}{r_b^2} + \frac{r_b'^2}{\varepsilon^2}\right)(x^2+y^2) - \frac{2r_b r_b'}{\varepsilon^2}(xx'+yy') - 1\right] \\ &= \frac{\lambda}{q2\pi^2 r_b^2}\delta\left[\frac{1}{2}(x'^2+y'^2) + \frac{1}{2r_b^2}\left(\frac{\varepsilon^2}{r_b^2} + r_b'^2\right)(x^2+y^2) - \frac{r_b'}{r_b}(xx'+yy') - \frac{\varepsilon^2}{2r_b^2}\right] \end{aligned}$$

- These forms are valid regardless of the amplitude of variation in $r_b(s)$ which also satisfies the envelope equation



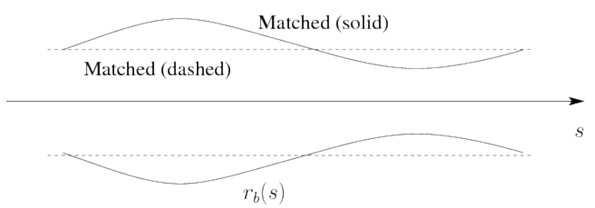
### Mismatched KV beam envelope:

Envelope $r_b = r_b(s)$ evolves consistently with the envelope equation:

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

from some specified initial condition

$$r_b(s = s_i) = r_{bi}$$
$$r_b'(s = s_i) = r_{bi}'$$




- For small amplitudes, the envelope will be oscillate harmonically with the period corresponding to the breathing mode wavelength as described in lectures on Transverse Centroid and Envelope Models of Beam Evolution

## S5: Stationary Equilibrium Distributions in Continuous Focusing Channels

Take

$$\boxed{\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}}$$

- Real transport channels have $s$-varying focusing functions
- For a rough correspondence to physical lattices take: $k_{\beta 0} = \sigma_0 / L_p$

A class of equilibrium can be constructed for any non-negative choice of function:

$$\boxed{f_\perp = f_\perp(H_\perp) \geq 0} \qquad H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2}$$

$\phi$ must be calculated consistently from the (generally nonlinear) Poisson equation:

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int d^2x_\perp'\; f_\perp(H_\perp)}$$

- Solutions generated will be steady-state $(\partial/\partial s = 0)$
- When $f_\perp = f_\perp(H_\perp)$ , the Poisson equation *only* has axisymmetric solutions with $\partial/\partial\theta = 0$ [see: Lund, PRSTAB **10**, 064203 (2007)]

The Hamiltonian is only equivalent to the Courant-Snyder invariant in continuous focusing (see: Transverse Particle Dynamics). In periodic focusing channels $\kappa_x(s)$ and $\kappa_y(s)$ vary in $s$ and the Hamiltonian is *not* a constant of the motion.



The axisymmetric Poisson equation simplifies to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{qn}{\epsilon_0} = -\frac{q}{\epsilon_0}\int d^2x_\perp'\; f_\perp(H_\perp)$$

For notational convenience, introduce an effective potential (add applied component and rescale) defined by:

$$\boxed{\psi(r) \equiv \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}} \qquad r = \sqrt{x^2+y^2}$$

then

$$\boxed{H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \psi}$$

and system axisymmetry can be exploited to calculate the beam density :

$$\boxed{n(r) = \int d^2x_\perp'\; f_\perp(H_\perp) \;= 2\pi\int_\psi^\infty dH_\perp\; f_\perp(H_\perp)}$$

Proof:

$$n(r) = \int d^2x_\perp'\; f_\perp(H_\perp) \;= \int_{-\pi}^{\pi} d\tilde{\theta}' \int_0^\infty d\tilde{r}'\;\tilde{r}' f_\perp\left(\frac{1}{2}\tilde{r}'^2 + \psi\right)$$

$$H_\perp = \frac{1}{2}\tilde{r}'^2 + \psi \qquad H_\perp|_{\tilde{r}'=0} = \psi \qquad = 2\pi\int_0^\infty d\tilde{r}'\;\tilde{r}' f_\perp\left(\frac{1}{2}\tilde{r}'^2+\psi\right)$$

$$dH_\perp = \tilde{r}'d\tilde{r}' \qquad H_\perp|_{\tilde{r}'\to\infty} \to \infty \qquad = 2\pi\int_\psi^\infty dH_\perp\; f_\perp(H_\perp)$$



The Poisson equation can then be expressed in terms of the effective potential as:

$$\boxed{\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) &= 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0\gamma_b^3\beta_b^2c^2}\int_{\psi(r)}^\infty dH_\perp\; f_\perp(H_\perp) \\ H_\perp &= \frac{1}{2}\mathbf{x}_\perp'^2 + \psi \\ -\frac{\partial\phi}{\partial r} &= m\gamma_b^3\beta_b^2c^2\left[k_{\beta 0}^2 r - \frac{\partial\psi}{\partial r}\right] \end{aligned}}$$

To characterize a choice of equilibrium function $f_\perp(H_\perp)$, the (transformed) Poisson equation must be solved

- Equation is, in general, *highly* nonlinear rendering the procedure difficult
  - Linear for 2 special cases: KV (covered) and Waterbag (section to follow)

Some general features of equilibria can still be understood:

- Apply rms equivalent beam picture and interpret in terms of moments
- Calculate equilibria for a few types of very different functions to understand the likely range of characteristics



## Moment properties of continuous focusing equilibrium distributions

Equilibria with *any* valid equilibrium $f_\perp(H_\perp)$ satisfy the stationary $(r_b = \text{const})$ rms equivalent envelope equation for a matched beam:

$$\boxed{k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0}$$

- Describes average radial force balance of particles
- Uses the result (see J.J. Barnard, Intro. Lectures): $\langle x\partial\phi/\partial x\rangle_\perp = -\lambda/(8\pi\epsilon_0)$

where

$$\boxed{\begin{aligned} Q &= \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const} \qquad \lambda = q\int d^2x_\perp\int d^2x_\perp'\; f_\perp(H_\perp) \\ r_b^2 &= 4\langle x^2\rangle_\perp = 2\langle r^2\rangle_\perp = \frac{\int_0^\infty dr\; r^3\int_\psi^\infty dH_\perp\; f_\perp(H_\perp)}{\int_0^\infty dr\; r\int_\psi^\infty dH_\perp\; f_\perp(H_\perp)} = \text{const} \\ \varepsilon^2 &= 2r_b^2\langle\mathbf{x}_\perp'^2\rangle_\perp = 2r_b^2\frac{\int_0^\infty dr\; r\int_\psi^\infty dH_\perp\;(H_\perp-\psi)f_\perp(H_\perp)}{\int_0^\infty dr\; r\int_\psi^\infty dH_\perp\; f_\perp(H_\perp)} = \text{const} \\ \langle\cdots\rangle_\perp &= \frac{\int d^2x_\perp\int d^2x_\perp'\;\cdots f_\perp(H_\perp)}{\int d^2x_\perp\int d^2x_\perp'\; f_\perp(H_\perp)} \end{aligned}}$$

Parameters used to define the equilibrium function

$$f_\perp(H_\perp)$$

should be cast in terms of (or ratios of)

$$k_{\beta 0}, \quad Q, \quad \varepsilon, \quad r_b$$

for use in accelerator applications. The rms equivalent beam equations can be used to carry out needed parameter eliminations. Such eliminations can be complicated due to the nonlinear structure of the equations.

A local (generally $r$ varying) kinetic temperature can also be calculated

$$T_x = \langle x'^2 \rangle_{\mathbf{x}'_\perp} \qquad \langle \cdots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \; \cdots f_\perp}{\int d^2x'_\perp \; f_\perp}$$

$$\boxed{n(r)T_x(r) = \frac{1}{2}\int d^2x'_\perp \; \mathbf{x}'^2_\perp f_\perp(H_\perp) \; = 2\pi \int_\psi^\infty dH_\perp \; (H_\perp - \psi) f_\perp(H_\perp)}$$

which is also related to the emittance,

$$\langle x'^2 \rangle_\perp = \frac{\int d^2x_\perp \; nT_x}{\int d^2x_\perp n} \qquad \boxed{\varepsilon^2 = 16\langle x^2\rangle_\perp \langle x'^2\rangle_\perp = 4r_b^2 \frac{\int d^2x_\perp \; nT}{\int d^2x_\perp \; n}}$$



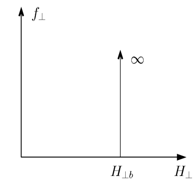
## Choices of continuous focusing equilibrium distributions:

Common choices for $f_\perp(H_\perp)$ analyzed in the literature:

1) KV (already covered)

$$\boxed{f_\perp \propto \delta(H_\perp - H_{\perp b})}$$

$$H_{\perp b} = \text{const}$$

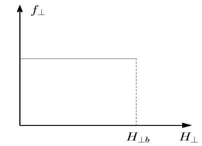



2) Waterbag (to be covered)

[see M. Reiser, *Charged Particle Beams*, (1994, 2008)]

$$\boxed{f_\perp \propto \Theta(H_{\perp b} - H_\perp)}$$

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x \end{cases}$$

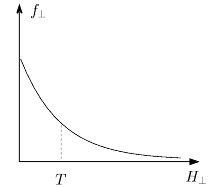



3) Thermal (to be covered)

[see M. Reiser; Davidson, *Nonneutral Plasmas*, 1990]

$$\boxed{f_\perp \propto \exp(-H_\perp/T)}$$

$$T = \text{const} > 0$$




Infinity of choices can be made for an infinity of papers!

- Fortunately, range of behavior can be understood with a few reasonable choices

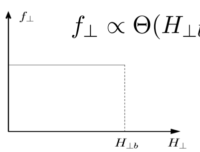


## Preview of what we will find: When relative space-charge is strong, all smooth equilibrium distributions expected to look similar

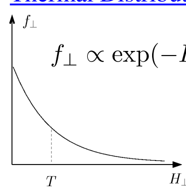
Constant charge and focusing: $Q = 10^{-4}$ $\quad k_{\beta 0}^2 = \text{const}$

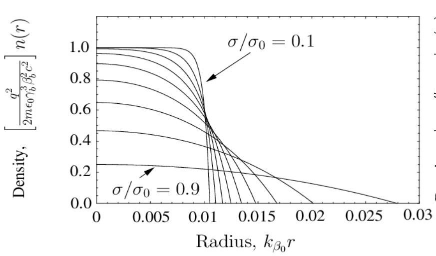
Vary relative space-charge strength: $\sigma/\sigma_0 = 0.1, \; 0.2, \; \cdots, \; 0.9$

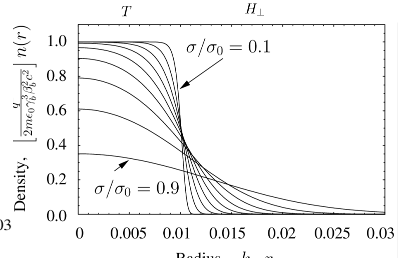
Waterbag Distribution




Thermal Distribution




Edge shape varies with distribution choice, but cores similar when $\sigma/\sigma_0$ small



## S6: Continuous Focusing: The Waterbag Equilibrium Distribution:

[Reiser, *Theory and Design of Charged Particle Beams*, Wiley (1994, 2008); and Review: Lund, Kikuchi, and Davidson, PRSTAB **12**, 114801 (2009), Appendix D]

Waterbag distribution:

$$\boxed{f_\perp(H_\perp) = f_0 \Theta(H_b - H_\perp) \qquad f_0 = \text{const} \qquad H_b = \text{const} \quad \text{Edge Hamiltonian} \qquad \Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}}$$

The physical edge radius $r_e$ of the beam will be related to the edge Hamiltonian:

$$\boxed{H_\perp|_{r=r_e} = H_b} \qquad \text{Note (generally):} \quad \begin{matrix} r_e \neq r_b \equiv 2\langle x^2\rangle_\perp^{1/2} \\ r_e > r_b \end{matrix}$$

Using previous formulas the equilibrium density can then be calculated as:

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \psi \qquad \psi = \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$\boxed{n(r) = \int d^2x'_\perp \; f_\perp = 2\pi \int_\psi^\infty dH_\perp \; f_\perp(H_\perp) = 2\pi f_0 \begin{cases} H_b - \psi(r), & \psi < H_b \\ 0, & \psi > H_b \end{cases}}$$

The transformed Poisson equation of the equilibrium

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0\gamma_b^3\beta_b^2 c^2}\int_{\psi(r)}^{\infty} dH_\perp \; f_\perp(H_\perp)$$

can be expressed within the beam $(r < r_e)$ as:

$$\boxed{\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - k_0^2\psi = 2k_{\beta 0}^2 - k_0^2 H_b \\ &k_0^2 \equiv \frac{2\pi q^2 f_0}{\epsilon_0 m\gamma_b^3\beta_b^2 c^2} = \text{const} \end{aligned}}$$

This is a modified Bessel function equation and the solution within the beam regular at the origin r = 0 and satisfying $\psi(r = r_e) = H_b$ is given by

$$\boxed{\psi(r) = H_b - 2\frac{k_{\beta 0}^2}{k_0^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right]}$$

where $I_\ell(x)$ denotes a modified Bessel function of order $\ell$



The density is then expressible within the beam $(r < r_e)$ as:

$$\boxed{\begin{aligned} n(r) &= 4\pi f_0 \frac{k_{\beta 0}^2}{k_0^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right] \\ &= \frac{2\epsilon_0 m\gamma_b^3\beta_b^2 c^2 k_{\beta 0}^2}{q^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right] \end{aligned}}$$

Similarly, the local beam temperature within the beam can be calculated as:

$$\boxed{\begin{aligned} T_x(r) &= \langle x'^2\rangle_{\mathbf{x}'_\perp} = \frac{k_{\beta 0}^2}{k_0^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right] \\ &\propto n(r) \end{aligned}}$$

The feature of a fixed proportionality between the temperature $T_x(r)$ and the density *n(r)* is a consequence of the waterbag equilibrium distribution choice and is *not* a general feature of continuous focusing.



The waterbag distribution expression can now be expressed as:

$$\boxed{f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp) = f_0 \Theta\left(2\frac{k_{\beta 0}^2}{k_0^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right] - \frac{1}{2}\mathbf{x}'^2_\perp\right)}$$

- The edge Hamiltonian value $H_b$ has been eliminated
- Parameters are:

$f_0$ .... distribution normalization

$k_0 r_e$ .... scaled edge radius

$k_{\beta 0}/k_0$ .... scaled focusing strength

Parameters preferred for accelerator applications:

$$k_{\beta 0}, \quad Q, \quad \varepsilon_x = \varepsilon_y = \varepsilon_b$$

Needed constraints to eliminate parameters in terms of our preferred set will now be derived.



## Parameters constraints for the waterbag equilibrium beam

First calculate the beam line-charge:

$$\lambda = 2\pi q\int_0^{r_e} dr\; rn(r) = 4\pi^2 q f_0 \frac{k_{\beta 0}^2}{k_0^2} r_e^2\left[1 - \frac{2}{k_0 r_e}\frac{I_1(k_0 r_e)}{I_0(k_0 r_e)}\right]$$

$$\boxed{\lambda = 2\pi q\int_0^{r_e} dr\; rn(r) = 4\pi^2 q f_0 \frac{k_{\beta 0}^2}{k_0^2} r_e^2 \frac{I_2(k_0 r_e)}{I_0(k_0 r_e)}}$$

here we have employed the modified Bessel function identities ( $\ell$ integer):

$$\frac{d}{dx}[x^\ell I_\ell(x)] = x^\ell I_{\ell-1}(x),$$

$$-\frac{2\ell}{x}I_\ell(x) = I_{\ell+1}(x) - I_{\ell-1}(x),$$

Similarly, the beam rms edge radius can be explicitly calculated as:

$$r_b^2 = 2\langle r^2\rangle_\perp = 2\frac{\int_0^{r_e} dr\; r^3 n(r)}{\int_0^{r_e} dr\; rn(r)}$$

$$\boxed{\left(\frac{r_b}{r_e}\right)^2 = \frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2}\left[2 + (k_0 r_e)\frac{I_3(k_0 r_e)}{I_2(k_0 r_e)}\right]}$$

The perveance is then calculated as:

$$\boxed{Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2} = (k_{\beta 0} r_e)^2 \frac{I_2(k_0 r_e)}{I_0(k_0 r_e)}}$$

The edge and perveance equations can then be combined to obtain a parameter constraint relating $k_0 r_e$ to desired system parameters:

$$\boxed{\frac{k_{\beta 0}^2 r_b^2}{Q} = \frac{I_0^2(k_0 r_e)}{I_2^2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2}\left[2\frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} + (k_0 r_e)\frac{I_0(k_0 r_e) I_3(k_0 r_e)}{I_2^2(k_0 r_e)}\right]}$$

Here, any of the 3 system parameters on the LHS may be eliminated using the matched beam envelope equation to effect alternative parameterizations:

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_b^2}{r_b^3} = 0 \quad \longrightarrow \quad \text{eliminate any of:} \quad k_{\beta 0}^2, \;\; r_b, \;\; Q$$

The rms equivalent beam concept can also be applied to show that:

$$\boxed{\frac{k_{\beta 0}^2 r_b^2}{Q} = \frac{1}{1 - (\sigma/\sigma_0)^2}}$$

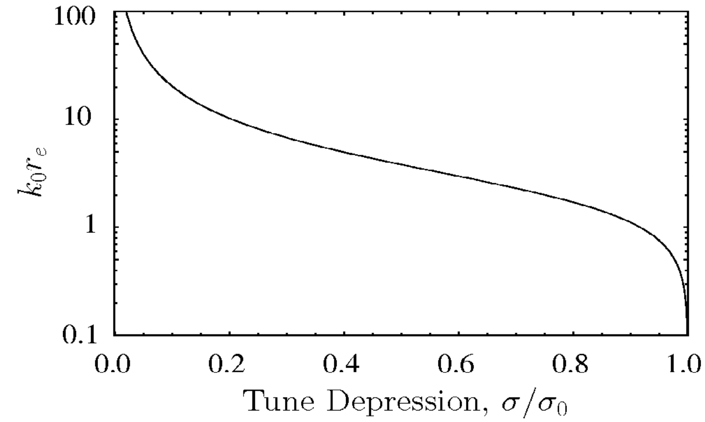
*rms equivalent* KV measure of $\sigma/\sigma_0$

- Space-charge really nonlinear and the Waterbag equilibrium has a spectrum of $\sigma$



The constraint is plotted over the full range of effective space-charge strength:

$$\boxed{\frac{1}{1 - (\sigma/\sigma_0)^2} = \frac{I_0^2(k_0 r_e)}{I_2^2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2}\left[2\frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} + (k_0 r_e)\frac{I_0(k_0 r_e) I_3(k_0 r_e)}{I_2^2(k_0 r_e)}\right]}$$




- Equilibrium parameter $k_0 r_e$ uniquely fixes effective space-charge strength



///Aside: Parameter choices and limits of the constraint equation

Some prefer to use an alternative space-charge strength measure to $\sigma/\sigma_0$ and use a so-called self-field parameter defined in terms of the on-axis plasma frequency of the distribution:

Self-field parameter:

$$s_b \equiv \frac{\hat{\omega}_p^2}{2\gamma_b^3\beta_b^2 c^2 k_{\beta 0}^2} \qquad \hat{\omega}_p^2 \equiv \frac{q^2 \hat{n}}{m\epsilon_0} \qquad \begin{aligned}\hat{n} &= n(r=0) \\ &= \text{on-axis plasma density}\end{aligned}$$

For a KV equilibrium, $s_b$ and $\sigma/\sigma_0$ are simply related:

$$s_b = 1 - \left(\frac{\sigma}{\sigma_0}\right)^2$$

For a waterbag equilibrium, $s_b$ and $k_0 r_e$ (from which $\sigma/\sigma_0$ can be calculated) are related by:

$$s_b = 1 - \frac{1}{I_0(k_0 r_e)}$$

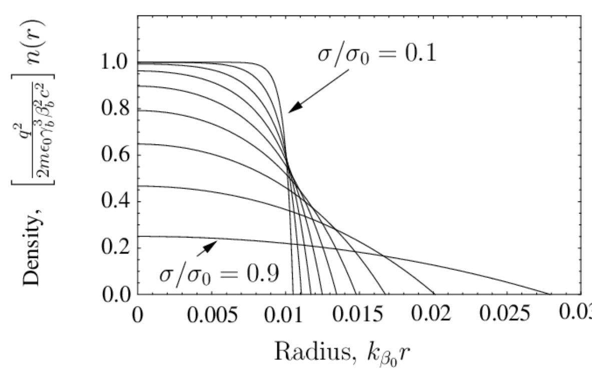
Generally, for smooth (non-KV) equilibria, $s_b$ is a logarithmically insensitive parameter for strong space-charge strength (see tables in S6 and S7) ///



## Use parameter constraints to plot properties of waterbag equilibrium

1) Density and temperature profile at fixed line charge and focusing strength

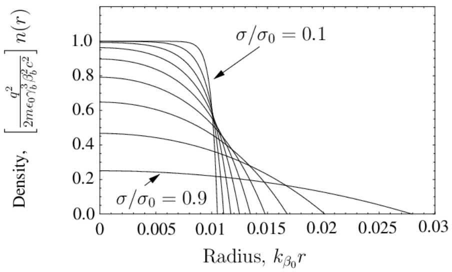
$Q = 10^{-4}$ $\qquad k_{\beta 0}^2 = \text{const}$



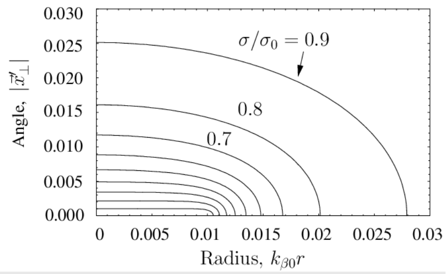

- Parabolic density for weak space-charge and flat in the core out to a sharp edge for strong space charge
- For the waterbag equilibrium, temperature $T(r)$ is proportional to density $n(r)$ so the same curves apply for $T(r)$

2) Phase-space boundary of distribution at fixed line charge and focusing strength

$Q = 10^{-4}$ $\qquad k_{\beta 0}^2 = \text{const}$




Density Profile




Edge of distribution in phase-space



3) Summary of scaled parameters for example plots:

| | | | | | $Q = 10^{-4}$ | |
|---|---|---|---|---|---|---|
| $\sigma/\sigma_0$ | $s_b$ | $\frac{k_{\beta 0}^2 r_b^2}{Q}$ | $k_0 r_e$ | $\frac{r_e}{r_b}$ | $\frac{k_0}{k_{\beta 0}}$ | $10^3 \times k_{\beta 0}\varepsilon_b$ |
| 0.9 | 0.2502 | 5.263 | 1.112 | 1.217 | 39.81 | 0.4737 |
| 0.8 | 0.4666 | 2.778 | 1.709 | 1.208 | 84.87 | 0.2222 |
| 0.7 | 0.6477 | 1.961 | 2.304 | 1.197 | 137.5 | 0.1373 |
| 0.6 | 0.7916 | 1.563 | 2.979 | 1.183 | 201.5 | 0.09375 |
| 0.5 | 0.8968 | 1.333 | 3.821 | 1.166 | 283.8 | 0.06667 |
| 0.4 | 0.9626 | 1.190 | 4.978 | 1.144 | 398.7 | 0.04762 |
| 0.3 | 0.9928 | 1.099 | 6.789 | 1.118 | 579.3 | 0.03297 |
| 0.2 | 0.9997 | 1.042 | 10.25 | 1.085 | 925.6 | 0.02083 |
| 0.1 | 1.0000 | 1.010 | 20.38 | 1.046 | 1938. | 0.01010 |



## S7: Continuous Focusing: The Thermal Equilibrium Distribution:

[Davidson, Physics of Nonneutral Plasma, Addison Wesley (1990),
Reiser, Theory and Design of Charged Particle Beams, Wiley (1994, 2008),
Review: Lund, Kikuchi, and Davidson, PRSTAB **12**, 114801 (2009), Appendix F]

In an infinitely long continuous focusing channel, collisions will eventually relax the beam to thermal equilibrium. The Fokker-Planck equation predicts that the unique Maxwell-Boltzmann distribution describing this limit is:

$$\lim_{s\to\infty} f_\perp \propto \exp\left(-\frac{H_{\text{rest}}}{T}\right)$$

$H_{\text{rest}} =$ single particle Hamiltonian of beam in rest frame (energy units)

$T = \text{const}$ Thermodynamic temperature (energy units)

Beam propagation time in transport channel is generally short relative to collision time, inhibiting full relaxation

- Collective effects may enhance relaxation rate
  - Wave spectrums likely large for real beams and enhanced by transient and nonequilibrium effects
  - Random errors acting on system may enhance and lock-in phase mixing



## Continuous focusing thermal equilibrium distribution

Analysis of the rest frame transformation shows that the 2D Maxwell-Boltzmann distribution (careful on frame for temperature definition!) is:

$$f_\perp(H_\perp) = \frac{m\gamma_b\beta_b^2c^2\hat{n}}{2\pi T}\exp\left(-\frac{m\gamma_b\beta_b^2c^2H_\perp}{T}\right)$$

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} = \frac{1}{2}\mathbf{x}'^2_\perp + \psi$$

$T = \text{const}$ Temperature (energy units, lab frame)

$n(r=0) = \hat{n} = \text{const}$ on-axis density

$\phi(r=0) = 0$ (reference choice)

The density can then be conveniently calculated in terms of a scaled stream function:

$$n(r) = \int d^2x'_\perp\, f_\perp = \hat{n}e^{-\tilde{\psi}}$$

$$\tilde{\psi}(r) \equiv \frac{m\gamma_b\beta_b^2c^2\psi}{T} = \frac{1}{T}\left(\frac{m\gamma_b\beta_b^2c^2k_{\beta 0}^2}{2}r^2 + \frac{q\phi}{\gamma_b^2}\right)$$

and the $x$- and $y$-temperatures are equal and spatially uniform with:

$$T_x = \gamma_b m\beta_b^2c^2\frac{\int d^2x'_\perp\, x'^2\, f_\perp}{\int d^2x'_\perp\, f_\perp} = T = \text{const} \qquad T_x = T_y$$

## Scaled Poisson equation for continuous focusing thermal equilibrium

To describe the thermal equilibrium density profile, the Poisson equation must be solved. In terms of the scaled effective potential, the Poisson equation is:

- Derivation details carried out in the problem sets

$$\boxed{\begin{gathered}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\tilde{\psi}}{\partial\rho}\right) = 1 + \Delta - e^{-\tilde{\psi}} \\ \tilde{\psi}(\rho = 0) = 0 \qquad \frac{\partial\tilde{\psi}}{\partial\rho}(\rho = 0) = 0\end{gathered}}$$

Here,

$$\lambda_D = \left(\frac{\epsilon_0 T}{q^2\hat{n}}\right)^{1/2}$$ Debye length formed from the peak, on-axis beam density

$$\rho = \frac{r}{\gamma_b\lambda_D}$$ Scaled radial coordinate in rel. Debye lengths

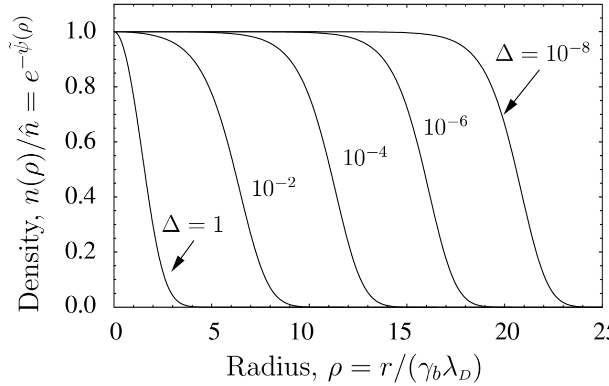
$$\hat{\omega}_p \equiv \left(\frac{q^2\hat{n}}{\epsilon_0 m}\right)^{1/2}$$ Plasma frequency formed from on-axis beam density $\longrightarrow$ $$\lambda_D = \left(\frac{T}{\hat{\omega}_p^2 m}\right)^{1/2}$$

$$\Delta = \frac{2\gamma_b^3\beta_b^2c^2k_{\beta 0}^2}{\hat{\omega}_p^2} - 1$$ Dimensionless parameter relating the ratio of applied to space-charge defocusing forces

- Equation is highly nonlinear, but can be solved (approximately) analytically
- Scaled solutions depend only on the single dimensionless parameter $\Delta$



## Numerical solution of scaled thermal equilibrium Poisson equation in terms of a normalized density




- Equation is highly nonlinear and must, in general, be solved numerically
  - Dependance on $\Delta$ is very sensitive
  - For small $\Delta$, the beam is nearly uniform in the core
- Edge fall-off is always in a few Debye lengths when $\Delta$ is small
  - Edge becomes very sharp at fixed beam line-charge



## /// Aside: Approximate Analytical Solution for the Thermal Equilibrium Density/Potential

Using the scaled density

$$N \equiv \frac{n}{\hat{n}} = e^{-\tilde{\psi}}$$

the equilibrium Poisson equation can be equivalently expressed as:

$$\boxed{\begin{gathered}\frac{\partial^2 N}{\partial\rho^2} - \frac{1}{N}\left(\frac{\partial N}{\partial\rho}\right)^2 + \frac{1}{\rho}\frac{\partial N}{\partial\rho} = N^2 - (1+\Delta)N \\ N(\rho = 0) = 1 \\ \left.\frac{\partial N}{\partial\rho}\right|_{\rho=0} = 0\end{gathered}}$$

This equation has been analyzed to construct limiting form analytical solutions for both large and small $\Delta$ [see: Startsev and Lund, PoP **15**, 043101 (2008)]

- *Large* $\Delta$ solution => warm beam => Gaussian-like radial profile
- *Small* $\Delta$ solution => cold beam => Flat core, bell shaped profile
  - Highly nonlinear structure, but approx solution has very high accuracy out to where the density becomes exponentially small!



### Large $\Delta$ solution:

$$N \simeq \exp\left[-\frac{1+\Delta}{4}\rho^2\right]$$

- Accurate for $\Delta \gtrsim 0.1$ [For full error spec. see: PoP **15**, 043101 (2008)]

### Small $\Delta$ solution:

$$N \simeq \frac{\left(1 + \frac{1}{2}\Delta + \frac{1}{24}\Delta^2\right)^2}{\left\{1 + \frac{1}{2}\Delta I_0(\rho) + \frac{1}{24}[\Delta I_0(\rho)]^2\right\}^2}$$

$I_0(x) =$ 0th order Modified Bessel Function of 1st kind

- Highly accurate for $\Delta \lesssim 0.1$ [For full error spec. see: PoP **15**, 043101 (2008)]

Special numerical methods have also been developed to calculate $N$ or $\tilde{\psi} = -\ln N$ to arbitrary accuracy for *any* value of $\Delta$, *however small*
[see: Lund, Kikuchi, and Davidson, PRSTAB, to be published, (2008) Appendices F, G]

- Extreme flatness of solution for small $\Delta \lesssim 10^{-8}$ creates numerical precision problems that require special numerical methods to address
- Method was used to verify accuracy of small $\Delta$ solution above

///

## Parameters constraints for the thermal equilibrium beam

Parameters employed in $f_\perp(H_\perp)$ to specify the equilibrium are (+ kinematic factors): $\hat{n}, \quad T, \quad \Delta$

Parameters preferred for accelerator applications:

$$k_{\beta 0}, \quad Q, \quad \varepsilon_x = \varepsilon_y = \varepsilon_b$$

Needed constraints can be calculated directly from the equilibrium:

$$\boxed{\begin{aligned} Q &= \left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) \int_0^\infty d\rho\rho\; e^{-\tilde{\psi}} \\ k_{\beta 0}^2 \varepsilon_b &= 4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right)\left[4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) + Q\right] \\ k_{\beta 0}^2 &= \left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) \frac{1+\Delta}{2(\gamma_b \lambda_D)^2} \end{aligned}}$$

Integral function of $\Delta$ only

Also useful,

$$\boxed{\begin{aligned} \varepsilon_b^2 &= 16 \frac{T}{\gamma_b m \beta_b^2 c^2} \langle x^2 \rangle_\perp = 4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) r_b^2 \\ r_b^2 &= 4\langle x^2 \rangle_\perp = \frac{1}{k_{\beta 0}^2}\left[4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) + Q\right] \end{aligned}}$$



Example of derivation steps applied to derive previous constraint equations:

Line charge:
$$\lambda = \frac{\gamma_b^2 T}{2q} \int_0^\infty d\rho\; \rho e^{-\tilde{\psi}}$$

rms edge radius:
$$r_b^2 = 4\langle x^2 \rangle_\perp = 2\gamma_b^2 \lambda_D^2 \frac{\int_0^\infty d\rho\; \rho^3 e^{-\tilde{\psi}}}{\int_0^\infty d\rho\; \rho e^{-\tilde{\psi}}}$$

rms edge emittance:
$$\varepsilon_b^2 = \varepsilon_x^3 = 16[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \cancelto{0}{\langle x x' \rangle_\perp^2}]$$
$$= 16 \frac{T}{\gamma_b m \beta_b^2 c^2} \langle x^2 \rangle_\perp = 4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) r_b^2$$

Matched envelope equation:
$$\cancelto{0}{r_b''} + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_b^2}{r_b^3} = 0$$



These constraints must, in general, be solved numerically

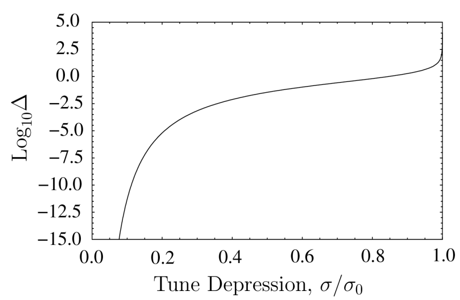
- Useful to probe system sensitivities in relevant parameters

Examples:

1) rms equivalent beam tune depression as a function of $\Delta$

$$\frac{\sigma}{\sigma_0} = \sqrt{1 - \frac{Q}{k_{\beta_0}^2 r_b^2}} = \left\{1 - \frac{[\int_0^\infty d\rho\; \rho e^{-\tilde{\psi}}]^2}{(1+\Delta)\int_0^\infty d\rho \rho^3 e^{-\tilde{\psi}}}\right\}^{1/2}$$

R.H.S function of $\Delta$ only




*rms equivalent* KV measure of $\sigma/\sigma_0$

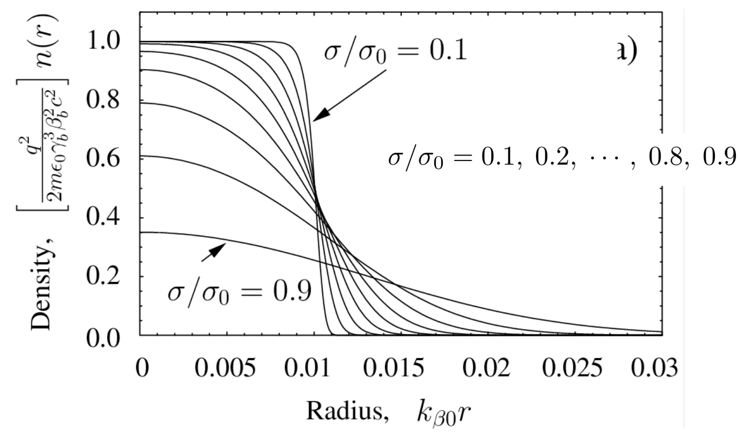
- Space-charge really nonlinear and the Thermal equilibrium has a spectrum of $\sigma$

- Small rms equivalent tune depression corresponds to *extremely* small values of $\Delta$
  - Special numerical methods generally must be employed to calculate equilibrium



2) Density profile at fixed line charge and focusing strength

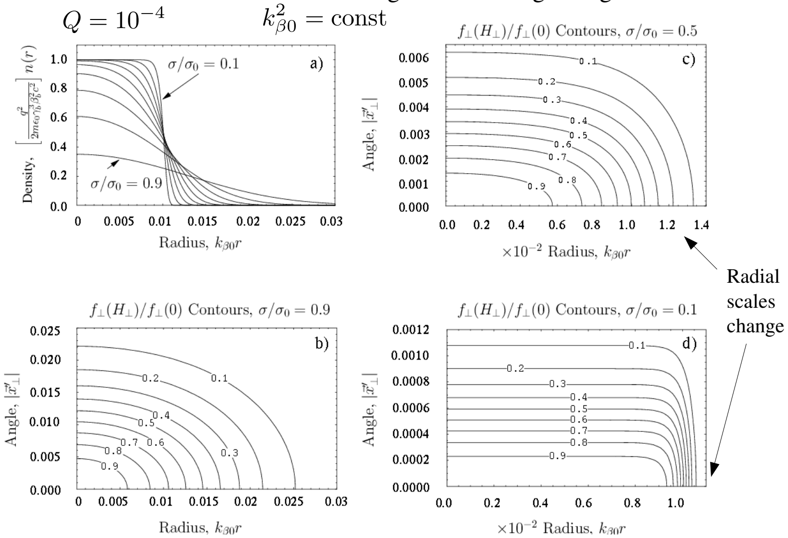
$Q = 10^{-4} \qquad k_{\beta 0}^2 = \text{const}$




- Density profile changes with scaled T
  - Low values yields a flat-top => $\sigma/\sigma_0 \to 0$
  - High values yield a Gaussian like profile => $\sigma/\sigma_0 \to 1$

3) Distribution contours at fixed line charge and focusing strength

$Q = 10^{-4}$ $\qquad k_{\beta 0}^2 = \text{const}$




- Particles will move approximately force-free till approaching the edge where it is rapidly bent back (see Debye screening analysis this lecture)



Scaled parameters for examples 2) and 3)

$Q = 10^{-4}$

| $\sigma/\sigma_0$ | $\Delta$ | $s_b$ | $k_{\beta 0}\gamma_b\lambda_D$ | $\frac{T}{m\gamma_b\beta_b^2c^2}$ | $10^3 \times k_{\beta 0}\varepsilon_b$ |
|---|---|---|---|---|---|
| 0.9 | 1.851 | 0.3508 | 12.33 | $1.065\times 10^{-4}$ | 0.4737 |
| 0.8 | $6.382\times 10^{-1}$ | 0.6104 | 6.034 | $4.444\times 10^{-5}$ | 0.2222 |
| 0.7 | $2.649\times 10^{-1}$ | 0.7906 | 3.898 | $2.402\times 10^{-5}$ | 0.1373 |
| 0.6 | $1.059\times 10^{-1}$ | 0.9043 | 2.788 | $1.406\times 10^{-5}$ | 0.09375 |
| 0.5 | $3.501\times 10^{-2}$ | 0.9662 | 2.077 | $8.333\times 10^{-6}$ | 0.06667 |
| 0.4 | $7.684\times 10^{-3}$ | 0.9924 | 1.549 | $4.762\times 10^{-6}$ | 0.04762 |
| 0.3 | $6.950\times 10^{-4}$ | 0.9993 | 1.112 | $2.473\times 10^{-6}$ | 0.03297 |
| 0.2 | $6.389\times 10^{-6}$ | 1.0000 | 0.7217 | $1.042\times 10^{-6}$ | 0.02083 |
| 0.1 | $4.975\times 10^{-12}$ | 1.0000 | 0.3553 | $2.525\times 10^{-7}$ | 0.01010 |



## Comments on continuous focusing thermal equilibria

From these results it is not surprising that the KV envelope model works well for real beams with strong space-charge (i.e, rms equivalent $\sigma/\sigma_0$ small) since the edges of a smooth thermal [and other smooth $f_\perp(H_\perp)$ distribution become sharp

- Thermal equilibrium likely overestimates the edge with since T = const, whereas a real distribution likely becomes colder near the edge

However, the beam edge contains strong nonlinear terms that will cause deviations from the KV model

- Nonlinear terms can radically change the stability properties (stabilize fictitious higher order KV modes)
- Smooth distributions for strong space-charge contain a broad spectrum of particle oscillation frequencies that are amplitude dependent which is stabilizing
  - Landau damping
  - Phase mixing
  - Less of distribution resonant with perturbations



## Frequency distribution in a thermal equilibrium beam

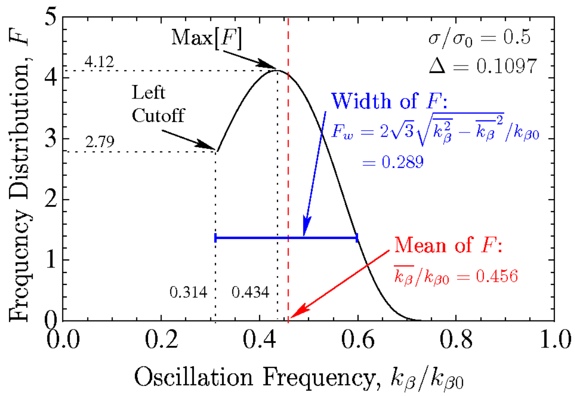
In 2D thermal equilibrium beam, frequency distribution is 2D. Orbits are closed in r and theta but not in x and y:

- Radial bounce frequency
- Azimuthal frequency

Simplified 1D (sheet beam) model developed to more simply calculate the frequency distribution in a thermal equilibrium beam to more simply illustrate the influence of space-charge in 1D

- Lund, Friedman, and Bazouin, PRSTAB **14**, 054201 (2011)
- Model shown to produce equilibria with same essential structure as higher dimensional (2D, 3D) models when appropriate "equivalent" parameters used

Result for space-charge canceling out ~ 1/2 applied focus strength




Mean: $\mu_F \equiv \overline{k_\beta}/k_{\beta0}$

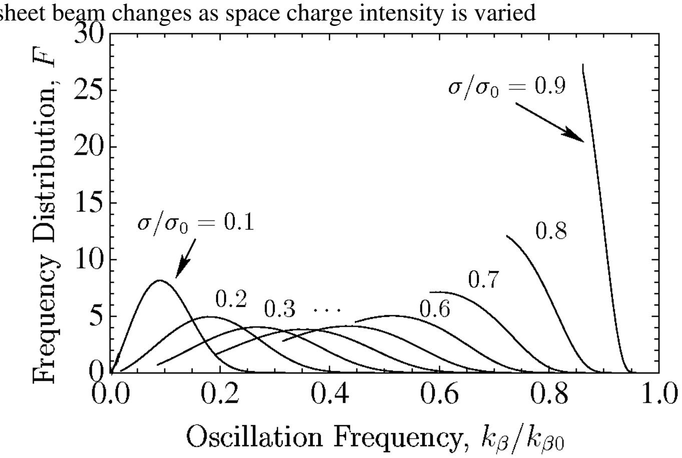
RMS: $\sigma_F \equiv \sqrt{\overline{(k_\beta - \overline{k_\beta})^2}}/k_{\beta0} = \sqrt{\overline{k_\beta^2} - \overline{k_\beta}^2}/k_{\beta0}$

Width: $F_w \equiv 2\sqrt{3}\sigma_k$

Relative Width: $F_w/\mu_F$ $\qquad \overline{\cdots} = \int_0^1 d(k_\beta/k_{\beta0}) \cdots F$



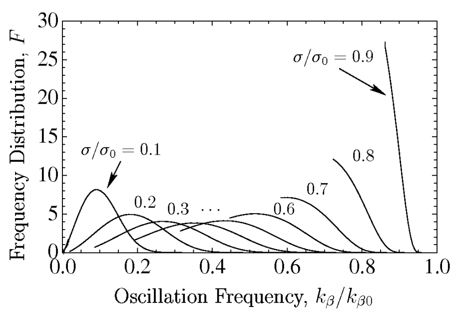
Superimposed results for values of $\sigma/\sigma_0$ show how the normalized distribution of oscillator frequencies F in the thermal equilibrium sheet beam changes as space charge intensity is varied




- Distribution becomes very broad as space-charge intensity becomes stronger!
  - KV model (single frequency) very poor
- Sharp for weak space-charge
  - KV model approximately right (single frequency shifted from applied focus)



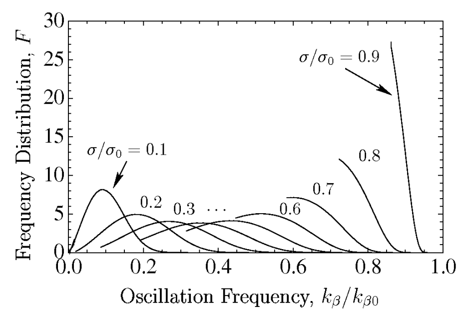
Frequency distribution, statistical measures:




| | | Statistical Measures | | | |
|---|---|---|---|---|---|
| | | Mean: | RMS: $\sigma_F =$ | Width: | Relative Width: |
| $\sigma/\sigma_0$ | $\Delta$ | $\mu_F = \overline{k_\beta}/k_{\beta0}$ | $\sqrt{\overline{k_\beta^2} - \overline{k_\beta}^2}/k_{\beta0}$ | $F_w = 2\sqrt{3}\sigma_F$ | $F_w/\mu_F$ |
| 0.9 | 2.879 | 0.886 | 0.0176 | 0.0610 | 0.0689 |
| 0.8 | 1.093 | 0.774 | 0.0354 | 0.123 | 0.159 |
| 0.7 | 0.5181 | 0.663 | 0.0531 | 0.184 | 0.277 |
| 0.6 | 0.2500 | 0.557 | 0.0696 | 0.241 | 0.433 |
| 0.5 | 0.1097 | 0.456 | 0.0833 | 0.289 | 0.634 |
| 0.4 | $3.780 \times 10^{-2}$ | 0.361 | 0.0915 | 0.317 | 0.878 |
| 0.3 | $7.562 \times 10^{-3}$ | 0.274 | 0.0898 | 0.311 | 1.14 |
| 0.2 | $3.649 \times 10^{-4}$ | 0.190 | 0.0750 | 0.260 | 1.37 |
| 0.1 | $5.522 \times 10^{-8}$ | 0.102 | 0.0465 | 0.161 | 1.58 |



Frequency distribution, extreme value measures:




| | | Extreme Measures | | | |
|---|---|---|---|---|---|
| | | At Max[$F$] | | At Left $F$ Cutoff | |
| $\sigma/\sigma_0$ | $\Delta$ | $F$ | $k_\beta/k_{\beta0}$ | $F$ | $k_\beta/k_{\beta0}$ |
| 0.9 | 2.879 | 27.3 | 0.862 | 27.3 | 0.862 |
| 0.8 | 1.093 | 12.1 | 0.723 | 12.1 | 0.723 |
| 0.7 | 0.5181 | 7.13 | 0.598 | 7.09 | 0.584 |
| 0.6 | 0.2500 | 5.03 | 0.515 | 4.47 | 0.447 |
| 0.5 | 0.1097 | 4.12 | 0.434 | 2.79 | 0.314 |
| 0.4 | $3.780 \times 10^{-2}$ | 3.83 | 0.352 | 1.58 | 0.191 |
| 0.3 | $7.562 \times 10^{-3}$ | 4.03 | 0.270 | 0.698 | 0.0866 |
| 0.2 | $3.649 \times 10^{-4}$ | 4.94 | 0.177 | 0.153 | 0.0191 |
| 0.1 | $5.522 \times 10^{-8}$ | 8.18 | 0.0912 | 0.00191 | 0.000235 |

## S8: Continuous Focusing: Debye Screening in a Thermal Equilibrium Beam

[Davidson, *Physics of Nonneutral Plasmas*, Addison Wesley (1990)]

We will show that space-charge and the applied focusing forces of the lattice conspire together to Debye screen interactions in the core of a beam with high space-charge intensity

- Will systematically derive the Debye length employed by J.J. Barnard in the Introductory Lectures
- Applied focusing forces are analogous to a stationary neutralizing species in a plasma
- 2D case is derived in class, 3D analogous will be covered in homework problem
  - Ironically, 3D case simpler to derive!

// Review:

Free-space field of a "bare" test line-charge $\lambda_t$ at the origin $r = 0$

$$\rho(r) = \lambda_t \frac{\delta(r)}{2\pi r} \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

solution (use Gauss' theorem) shows long-range interaction

$$\boxed{\begin{aligned} \phi &= -\frac{\lambda_t}{2\pi\epsilon_0}\ln(r) + \text{const} \\ E_r &= -\frac{\partial \phi}{\partial r} = \frac{\lambda_t}{2\pi\epsilon_0 r} \end{aligned}}$$

//



Place a *small* test line charge at $r = 0$ in a thermal equilibrium beam:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \underbrace{-\frac{q}{\epsilon_0}\int d^2x'_\perp\, f_\perp(H_\perp)}_{\text{Thermal Equilibrium}} \underbrace{- \frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}}_{\text{Test Line-Charge}}$$

Set:

$$\phi = \phi_0 + \delta\phi \qquad \begin{aligned} \phi_0 &= \text{ Thermal Equilibrium potential with no test line-charge} \\ \delta\phi &= \text{ Perturbed potential from test line-charge} \end{aligned}$$

Assume thermal equilibrium adapts adiabatically to the test line-charge:

$$n(r) = \int d^2x'_\perp\, f_\perp(H_\perp) = \hat{n}e^{-\tilde{\psi}} \simeq \hat{n}e^{-\tilde{\psi}_0(r)}e^{-q\delta\phi/(\gamma_b^2 T)} \qquad \left|\frac{q\delta\phi}{\gamma_b^2 T}\right| \ll 1$$

$$\simeq \hat{n}e^{-\tilde{\psi}_0(r)}\left(1 - \frac{q\delta\phi}{\gamma_b^2 T}\right) \qquad \tilde{\psi} \equiv \text{ const } \cdot r^2 + \frac{q\phi}{\gamma_b^2 T}$$

Yields:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) = -\frac{q^2}{\epsilon_0\gamma_b^2 T}\hat{n}e^{-\tilde{\psi}_0(r)}\delta\phi - \frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

Assume a relatively cold beam so the density is flat near the test line-charge:

- Should be good approximation

$$\hat{n}e^{-\tilde{\psi}_0(r)} \simeq \hat{n}$$



This gives:

$$\boxed{\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) - \frac{\delta\phi}{\gamma_b^2\lambda_D^2} = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r} \\ &\lambda_D = \left(\frac{\epsilon_0 T}{q^2\hat{n}}\right)^{1/2} = \text{ Debye radius formed from peak, on-axis beam density} \end{aligned}}$$

Derive a general solution by connecting solution very near the test charge with the general solution for r nonzero:

Near solution: $(r \rightarrow 0)$

$$\frac{\delta\phi}{\gamma_b^2\lambda_D^2} \quad \text{Negligible --->} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

The free-space solution can be immediately applied:

$$\boxed{\begin{aligned} &\delta\phi \simeq -\frac{\lambda_t}{2\pi\epsilon_0}\ln(r) + \text{const} \\ &r \rightarrow 0 \end{aligned}}$$



General Exterior Solution: $(r \neq 0)$

The delta-function term vanishes giving: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) - \frac{\delta\phi}{\gamma_b^2\lambda_D^2} = 0$ rescale r:

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \delta\phi}{\partial \rho}\right) - \delta\phi = 0 \qquad \rho \equiv \frac{r}{\gamma_b\lambda_D}}$$

This is a modified Bessel equation of order 0 with general solution:

$$\boxed{\delta\phi = C_1 I_0(\rho) + C_2 K_0(\rho) \qquad \begin{aligned} I_0(x) &= \text{ Modified Bessel Func, 1}^{\text{st}}\text{ kind} \\ K_0(x) &= \text{ Modified Bessel Func, 2}^{\text{nd}}\text{ kind} \\ C_1,\quad C_2 &= \text{ constants} \end{aligned}}$$

Connection and General Solution:

Use limiting forms:

$\rho \ll 1$

$$I_0(\rho) \rightarrow 1 + \Theta(\rho^2)$$
$$K_0(\rho) \rightarrow -[\ln(\rho/2) + 0.5772\cdots + \Theta(\rho^2)]$$

$\rho \gg 1$

$$I_0(\rho) \rightarrow \frac{e^\rho}{\sqrt{2\pi\rho}}[1 + \Theta(1/\rho)]$$
$$K_0(\rho) \rightarrow \sqrt{\frac{\pi}{2\rho}}[1 + \Theta(1/\rho)]$$

Comparison shows that we must choose for connection to the near solution and regularity at infinity:

$$\boxed{\begin{aligned} C_1 &= 0 \\ C_2 &= \frac{\lambda_t}{2\pi\epsilon_0} \end{aligned}}$$

General solution shows Debye screening of test charge in the core of the beam:

$$\delta\phi = \frac{\lambda_t}{2\pi\epsilon_0} K_0\left(\frac{r}{\gamma_b \lambda_D}\right) \qquad K_0(x) \quad \text{Order Zero Modified Bessel Function}$$

$$\simeq \frac{\lambda_t}{2\sqrt{2\pi}\epsilon_0} \frac{1}{\sqrt{r/(\gamma_b\lambda_D)}} e^{-r/(\gamma_b\lambda_D)} \qquad r \gg \gamma_b \lambda_D$$

- Screened interaction does not require overall charge neutrality!
  - Beam particles redistribute to screen bare interaction
  - Beam behaves as a plasma and expect similar collective waves etc.
- Same result for all smooth thermal equilibrium distributions and in 1D, 2D, and 3D
  - Reason why lower dimension models can get the "right" answer for collective interactions in spite of the Coulomb force varying with dimension
  - See table on next slide and Homework problem for 3D (easier than 2D case!)
- Explains why the radial density profile in the core of space-charge dominated beams are expected to be flat: space-charge cancels (linear) applied focus out to charge limit



## Debye screened potential for a test charge inserted in a thermal equilibrium beam essentially the same in 1D, 2D, and 3D

Test Charge:

1D:
Sheet Charge Density: $\Sigma_t$

2D:
Line Charge Density: $\lambda_t$

3D: (physical case)
Point Charge: $q_t$

All Cases:

$$\lambda_D = \left(\frac{\epsilon_0 T}{q^2 \hat{n}}\right)^{1/2}$$

| Dimension | Distance Measure | Test Charge Density $\rho =$ | Screened Potential $\delta\phi \simeq$ |
|---|---|---|---|
| 1D | $\lvert x \rvert$ | $\Sigma_t \delta(x)$ | $\frac{\gamma_b \lambda_D \Sigma_t}{2\epsilon_0} e^{-\lvert x \rvert/(\gamma_b\lambda_D)}$ |
| 2D | $r = \sqrt{x^2+y^2}$ | $\lambda_t \frac{\delta(r)}{2\pi r}$ | $\frac{\lambda_t}{2\sqrt{2\pi}\epsilon_0} \frac{1}{\sqrt{r/(\gamma_b\lambda_D)}} e^{-r/(\gamma_b\lambda_D)}, \quad r \gg \gamma_b\lambda_D$ |
| 3D | $r = \sqrt{x^2+y^2+z^2}$ | $q_t \delta(x)\delta(y)\delta(z)$ | $\frac{q_t}{4\pi\epsilon_0 r} e^{-r/(\gamma_b\lambda_D)}$ |

References for Calculation:

1D: Lund, Friedman, Bazouin, PRSTAB **14**, 054201 (2011)
2D: These Lectures
3D: Davidson, *Theory of Nonneutral Plasmas*, Addison-Wesley 1989



## S9: Continuous Focusing: The Density Inversion Theorem

Shows that in an equilibrium distribution the x and x' dependencies are strongly connected due to the form of $f_\perp(H_\perp)$ and Poisson's equation

For:

$$f_\perp = f_\perp(H_\perp) \qquad H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$= \frac{1}{2}\mathbf{x}_\perp'^2 + \psi(r) \qquad \psi \equiv \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

calculate the beam density

$$n(r) = \int d^2x_\perp' \; f_\perp(H_\perp) = 2\pi \int_0^\infty dU \; f_\perp(U + \psi(r)) \qquad U \equiv \frac{1}{2}\mathbf{x}_\perp'^2 \qquad H_\perp = U + \psi$$

differentiate:

$$\frac{\partial n}{\partial \psi} = 2\pi \int_0^\infty dU \; \frac{\partial}{\partial \psi} f_\perp(U+\psi) = 2\pi \int_0^\infty dU \; \frac{\partial}{\partial U} f_\perp(U+\psi) \qquad \frac{\partial f_\perp}{\partial H_\perp} = \frac{\partial f_\perp}{\partial U} = \frac{\partial f_\perp}{\psi}$$

$$= 2\pi \lim_{U\to\infty} \underbrace{f_\perp(U+\psi)}_{0} - 2\pi f_\perp(U+\psi)|_{U=0} = -2\pi f_\perp(\psi)$$

bounded distribution

$$\Longrightarrow \quad \boxed{f_\perp(H_\perp) = -\frac{1}{2\pi} \frac{\partial n}{\partial \psi}\bigg|_{\psi = H_\perp}}$$



Assume that $n(r)$ is specified, then the Poisson equation can be integrated:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) = -\frac{q n(r)}{\epsilon_0}$$

Giving

$$\phi(r) = \phi(r=0) - \frac{q}{\epsilon_0} \int_0^r \frac{d\tilde{r}}{\tilde{r}} \int_0^{\tilde{r}} d\tilde{\tilde{r}} \; \tilde{\tilde{r}} \; n(\tilde{\tilde{r}})$$

Calculate the effective potential: $\psi(r) = \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi(r)}{m\gamma_b^3\beta_b^2c^2}$

$$\boxed{\psi(r) - \frac{q\phi(r=0)}{m\gamma_b^3\beta_b^2c^2} = \frac{1}{2}k_{\beta 0}^2 r^2 - \frac{q}{m\gamma_b^3\beta_b^2c^2\epsilon_0} \int_0^r \frac{d\tilde{r}}{\tilde{r}} \int_0^{\tilde{r}} d\tilde{\tilde{r}} \; \tilde{\tilde{r}} \; n(\tilde{\tilde{r}})}$$

For $n(r)$ = const $\quad \int_0^r \frac{d\tilde{r}}{\tilde{r}} \int_0^{\tilde{r}} d\tilde{\tilde{r}} \; \tilde{\tilde{r}} \; n(\tilde{\tilde{r}}) \quad \propto r^2$

This suggests that $\psi(r)$ is monotonic in $r$ when $d\, n(r)/dr$ is monotonic. Apply the chain rule:

**Density Inversion Theorem**

$$\boxed{\begin{aligned} f_\perp(H_\perp) &= -\frac{1}{2\pi}\frac{\partial n}{\partial \psi}\bigg|_{\psi=H_\perp} = -\frac{1}{2\pi}\frac{\partial n(r)/\partial r}{\partial \psi(r)/\partial r}\bigg|_{\psi=H_\perp} \\ \psi(r) &= \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \end{aligned}}$$

For specified monotonic $n(r)$ the density inversion theorem can be applied with the Poisson equation to calculate the corresponding equilibrium $f_\perp(H_\perp)$

Comments on density inversion theorem:

- Shows that the $x$ and $x'$ dependence of the distribution are *inextricably linked* for an equilibrium distribution function $f_\perp(H_\perp)$
  - Not so surprising -- equilibria are highly constrained
- If $df_\perp(H_\perp)/dH_\perp \leq 0$ then the kinetic stability theorem (see: S.M. Lund, lectures on Transverse Kinetic Stability) shows that the equilibrium generated is also stable to small amplitude perturbations (this generalizes to nonlinear stability)
- The beam density profile $n(r)$ can be measured in the lab using several methods, but full 4D $x,y$ $x',y'$ phase-space is typically more difficult to measure. But insofar as the beam is near equilibrium form, the inversion theorem can be applied to infer the full distribution phase-space from measurement of the beam density profile.

  Real beams have s-varying focusing – but canonoical transforms can be applied for variables that appear closer to continuous focusing to allow approximate use of methodology developed here.



// Example: Application of the inversion theorem to the KV equilibrium

$$n = \begin{cases} \hat{n}, & 0 \leq r < r_b \\ 0, & r_b < r \end{cases} \quad \Longrightarrow \quad \frac{\partial n}{\partial r} = -\hat{n}\delta(r - r_b)$$

$$\begin{aligned} \frac{\partial n}{\partial \psi} &= \frac{\partial n/\partial r}{\partial \psi/\partial r} \\ &= -\frac{\hat{n}\delta(r-r_b)}{\partial\psi/\partial r} \\ &= -\frac{\hat{n}\delta(r-r_b)}{\partial\psi/\partial r|_{r=r_b}} \\ &= -\hat{n}\delta(\psi(r) - \psi(r_b)) \end{aligned}$$

property of delta-function:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x=x_i}}$$

$$f(x_i) = 0$$

$x_i$ is root of $f$

use: $\psi(r_b) = H_\perp|_{\mathbf{x}'_\perp=0} = H_{\perp b}$

$$\Longrightarrow \quad \boxed{f_\perp(H_\perp) = -\frac{1}{2\pi}\frac{\partial n}{\partial \psi}\bigg|_{\psi=H_\perp} = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})}$$

Expected KV form

//

Steps in this example can be used to "derive" the delta-function form required for the elliptical beam KV distribution in the more general elliptical beam case:

- Use canonical transforms (Appendix B) to express elliptical beam in axisymmetric form
- Apply inversion theorem as outlined above in transformed variables
- Transform back to regular variables to obtain KV distribution for an elliptical beam

These steps also imply that the KV form is unique



## S10: Comments on the Plausibility of Smooth Vlasov Equilibria in Periodic Transport Channels

### S10A: Introduction

The KV and continuous models are the only (or related to simple transforms thereof) known exact beam equilibria. Both suffer from idealizations that render them inappropriate for use as initial distribution functions for detailed modeling of stability in real accelerator systems:

- KV distribution has an unphysical singular structure giving rise to collective instabilities with unphysical manifestations
  - Low order properties (envelope and some features of low-order plasma modes) are physical and very useful in machine design
- Continuous focusing is inadequate to model real accelerator lattices with periodic or $s$-varying focusing forces
  - Focusing force cannot be realized (massive partially neutralizing background charge)
  - Kicked oscillator intrinsically different than a continuous oscillator

There is much room for improvement in this area, including study if smooth equilibria exist in periodic focusing and implications if no exact equilibria exist.



If exact smooth "equilibrium" beam distributions exist for periodic focusing, they are highly nontrivial.

Would a nonexistence of an equilibrium distribution be a problem?

- Real beams are born off a source that can be simulated
  - Propagation length can be relatively small in linacs
- Transverse confinement can exist without an equilibrium
  - Particles can turn at large enough radii forming an edge
  - Edge can oscillate from lattice period to lattice period without pumping to large excursions

$\longrightarrow$ Might not preclude long propagation with preserved statistical beam quality

Even approximate equilibria would help sort out complicated processes:

- Reduce transients and fluctuations can help understand processes in simplest form
  - Allows more "plasma physics" type analysis and advances
- Beams in Vlasov simulations are often observed to "settle down" to a fairly regular state after an initial transient evolution
  - Phase mixing can rapidly lead to an effective relaxation

Recent progress has been made in better understanding whether smooth equilibria exist in periodic focusing lattices. Results suggest that they are at least classes of distributions that are very near equilibria:

- M. Dorf *et. al*: Carried out systematic simulations adiabatically changing continuous focusing to periodic quadrupole at low $\sigma_0$ and find nearly self-similar periodic beams with small residual oscillations
  Dorf, Davidson, Startsev, Qin, Phys. Plasmas **16**, 123107 (2009)
- S. Lund *et. al*: Guessed a primitive construction taking continuous focusing distributions and applying KV canonical transforms to better match to periodic focusing. Procedure implemented in WARP code and shown to produce excellent results up to near stability limits in $\sigma_0$
  Lund, Kikuchi, Davidson, PRSTAB **12**, 114801 (2009)
- E. Startsev *et. al*: Developed systematic Hamiltonian averaged perturbation theories showing near equilibrium structure for low $\sigma_0$
  Startsev, Davidson, Dorf, PRSTAB **13**, 064402 (2010) + Extension papers
- K. Sonnad *et. al*: Developed a canonical transform theory including space-charge which promises increased insight with a high degree of flexibility
  K. Sonnad and J. Cary, PRE **69**, 056501(2004)
  K. Sonnad and J. Cary, Physics of Plasmas **22** 043120 (2015)

Details of perturbative theories beyond scope of class: Much remains to be done!



## S10B: Simple Approximate Pseudo-Equilibrium Distributions to Model a Smooth Equilibrium

Lund, Kikuchi, Davidson, PRSTAB **12**, 114801 (2009)

Simple "pseudo-equilibrium" initial distribution to represent an intense beam:

1) Use rms equivalent measures to specify the beam
   - Natural set of parameters for accelerator applications
2) Map rms equivalent beam to a smooth, continuous focused matched beam
   - Use smooth core models that are stable in continuous focusing:
     - Waterbag Equilibrium
     - Parabolic Equilibrium
     - Thermal Equilibrium
     - ⋮

     See: S5, S6, S7
3) Transform continuous focused beam for rms equivalency with initial spec
   - Use KV transforms that preserve uniform beam Courant-Snyder invariants

Procedure applies to any s-varying focusing channel

- Focusing channel need not be periodic
- Beam can be initially rms equivalent matched or mismatched if launched in a periodic transport channel
- Can apply to both 2D transverse and 3D beams



## 4-Step Procedure for Initial Distribution Specification

Assume focusing lattice is given:

$\boxed{\kappa_x(s), \quad \kappa_y(s) \quad \text{specified}}$

Strength usually set by specifying undepressed phase advances $\sigma_{0x}, \quad \sigma_{0y}$

Step 1:
For each particle (3D) or slice (2D) specify 2[nd] order rms properties at axial coordinate *s*

Envelope coordinates/angles: (specify beam envelope)

$$\boxed{\begin{aligned} r_x(s) &= 2\langle x^2\rangle_\perp^{1/2} & r_x'(s) &= 2\langle xx'\rangle_\perp/\langle x^2\rangle_\perp^{1/2} \\ r_y(s) &= 2\langle y^2\rangle_\perp^{1/2} & r_y'(s) &= 2\langle yy'\rangle_\perp/\langle y^2\rangle_\perp^{1/2} \end{aligned}}$$

RMS Emittances: (specify phase-space area)

$$\boxed{\begin{aligned} \varepsilon_x(s) &= 4[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2} \\ \varepsilon_y(s) &= 4[\langle y^2\rangle_\perp\langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2]^{1/2} \end{aligned}}$$

Perveance: (specify space-charge intensity)

$$\boxed{Q = \frac{q\lambda(s)}{2\pi\epsilon_0 m\gamma_b^3(s)\beta_b^2(s)c^2}}$$



## Procedure for Initial Distribution Specification (2)

If the beam is rms matched, we take:

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

$$\kappa_x(s + L_p) = \kappa_x(s)$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

$$r_x(s + L_p) = r_x(s)$$
$$r_y(s + L_p) = r_y(s)$$

- Not necessary to match even for periodic lattices
  - Procedure applies to mismatched beams

## Procedure for Initial Distribution Specification (3)

Step 2:

Define an rms matched, continuously focused beam in each transverse $s$-slice:

| Continuous | s-Varying | |
|---|---|---|
| $r_b(s) =$ | $\sqrt{r_x(s) r_y(s)}$ | Envelope Radius |
| $\varepsilon_b(s) =$ | $\sqrt{\varepsilon_x(s)\varepsilon_y(s)}$ | Emittance |
| $Q(s) =$ | $Q(s)$ | Perveance |

Define a (local) matched beam focusing strength in continuous focusing consistent with the rms beam envelope:

$$\cancelto{0}{r_b''} + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_b^2}{r_b^2} = 0$$

$$\boxed{k_{\beta 0}^2(s) = \frac{Q(s)}{r_b^2(s)} + \frac{\varepsilon_b^2(s)}{r_b^4(s)}}$$



## Procedure for Initial Distribution Specification (4)

Step 3:

Specify an rms matched continuously focused equilibrium consistent with step 2:

Specify an equilibrium function:

$$\boxed{f_\perp(x, y, x', y') = f_\perp(H_\perp)} \qquad H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3 \beta_b^2 c^2}$$

and constrain parameters used to define the equilibrium function $f_\perp(H_\perp)$ with:

$$\lambda = q\int d^2x \int d^2x' \, f_\perp(H_\perp) \qquad \text{Line Charge <--> Perveance}$$

$$r_b^2 = \frac{4\int d^2x \int d^2x' \, x^2 f_\perp(H_\perp)}{\int d^2x \int d^2x' \, f_\perp(H_\perp)} \qquad \text{rms edge radius}$$

$$\frac{\varepsilon_b^2}{r_b^2} = \frac{4\int d^2x \int d^2x' \, x'^2 f_\perp(H_\perp)}{\int d^2x \int d^2x' \, f_\perp(H_\perp)} \qquad \text{rms edge emittance}$$

- This can be rms equivalence with a *smooth* distribution NOT a KV distribution!
- Constraint equations are generally highly nonlinear and must be solved numerically
  - Allows specification of beam with natural accelerations variables
  - Procedures to implement this can be involved (research problem)



## Procedure for Initial Distribution Specification (5)

Step 4:

Transform the continuous focused beam coordinates to rms equivalency in the system with $s$-varying focusing:

$$x = \frac{r_x}{r_b} x_i \qquad y = \frac{r_y}{r_b} y_i$$

$$x' = \frac{\varepsilon_x}{\varepsilon_b}\frac{r_b}{r_x} x_i' + \frac{r_x'}{r_b} x_i \qquad y' = \frac{\varepsilon_y}{\varepsilon_b}\frac{r_b}{r_y} y_i' + \frac{r_y'}{r_b} y_i$$

Here, $\{x_i\}$, $\{y_i\}$, $\{x_i'\}$, $\{y_i'\}$ are coordinates of the continuous equilibrium

- Transform reflects structure of linear field Courant-Snyder invariants but applied to the nonuniform beam
  - Approximation effectively treats Hamiltonian as Courant-Snyder invariant
  - Properties of beam nonuniform distribution retained in transform
  - Expect errors to be largest near beam radial "edge" at high space-charge intensity
- If applied to simulations using macroparticles (e.g., PIC codes), then details of transforms must be derived to weight macroparticles
  - Details in: Lund, Kikuchi, Davidson, PRSTAB **12**, 114801 (2009)



## Procedure for Initial Distribution Specification (6)

Load N particles in x,y,x',y' phase space consistent with continuous focusing equilibrium distribution $f_\perp(H_\perp)$

Step A (set particle coordinates):

Calculate beam radial number density $n(r)$ by (generally numerically) solving the Poisson/stream equation and load particle $x,y$ coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

- Radial coordinates $r$: Set by transforming uniform deviates consistent with $n(r)$
- Azimuthal angles $\theta$: Distribute randomly or space for low noise

Step B (set particle angles):

Evaluate $f_\perp(U, r)$ with $U = \sqrt{x'^2 + y'^2}$ at the particle $x$, $y$ coordinates loaded in step A to calculate the angle probability distribution function and load $x'$, $y'$ coordinates:

$$x' = U\cos\xi$$

$$y' = U\sin\xi$$

- Radial coordinate $U$: Set by transforming uniform deviates consistent with $f_\perp(U, r)$
- Azimuthal coordinate $\xi$: Distribute randomly or space for low noise

## Procedure for Initial Distribution Specification (7)

Step 4:

Transform continuous focused beam coordinates to rms equivalency in the system with $s$-varying focusing:

$$x = \frac{r_x}{r_b} x_i \qquad\qquad y = \frac{r_y}{r_b} y_i$$

$$x' = \frac{\varepsilon_x}{\varepsilon_b}\frac{r_b}{r_x} x_i' + \frac{r_x'}{r_b} x_i \qquad\qquad y' = \frac{\varepsilon_y}{\varepsilon_b}\frac{r_b}{r_y} y_i' + \frac{r_y'}{r_b} y_i$$

Here, $\{x_i\}, \{y_i\}, \{x_i'\}, \{y_i'\}$ are coordinates of the continuous equilibrium loaded

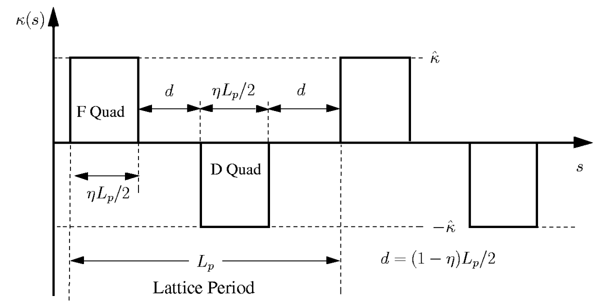
- Transform reflects structure of Courant-Snyder invariants



## Carry out numerical Vlasov simulations of the initial Pseudoequlibrium distributions to check how procedure works

Use the Warp (PIC) Vlasov code to advance an initial pseudoequilibrium distribution in a periodic FODO lattice to check how significant transient evolutions are period by period:

- Little evolution => suggests near relaxed equilibrium structure




$\sigma_0 =$ specified, $\quad L_p = 0.5$ m, $\quad \varepsilon_x = \varepsilon_y = 50$ mm-mrad

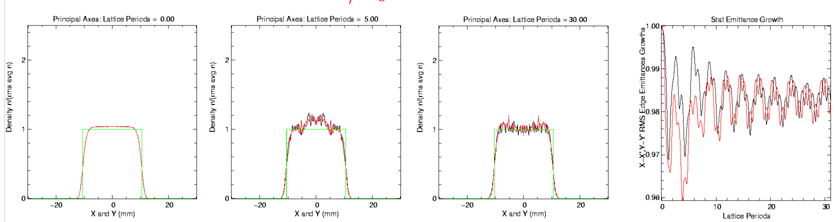
$\sigma/\sigma_0$ adjusted to fix $Q$



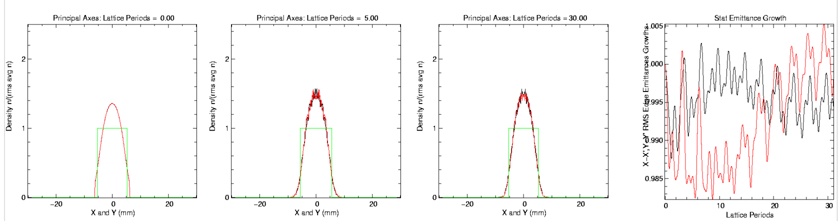
## Warp PIC Simulation – Pseudo Thermal Equilibrium

$\sigma_0 = 70^\circ, \quad L_p = 0.5$ m, $\quad \varepsilon_x = \varepsilon_y = 50$ mm-mrad

$\sigma/\sigma_0 = 0.2$




$\sigma/\sigma_0 = 0.7$



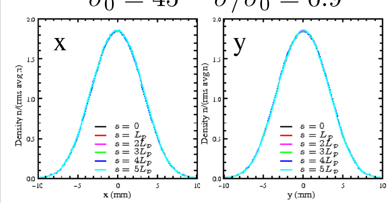



## Transient evolution of initial pseudo-equilibrium distributions with thermal core form in a FODO quadrupole focusing lattice

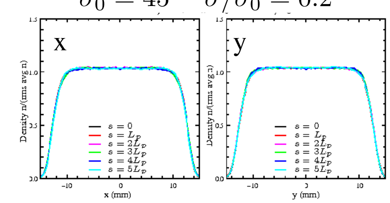
Density profiles along x and y axes

Snapshots at lattice period intervals over 5 periods

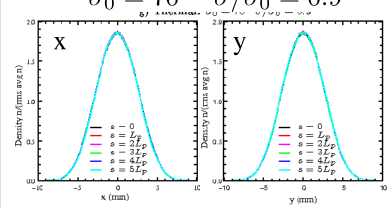
$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$



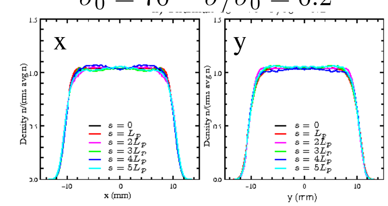

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$




$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$



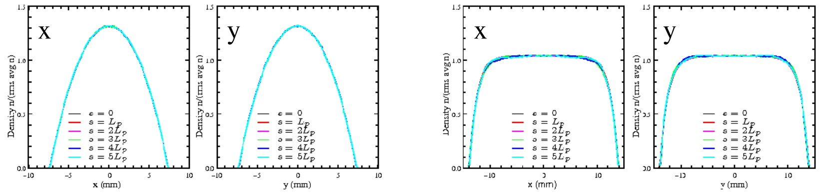

$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$



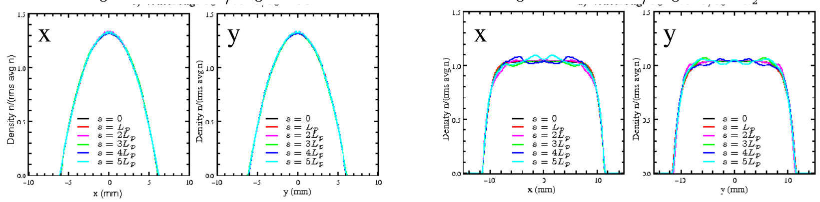
## Transient evolution of initial pseudo-equilibrium distributions with waterbag core form in a FODO quadrupole focusing lattice

Density profiles along x and y axes
Snapshots at lattice period intervals over 5 periods

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$ $\qquad\qquad$ $\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$




$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$ $\qquad\qquad$ $\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$



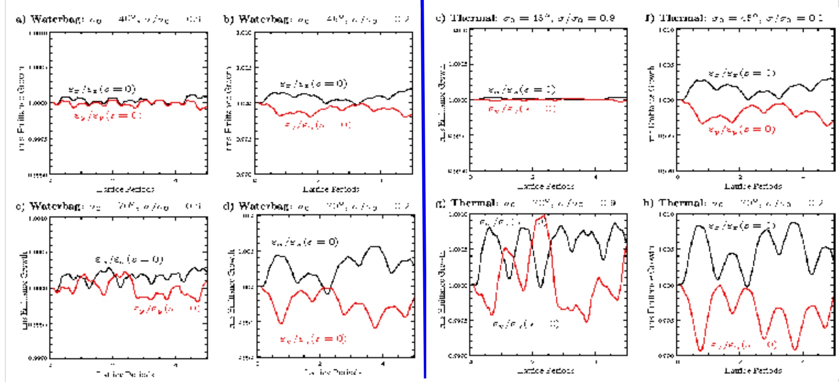



## The beam phase-space area (rms emittance measure) changes little during the evolutions indicating near equilibrium form

$$\varepsilon_x = 4\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right]^{1/2} \qquad \text{Plot: } \varepsilon_x(s)/\varepsilon_x(s_i)$$

$$\varepsilon_y = 4\left[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2\right]^{1/2} \qquad \text{Plot: } \varepsilon_y(s)/\varepsilon_y(s_i)$$

Waterbag Form $\qquad$ Gaussian/Thermal Form



Max Ranges 0.1 % Variation!!

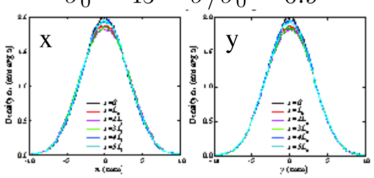


## Compare pseudo-equilibrium loads with other accelerator loads

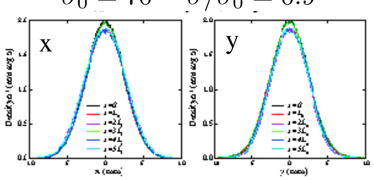
Comparison distribution from linear-field Courant-Snyder invariants
Batygin, Nuc. Inst. Meth. A **539**, 455 (2005)
Thermal/Gaussian forms with weak space-charge

Linear-Field Courant-Snyder:

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$

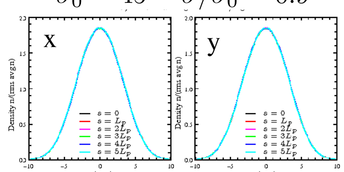



$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$

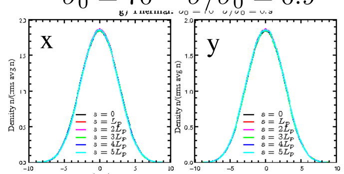



Pseudo-Equilibrium

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$




$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$



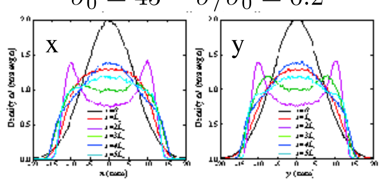



## Compare pseudo-equilibrium loads with other accelerator loads

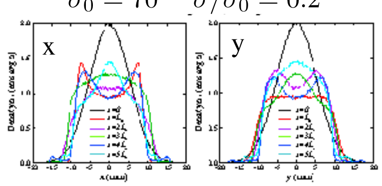
Comparison distribution from linear-field Courant-Snyder invariants
Batygin, Nuc. Inst. Meth. A **539**, 455 (2005)
Thermal/Gaussian forms with strong space-charge

Linear-Field Courant-Snyder:

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$

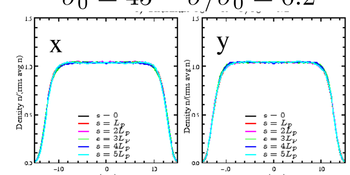



$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$

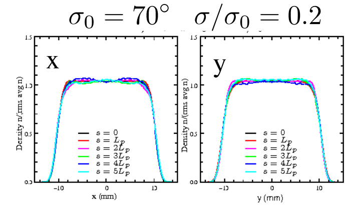



Pseudo-Equilibrium

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$




$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$



## Summary: Results suggest near equilibrium structure with good quiescent transport can be obtained for a broad range of beam parameters with a smooth distribution core loaded using the pseudoequilibrium construction

Find:

- Works well for quadrupole transport for $\sigma_0 \lesssim 85^\circ$
  - Should not work where beam is unstable and all distributions are expected to become unstable for $\sigma_0 >\sim 85^\circ$ see lectures on Transverse Kinetic Stability:

    Experiment: Tiefenback, Ph.D. Thesis, U.C. Berkeley (1986)
    Theory: Lund and Chawla, Proc. 2005 Part. Accel. Conf.

- Works better when matched envelope has less "flutter":
  - Solenoids: larger lattice occupancy $\eta$
  - Quadrupoles: smaller $\sigma_0$
  - Not surprising since less flutter" corresponds to being closer to continuous focusing



## Comments on Procedure for Initial Distribution Specification

- Applies to both 2D transverse and 3D beams
- Easy to generalize procedure for beams with centroid offsets
- Generates a charge distribution with elliptical symmetry
  - Sacherer's results on rms equivalency apply
  - Distribution will reflect self-consistent Debye screening
- Equilibria are only pseudo-equilibria since transforms are not exact
  - Nonuniform space-charge results in errors
  - Transform consistent with preserved Courant-Snyder invariants for uniform density beams
  - Errors largest near the beam edge - expect only small errors for very strong space charge where Debye screening leads to a flat density profile with rapid fall-off at beam edge
- Many researchers have presented or employed aspects of the improved loading prescription presented here, including:

  I. Hofmann, GSI    M. Reiser, U. Maryland    M. Ikigami, KEK
  E. Startsev, PPPL    Y. Batygin, SLAC



## Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2017

Redistributions of class material welcome. Please do not remove author credits.



## References: For more information see:

These course notes are posted with updates, corrections, and supplemental material at:
https://people.nscl.msu.edu/~lund/uspas/bpisc_2017

Materials associated with previous and related versions of this course are archived at:

JJ Barnard and SM Lund, *Beam Physics with Intense Space-Charge,* USPAS:
https://people.nscl.msu.edu/~lund/uspas/bpisc_2015 2015 Version
http://hifweb.lbl.gov/USPAS_2011 2011 Lecture Notes + Info
http://uspas.fnal.gov/programs/past-programs.shtml (2008, 2006, 2004)

JJ Barnard and SM Lund, *Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields*, UC Berkeley, Nuclear Engineering NE290H
http://hifweb.lbl.gov/NE290H 2009 Lecture Notes + Info

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S. Lund, T. Kikuchi, and R. Davidson, Review Article: "Generation of initial kinetic distributions for simulation of long-pulse charged particle beams with high space-charge intensity," PRSTAB **12**, 114801 (2009)

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S. Lund, A. Friedman, and G. Bazouin, "Sheet beam model for intense space charge: Application to Debye screening and the distribution of particle oscillation frequencies in a thermal equilibrium beam," PRSTAB **14**, 054201 (2011)




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