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USPAS
June 12-23, 2017
Lisle, Illinois

Injectors and longitudinal physics -- I

1. Fluid equations
2. Child-Langmuir Law
   (Reiser 2.5.2, Appendix 1)
3. Pierce electrodes
4. Transients in injectors
5. Injector choices

I) FLUID EQUATIONS

START WITH VLASOV EQUATION FOR $f(\underline{x}, \underline{p}, t)$

$$\frac{\partial f}{\partial t}(\underline{x}, \underline{p}, t) + \dot{\underline{x}} \cdot \frac{\partial f}{\partial \underline{x}}(\underline{x}, \underline{p}, t) + \dot{\underline{p}} \cdot \frac{\partial f}{\partial \underline{p}}(\underline{x}, \underline{p}, t) = 0$$

HERE $\dot{\underline{x}} = \frac{d\underline{x}}{dt} = \frac{\underline{p}}{\gamma m}$

$$\dot{\underline{p}} = \frac{d\underline{p}}{dt} = q\left(\underline{E}(\underline{x}, t) + \frac{\underline{p}}{\gamma m} \times \underline{B}(\underline{x}, t)\right)$$

$$\gamma^2 = (p/mc)^2 + 1$$

INTEGRATE OVER MOMENTUM AND MULTIPLY BY POWERS OF $\underline{p}$

a) CONTINUITY EQUATION

$$\int d^3p \left\{ \overset{①}{\frac{\partial f}{\partial t}} + \overset{②}{\dot{\underline{x}} \cdot \frac{\partial f}{\partial \underline{x}}} + \overset{③}{\left(q\underline{E}(\underline{x}, t) + \frac{q\underline{p}}{\gamma m} \times B(\underline{x}, t)\right) \frac{\partial f}{\partial \underline{p}}} \right\} = 0$$

DEFINE $n(\underline{x}, t) = \int f(\underline{x}, \underline{p}, t)\, d^3p$

$$n(\underline{x}, t)\, \underline{\bar{v}}(\underline{x}, t) = \int \frac{\underline{p}}{\gamma m} f(\underline{x}, \underline{p}, t)\, d^3p$$

① FIRST INTEGRAL

$$\int d^3p \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int f\, d^3p = \frac{\partial n(\underline{x}, t)}{\partial t}$$

② SECOND INTEGRAL

$$\int d^3p\, \dot{\underline{x}} \cdot \frac{\partial f}{\partial \underline{x}} = \int d^3p \frac{\underline{p}}{\gamma m} \cdot \frac{\partial f}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}} \cdot \int d^3p \frac{\underline{p}}{\gamma m} f$$

$$= \frac{\partial}{\partial \underline{x}} \cdot n\underline{\bar{v}}$$

TOPS.
35500

③ THIRD INTEGRAL

$$\int d^3p \left(q\underline{E} + \frac{q}{\gamma m} \underline{p} \times \underline{B} \right) \cdot \frac{\partial f}{\partial \underline{p}}$$

$$\underbrace{q E f \Big|_{p=-\infty}^{p=+\infty}}_{=0} \qquad \int \frac{q}{\gamma m} (p_y B_z - p_z B_y) \frac{\partial f}{\partial p_x} \, dp_x dp_y dp_z + \cdots$$

$$\boxed{\begin{aligned} &\int_{-\infty}^{\infty} u v' \, dp_x = u v \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u' v \, dp_x \\ &u = \frac{q}{\gamma m} (p_y B_z - p_z B_y) \\ &u' = \frac{-q}{\gamma^2 m} (p_y B_z - p_z B_y) \frac{\partial \gamma}{\partial p_x} \\ &v \quad f \\ &v' \quad \frac{\partial f}{\partial p_x} \\ &\gamma^2 = \frac{p_x^2 + p_y^2 + p_z^2}{m^2 c^2} + 1 \\ &\Rightarrow 2\gamma \frac{\partial \gamma}{\partial p_x} = \frac{2 p_x}{m^2 c^2} \end{aligned}}$$

$$\int \Big[\frac{q}{\gamma^3 m^2 c^2} (p_y B_z - p_z B_y) p_x + \frac{q}{\gamma^3 m^3 c^2} (p_z B_x - p_x B_z) p_y + \frac{q}{\gamma^3 m^3 c^2} (p_x B_y - p_y B_x) p_z \Big]' \cdot f \, d^3p$$

$$= 0 \quad !$$

So $\displaystyle \int d^3p \left\{ \frac{\partial f}{\partial t} + \dot{\underline{x}} \cdot \frac{\partial f}{\partial \underline{x}} + \dot{\underline{p}} \cdot \frac{\partial f}{\partial \underline{p}} \right\} = 0$

$$\Rightarrow \boxed{\frac{\partial n(\underline{x},t)}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot n(\underline{x},t) \, \underline{V}(\underline{x},t) = 0}$$

CONTINUITY EQUATION

$q n(\underline{x},t) \underline{V}(\underline{x},t) = \underline{J}(\underline{x},t)$

CURRENT DENSITY

ALTERNATIVELY $\displaystyle \frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$

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b) MOMENTUM EQUATION

(FOR SIMPLICITY : ASSUME NON-RELATIVISTIC)

$$\dot{x} = \frac{\underline{p}}{m} \qquad \underline{\dot{p}} = q\left(\underline{E}(\underline{x},t) + \frac{\underline{p}}{m} \times \underline{B}(\underline{x},t)\right)$$

MUTIPLY BY $\dot{x}$ & INTEGRATE OVER MOMENTUM (d^3p)

$$\int d^3p \left\{ \overset{①}{\underline{\dot{x}}\frac{\partial f}{\partial t}} + \overset{②}{\underline{\dot{x}}\,\underline{\dot{x}}\cdot\frac{\partial f}{\partial \underline{x}}} + \underline{\dot{x}}\left(q\underline{E} + \overset{③}{\frac{\underline{p}}{m}} \times \underline{B}\right)\cdot\frac{\partial f}{\partial \underline{p}} \right\} = 0$$

DEFINE $\underline{\underline{P}} \equiv m \int d^3\underline{p}\, (\underline{\dot{x}} - \underline{\bar{V}})(\underline{\dot{x}} - \underline{\bar{V}})\, f(x,p,t)$

($\underline{\underline{P}} \equiv$ pressure tensor)

$$= m \int d^3p\, \underline{\dot{x}}\,\underline{\dot{x}}\, f - 2m\underline{\bar{V}} \int \underline{\dot{x}} f\, d^3p + m\underline{\bar{V}}\,\underline{\bar{V}} \int f\, d^3p$$

$$= m \int d^3p\, \underline{\dot{x}}\,\underline{\dot{x}}\, f - mn\underline{\bar{V}}\,\underline{\bar{V}}$$

① FIRST INTEGRAL:

$$\int d^3p\, \underline{\dot{x}}\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int d^3p\, \underline{\dot{x}} f = \frac{\partial}{\partial t} n\underline{\bar{V}}$$

② SECOND INTEGRAL:

$$\int d^3p\, \underline{\dot{x}}\,\underline{\dot{x}}\cdot\frac{\partial f}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}}\cdot\int d^3p\, \underline{\dot{x}}\,\underline{\dot{x}}\, f$$

$$= \frac{1}{m}\frac{\partial}{\partial \underline{x}}\cdot\underline{\underline{P}} + \frac{\partial}{\partial \underline{x}}\cdot n\underline{\bar{V}}\,\underline{\bar{V}}$$

$$= \frac{1}{m}\frac{\partial}{\partial \underline{x}}\cdot\underline{\underline{P}} + \left(\frac{\partial}{\partial \underline{x}}\cdot n\underline{v}\right)\underline{V} + n\underline{v}\cdot\frac{\partial \underline{v}}{\partial \underline{x}}$$

③. THIRD INTEGRAL:

$$\int d^3p\, \frac{\underline{p}}{m}\left(q\underline{E} + q\frac{\underline{p}}{m}\times\underline{B}\right)\cdot\frac{\partial f}{\partial \underline{p}}$$

$$= \underbrace{f\,\frac{p_x}{m}\left(qE + \frac{q\underline{p}}{m}\times B\right)\Big|_{-\infty}^{\infty}}_{=0} - \int \frac{d^3\underline{p}}{m}\left(q\underline{E} + \frac{q\underline{p}}{m}\times\underline{B}\right)f$$

$$= -\frac{n(x,t)}{m}\left(q\underline{E} + q\underline{\bar{V}}\times\underline{B}\right)$$

$u \quad \frac{p_x}{m}\left(q\underline{E} + \frac{\underline{p}}{m}\times B\right)$

$u' \quad \frac{1}{m}\left(q\underline{E} + \frac{\underline{p}}{m}\times B\right)$

$v \quad f$

$v' \quad \frac{\partial f}{\partial p_x}$

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b) MOMENTUM EQUATION

(FOR SIMPLICITY : ASSUME NON-RELATIVISTIC)

$$\dot{x} = \frac{\underline{p}}{m} \qquad \underline{\dot{p}} = q\left(\underline{E}(x,t) + \frac{\underline{p}}{m} \times \underline{B}(\underline{x},t)\right)$$

MULTIPLY BY $\dot{x}$ & INTEGRATE OVER MOMENTUM (d^3p)

$$\int d^3p \left\{ \overset{①}{\underline{\dot{x}} \frac{\partial f}{\partial t}} + \overset{②}{\underline{\dot{x}}\,\underline{\dot{x}} \cdot \frac{\partial f}{\partial \underline{x}}} + \underline{\dot{x}}\left(q\underline{E} + \overset{③}{\frac{\underline{p}}{m}} \times \underline{B}\right) \cdot \frac{\partial f}{\partial \underline{p}} \right\} = 0$$

DEFINE $\underline{\underline{P}} \equiv m \int d^3\underline{p}\, (\underline{\dot{x}} - \underline{\bar{V}})(\underline{\dot{x}} - \underline{\bar{V}})\, f(x,p,t)$

($\underline{\underline{P}}$ ≡ pressure tensor)

$$= m \int d^3p\, \underline{\dot{x}}\,\underline{\dot{x}}\, f - 2m\underline{\bar{V}} \int \underline{\dot{x}}\, f\, d^3p + m\underline{\bar{V}}\,\underline{\bar{V}} \int f\, d^3p$$

$$= m \int d^3p\, \underline{\dot{x}}\,\underline{\dot{x}}\, f - mn\underline{\bar{V}}\,\underline{\bar{V}}$$

① FIRST INTEGRAL:

$$\int d^3p\, \underline{\dot{x}} \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int d^3p\, \underline{\dot{x}}\, f = \frac{\partial}{\partial t} n\underline{\bar{V}}$$

② SECOND INTEGRAL:

$$\int d^3p\, \underline{\dot{x}}\,\underline{\dot{x}} \cdot \frac{\partial f}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}} \cdot \int d^3p\, \underline{\dot{x}}\,\underline{\dot{x}}\, f$$

$$= \frac{1}{m} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}} + \frac{\partial}{\partial \underline{x}} \cdot n\underline{\bar{V}}\,\underline{\bar{V}}$$

$$= \frac{1}{m} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}} + \left(\frac{\partial}{\partial \underline{x}} \cdot n\underline{v}\right)\underline{V} + n\underline{v} \cdot \frac{\partial \underline{v}}{\partial \underline{x}}$$

③. THIRD INTEGRAL:

$$\int d^3p\, \frac{\underline{p}}{m}\left(q\underline{E} + q\frac{\underline{p}}{m} \times \underline{B}\right) \cdot \frac{\partial f}{\partial \underline{p}}$$

$$= \underbrace{f\, \frac{p_x}{m}\left(qE + q\frac{p}{m} \times B\right)\Big|_{-\infty}^{\infty}}_{=0} - \int \frac{d^3\underline{p}}{m}\left(q\underline{E} + \frac{q\underline{p}}{m} \times \underline{B}\right) f$$

$$= -\frac{n(x,t)}{m}\left(q\underline{E} + q\underline{\bar{V}} \times \underline{B}\right)$$

$u \quad \frac{p_x}{m}\left(q\underline{E} + \frac{p}{m} \times B\right)$

$u' \quad \frac{1}{m}\left(q\underline{E} + \frac{p}{m} \times B\right)$

$v \quad f$

$v' \quad \frac{\partial f}{\partial p_x}$

ADDING THE INTEGRALS TOGETHER:

$$\frac{\partial}{\partial t} n\underline{\bar{v}} + \left(\frac{\partial}{\partial \underline{x}} \cdot n\underline{\bar{v}}\right)\underline{\bar{v}} + n\underline{\bar{v}} \cdot \frac{\partial \underline{\bar{v}}}{\partial \underline{x}} = n\frac{q}{m}\left(\underline{E} + \underline{\bar{v}} \times \underline{B}\right) - \frac{1}{m}\frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}}$$

$$n\frac{\partial \underline{\bar{v}}}{\partial t} + \underbrace{\frac{\partial n}{\partial t}\underline{\bar{v}} + \left(\frac{\partial}{\partial x} \cdot n\underline{\bar{v}}\right)\underline{\bar{v}}}_{\substack{=0 \\ \text{BY CONTINUITY} \\ \text{EQUATION}}} + n\underline{\bar{v}} \cdot \frac{\partial \underline{\bar{v}}}{\partial \underline{x}} = \quad '' \quad\quad ''$$

$$\rho(\underline{x},t) = m\,n(\underline{x},t)$$

DIVIDING BY n:

$$\boxed{\frac{\partial \underline{\bar{v}}}{\partial t} + \underline{\bar{v}} \cdot \frac{\partial \underline{\bar{v}}}{\partial \underline{x}} = \frac{q}{m}\left(\underline{E} + \underline{\bar{v}} \times \underline{B}\right) - \frac{1}{\rho}\frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}}}$$ (NON-RELATIVISTIC)

↑MOMENTUM EQUATION↑

NOTE THAT $\frac{\partial \underline{\bar{v}}}{\partial t} + \underline{\bar{v}} \cdot \frac{\partial \underline{v}}{\partial \underline{x}} = \frac{d}{dt}\underline{\bar{v}}\Big|_{\text{ALONG A TRAJECTORY}}$

$$\Rightarrow \boxed{\frac{d\underline{\bar{v}}}{dt} = \frac{q}{m}\left(\underline{E} + \underline{\bar{v}} \times \underline{B}\right) - \frac{1}{\rho}\frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}}}$$ (NON-RELATIVISTIC)

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Summary of fluid equations

Let $n(\underline{x},t) = \int d^3p \; f(\underline{x},\underline{p},t)$ PARTICLE DENSITY

$\underline{\bar{v}}(\underline{x},t) \equiv \frac{1}{n(\underline{x},t)} \int d^3p \; \frac{\underline{p}}{\gamma m} \; f(\underline{x},\underline{p},t)$ FLUID VELOCITY

$\underline{\bar{p}}(\underline{x},t) \equiv \frac{1}{n(\underline{x},t)} \int d^3p \; \underline{p} \; f(\underline{x},\underline{p},t)$ FLUID MOMENTUM

$\underline{\underline{\mathbb{P}}}(\underline{x},t) \equiv \int d^3p \, (\underline{p} - \underline{\bar{p}})\left(\frac{\underline{p}}{\gamma m} - \underline{\bar{v}}\right) f(\underline{x},\underline{p},t)$ PRESSURE TENSOR

$\frac{d\underline{x}}{dt} = \frac{\underline{p}}{\gamma m}$ $\quad \frac{d\underline{p}}{dt} = q\left(\underline{E}(\underline{x},t) + \frac{\underline{p}}{\gamma m} \times B(\underline{x},t)\right)$ $\quad \gamma^2 = \frac{\underline{p}\cdot\underline{p}}{(mc)^2} + 1$

CONTINUITY EQUATION:
$$\frac{\partial n(\underline{x},t)}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot n(\underline{x},t)\underline{\bar{v}}(x,t) = 0$$

MOMENTUM EQUATION:
$$\frac{\partial \underline{\bar{p}}}{\partial t} + \underline{v} \cdot \frac{\partial \underline{\bar{p}}}{\partial \underline{x}} = q(\underline{E} + \underline{\bar{v}} \times \underline{B}) - \frac{1}{n(\underline{x},t)} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{\mathbb{P}}} = 0$$

THE ABOVE EQUATIONS ARE RELATIVISTICALLY CORRECT.
IN THE NON-RELATIVISTIC LIMIT THE CONTINUITY EQUATION REMAINS UNCHANGED & THE MOMENTUM EQUATION MAY BE WRITTEN:

NON RELATIVISTIC →
$$\frac{\partial \underline{\bar{v}}}{\partial t} + \underline{\bar{v}} \cdot \frac{\partial \underline{\bar{v}}}{\partial \underline{x}} = \frac{q}{m}(\underline{E} + \underline{\bar{v}} \times \underline{B}) - \frac{1}{mn} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{\mathbb{P}}}$$

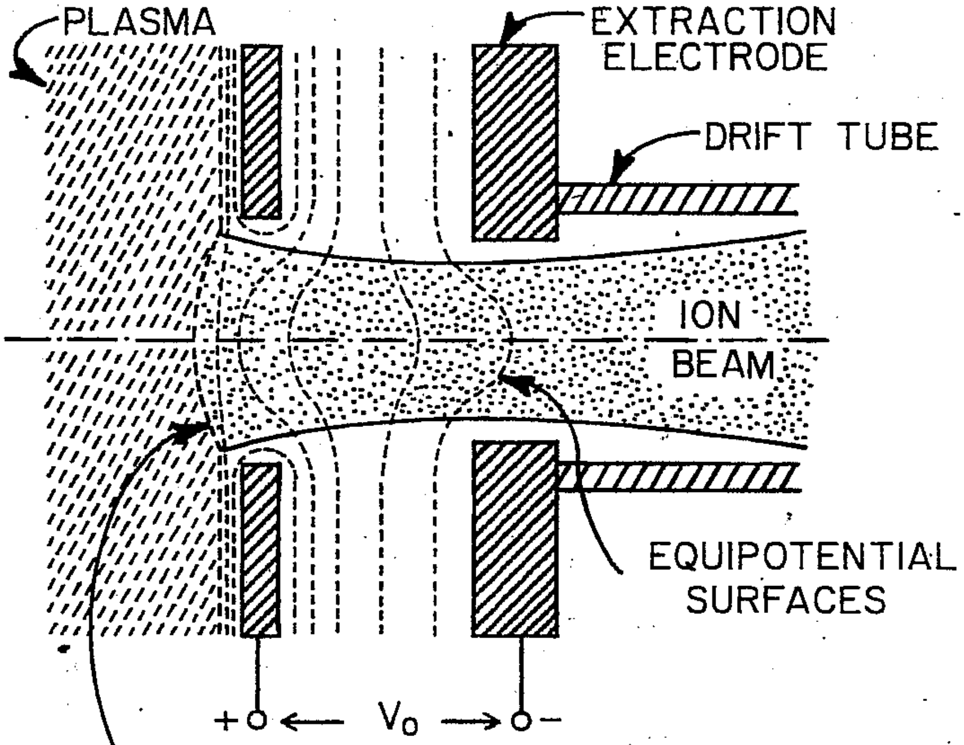
THESE EQUATIONS ARE SUPPLEMENTED WITH MAXWELL'S EQUATIONS: for $\underline{E}(\underline{x},t)$ & $\underline{B}(\underline{x},t)$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{E} = \frac{q\,n(\underline{x},t)}{\varepsilon_0} \qquad \frac{\partial}{\partial \underline{x}} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{B} = 0 \qquad \frac{\partial}{\partial \underline{x}} \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \qquad \underline{J}(\underline{x},t) = q\,n(\underline{x},t)\,\underline{\bar{v}}(\underline{x}$$

NEED ADDITIONAL EQUATIONS SUCH AS $\underline{\underline{\mathbb{P}}} = 0$ OR ENERGY EQUATION TO TERMINATE SET OF EQUATIONS.

INJECTORS

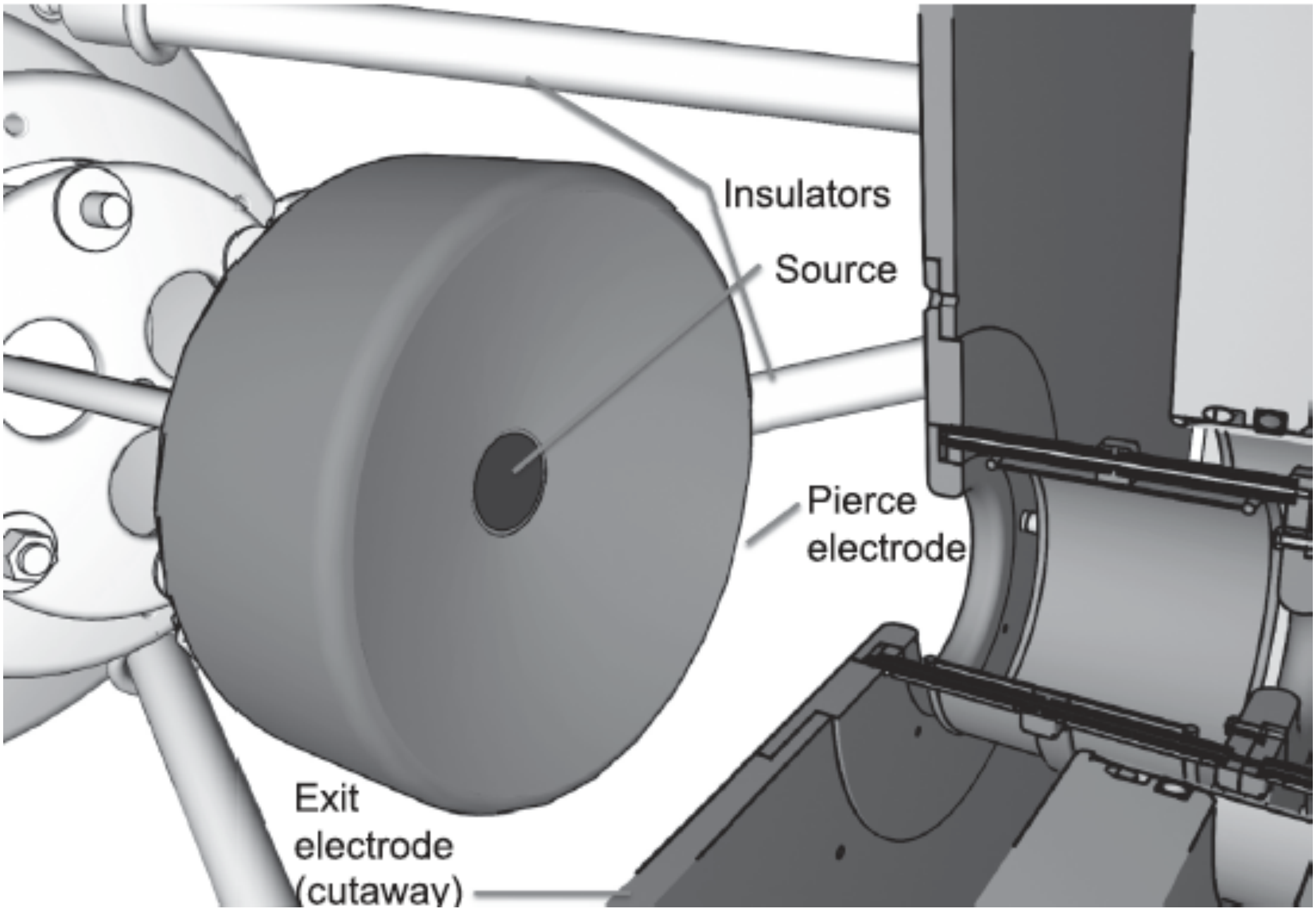


EMITTING SURFACE
(PLASMA "SHEATH" OR "MENISCUS")

OR "HOT PLATE"

- DOPED TUNGSTEN
- ALUMINO-SILICATE

REISER, FIGURE 1.2



A mechanical drawing of a hot-plate diode used on the NDCX-1 experiment at LBNL. One quarter of the exit electrode is cut away for viewing the source geometry

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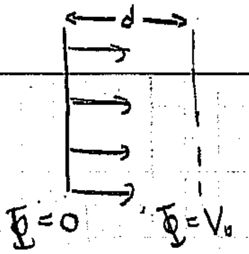
I CHILD-LANGMUIR EMISSION

ASSUME EMISSION IS PLANAR 1-D:



$$J = \rho v_z \qquad (1)$$

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} = q\frac{\partial \Phi}{\partial z} \Rightarrow \frac{1}{2} m v_z^2 = q\Phi(z) \qquad (2) \qquad \Rightarrow \quad v_z = \left(\frac{2q\Phi}{m}\right)^{1/2}$$

$$\frac{d^2\Phi}{dz^2} = \frac{\rho}{\varepsilon_0} \qquad (3)$$

(NOTE $\Phi = -\phi$ ← ACTUAL e.s. potential)

CONTINUITY EQUATION (1 D) $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}\rho v_z = 0$ (4)

for time steady emission ρv_z = constant = J

$$\Rightarrow \frac{d^2\Phi}{dz^2} = \frac{J}{\varepsilon_0 v_z} = \frac{J}{\varepsilon_0\left(\frac{2q\Phi}{m}\right)^{1/2}}$$

MULTIPLYING BY $\frac{d\Phi}{dz}$ AND INTEGRATING:

$$\frac{\Phi'^2}{2} = \frac{J m^{1/2}}{\varepsilon_0 (2q)^{1/2}} 2\Phi^{1/2} + \text{Const}$$

Assume $\Phi' = 0$ at $z = 0$ (Space-charge limited emission)

$\Phi = 0$ at $z=0$ ⇒ const = 0

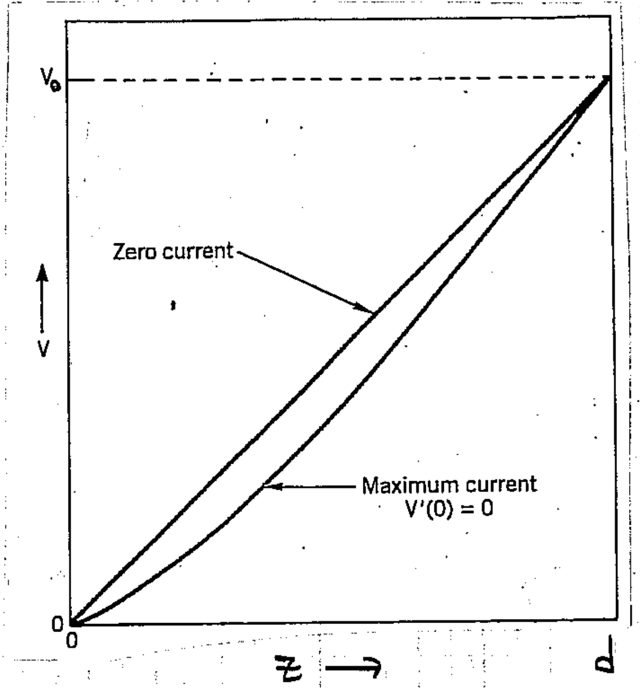
$$\frac{d\Phi}{dz} = \left(\frac{4J}{\varepsilon_0}\right)^{1/2}\left(\frac{m\Phi}{2q}\right)^{1/4}$$

$$\Rightarrow \frac{4}{3}\Phi^{3/4} = \left(\frac{4J}{\varepsilon_0}\right)^{1/2}\left(\frac{m}{2q}\right)^{1/4} z \Rightarrow \boxed{\Phi(z) = \left(\frac{3}{4}\right)^{4/3}\left(\frac{4J}{\varepsilon_0}\right)^{2/3}\left(\frac{m}{2q}\right)^{1/3} z^{4/3}}$$

If $\Phi = V_0$ at $z = d$ $\Rightarrow$ $\Phi = V_0 \left(\frac{z}{d}\right)^{4/3}$

$$\Rightarrow \quad V_0 = \left(\frac{3}{4}\right)^{4/3} \left(\frac{4J}{\varepsilon_0}\right)^{2/3} \left(\frac{m}{2q}\right)^{1/3} d^{4/3}$$

$$\text{or} \quad J = \frac{4}{9}\varepsilon_0 \left(\frac{2q}{m}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$$



NOTE THAT IF WE MULTIPLY J BY THE BEAM AREA πr_b^2, AND DIVIDE BY $v = \left(\frac{2qV_0}{m}\right)^{1/2}$

$$\Rightarrow \lambda = \frac{4\pi\varepsilon_0 V}{9} \left(\frac{r_b^2}{d^2}\right)$$

RECALL $Q \equiv \frac{\lambda}{4\pi\varepsilon_0 V}$ (NON-RELAT.)

$$\Rightarrow Q = \frac{1}{9}\left(\frac{r_b^2}{d^2}\right)$$

or as a function of z: $Q(z) = \frac{1}{9}\left(\frac{r_b^2}{z^2}\right)$

<u>PERVEANCE</u>

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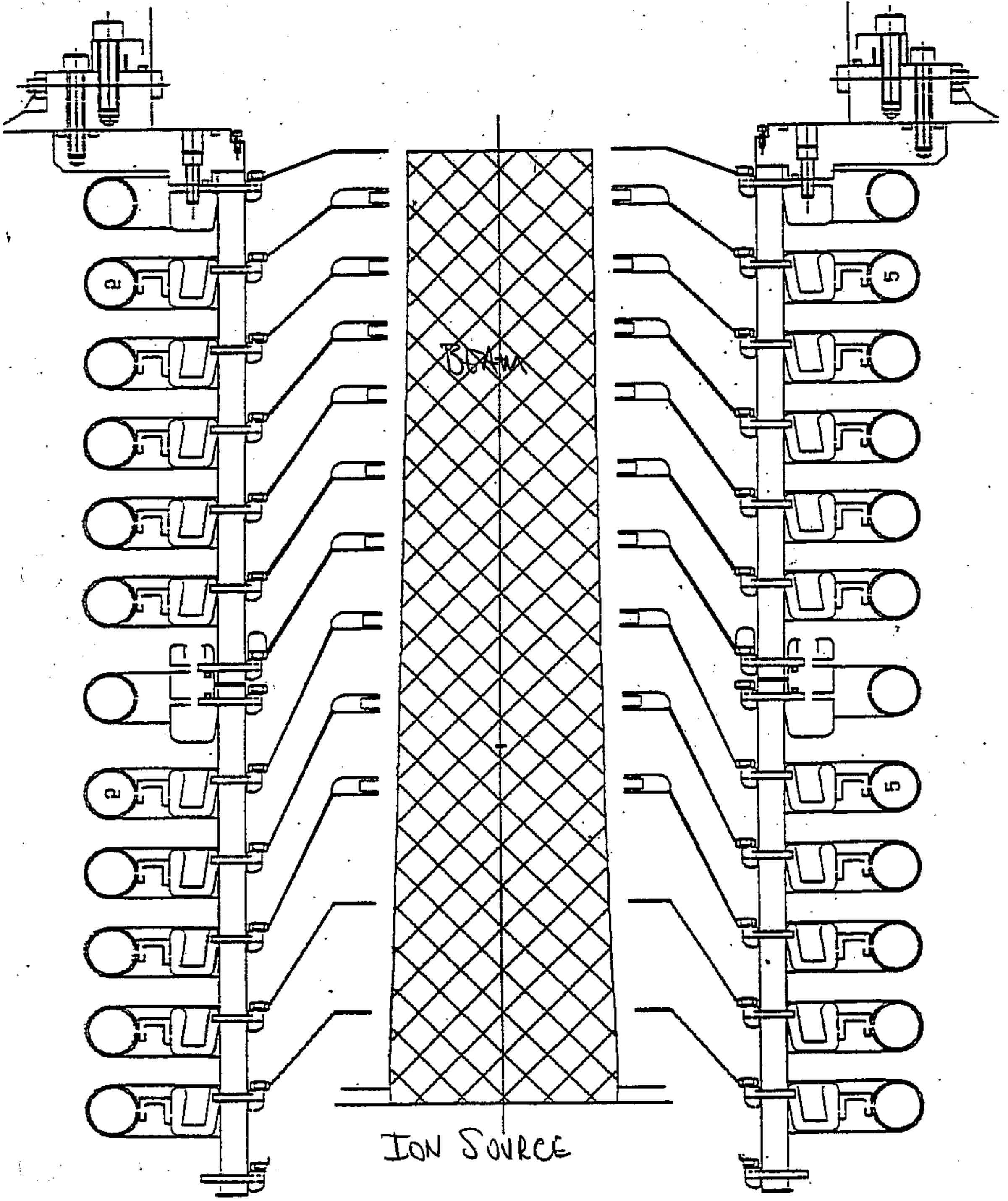
PIERCE COLUMN

$V \sim z^{4/3}$

$E \sim z^{1/3}$



WE DERIVED THE PARAXIAL RAY EQUATION FOR PARTICLES IN AXISYMMETRIC SYSTEMS:

$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma} r}_{E_r} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{V_\theta B_z \text{ − CENTRIFUGAL}} - \underbrace{\left(\frac{P_\theta}{\gamma\beta m c}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}}_{\text{SELF-FIELD}} =$$

$$\theta' = \frac{P_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma\beta c}$$ ← CONSTANCY & DEFINITION OF CANONICAL MOMENTUM

ENVELOPE EQUATION FOR AXISYMMETRIC BEAM

$$r_b'' + \frac{\gamma' r_b'}{\beta^2\gamma} + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle P_\theta\rangle^2}{(\gamma m\beta c)^2 r_b^3} - \frac{\varepsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0$$

$$\varepsilon_r^2 \equiv 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle rr'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$$

RETURNING TO PARAXIAL ENVELOPE EQUATION:

(for $\beta << 1$) $$r_b'' + \frac{\beta'}{\beta} r_b' + \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b - \frac{Q}{r_b} = 0$$

For $r_b'' = \frac{\beta'}{\beta} r_b' = 0$

If $\Phi = V_0 (z/d)^{4/3}$

$$v = C z^{2/3}$$
$$v' = \frac{2}{3} C z^{-1/3}$$
$$v'' = \frac{-2}{9} C z^{-4/3}$$

$$\Rightarrow \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b^2 = Q$$

$$\left[\overset{\shortparallel}{\frac{2}{9}\frac{1}{z^2}} \qquad \overset{\shortparallel}{-\frac{1}{9}\frac{1}{z^2}}\right]$$

$$\Rightarrow Q(z) = \frac{1}{9}\frac{r_b^2}{z^2}$$

So Child-Langmuir flow satisfies the PARAXIAL ENVELOPE EQUATION FOR A CONSTANT BEAM RADIUS (AS IT SHOULD!)

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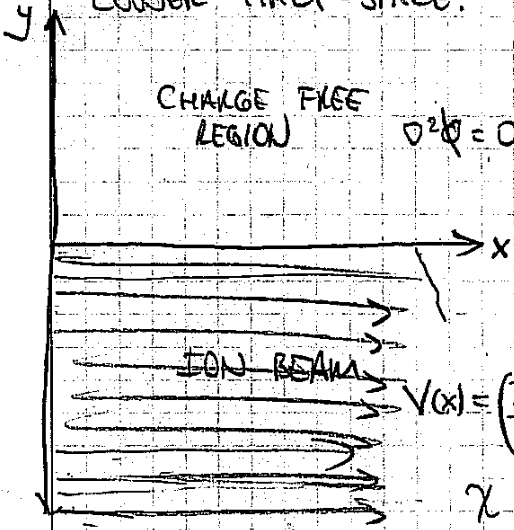
PIERCE ELECTRODES: GOING BEYOND 1D

CONSIDER THE CASE A BEAM WHICH FILLS THE LOWER HALF-SPACE.



$\nabla^2\phi = 0$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$V(x) = \left(\frac{J}{\chi}\right)^{2/3} x^{4/3}$$

$$\chi = \left(\frac{4\varepsilon_0}{9}\right)\sqrt{\frac{2q}{m}}$$

FIND SOLUTION SUCH THAT

$$\frac{\partial\phi}{\partial y}(x, y=0) = 0$$

\$ $\phi(x, y=0) = V(x)$

PIERCE'S SOLUTION: LET THE POTENTIAL BE THE REAL PART OF

$$\phi + iW = V(x+iy) \equiv V(z) \qquad z \equiv x+iy$$

NOTE THAT FOR ANY $V(z)$ WITH derivatives that exist independent of direction (analytic) the real part of $V(z)$ satisfies Laplace's equation: $\frac{\partial^2 \mathrm{Re}[V]}{\partial x^2} + \frac{\partial^2 \mathrm{Re}[V]}{\partial y^2} = 0$

$$\frac{\partial\phi}{\partial x} = \mathrm{Re}\left[\frac{dV}{dz}\right] \qquad \frac{\partial\phi}{\partial y} = \mathrm{Re}\left[i\frac{dV}{dz}\right]$$

$$\frac{\partial^2\phi}{\partial x^2} = \mathrm{Re}\left[\frac{d^2V}{dz^2}\right] \qquad \frac{\partial^2\phi}{\partial y^2} = -\mathrm{Re}\left[\frac{d^2V}{dz^2}\right] \Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \mathrm{Re}\left[\frac{d^2V(z)}{dz^2}\right] - \mathrm{Re}\left[\frac{d^2V(z)}{dz^2}\right] = 0$$

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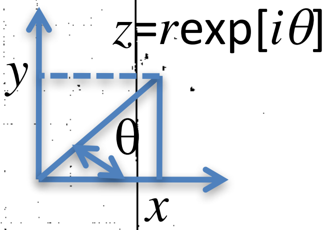
CHOOSE $V(z) = \left(\frac{J}{\chi}\right)^{2/3} (x+iy)^{4/3}$

By conctuction $\phi(x, y=0) = V(x)$ ✓



$$\phi = \text{Re}\left[\left(\frac{J}{\chi}\right)^{2/3} (x+iy)^{4/3}\right]$$

Let $x+iy = r\exp[i\theta]$

$(x+iy)^{4/3} = r^{4/3}\exp\left[\frac{4i\theta}{3}\right]$

$$= \left(\frac{J}{\chi}\right)^{2/3} (x^2+y^2)^{2/3} \text{Re}\left[\exp\left[i\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]\right]$$

$$\boxed{\phi(x,y) = \left(\frac{J}{\chi}\right)^{2/3} (x^2+y^2)^{2/3} \cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]}$$

Note that $\phi(x,y) = \phi(x,-y) \Rightarrow \frac{\partial\phi}{\partial y}(x, y=0) = 0$

$\phi = 0$ EQUIPOTENTIAL:

$$\Rightarrow 0 = \cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{3}{4}\left(\frac{\pi}{2}\right) = 67.5°$$

So line θ = 67.5° is a ϕ=0 equipotential.

FOR A GENERAL EQUIPOTENTIAL PASSING THROUGH X_0:

$$x_0^{4/3} = (x^2+y^2)^{2/3}\cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]$$

← Equipotential that passes through point $(x_0,0)$

$$\phi(x_0) = \left(\frac{J}{\chi}\right)^{2/3} x_0^{4/3}$$

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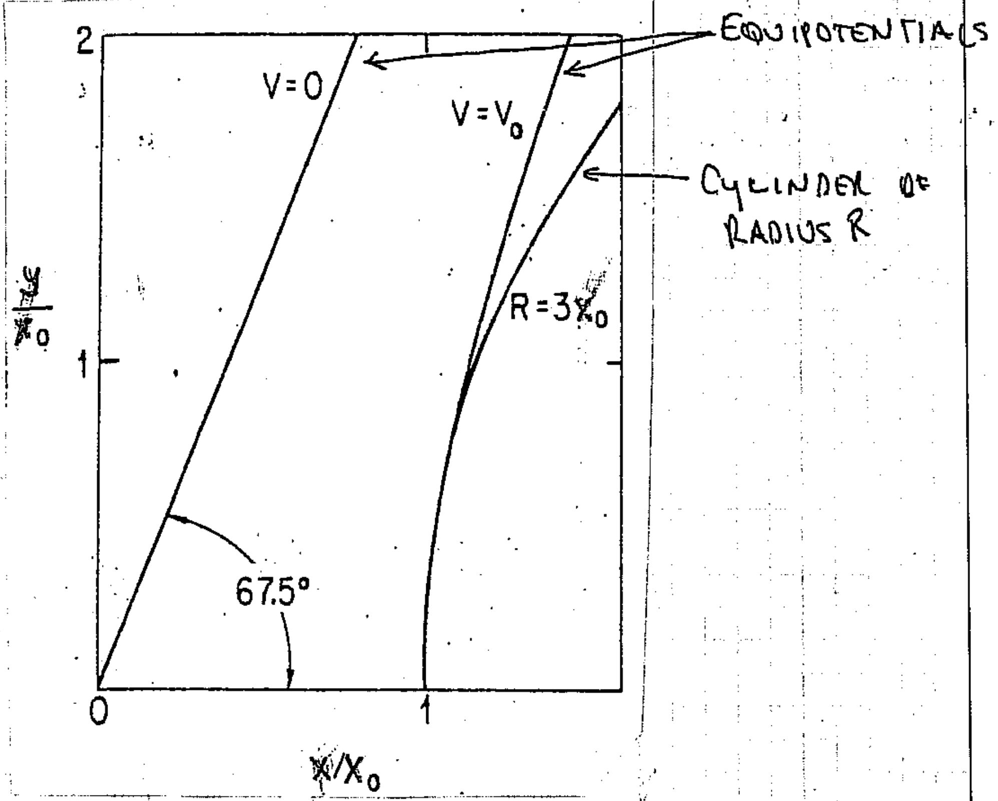
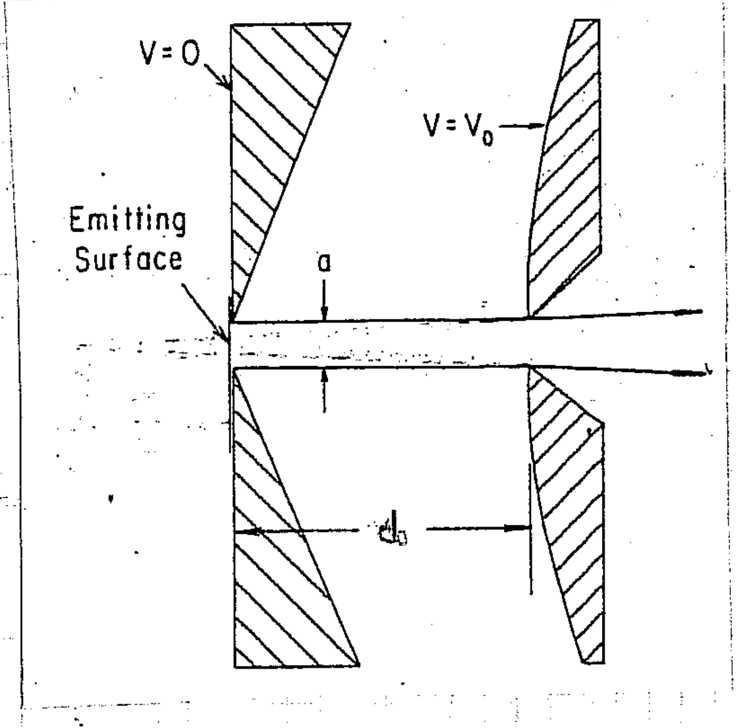
EQUIPOTENTIALS FOR MATCHING TO A PLANAR BOUNDARY





ELECTRODES FOR A LONG SLIT ION BEAM SYSTEM

FIGURES FROM
"LARGE ION BEAMS,
FUNDAMENTALS OF
GENERATION AND
PROPAGATION"
BY A.T. FORRESTER,
Wiley, NY, 1988.

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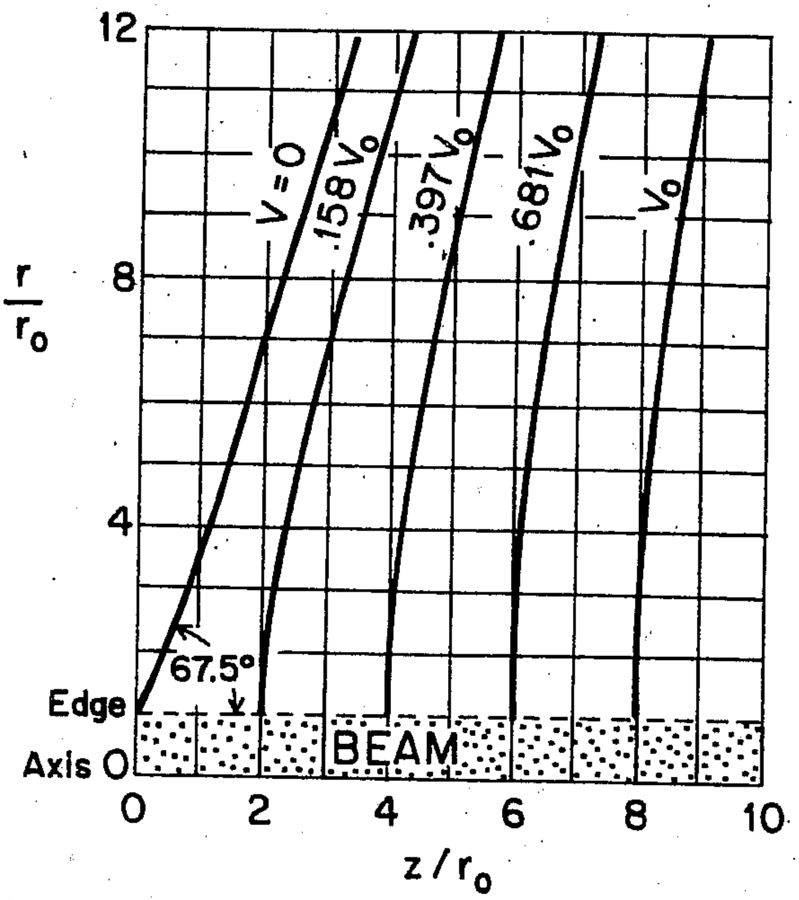
PIERCE ELECTRODES FOR CIRCULAR BEAMS



- SOLUTION is similar, but must be done numerically
- $\phi = 0$ is same as planar case

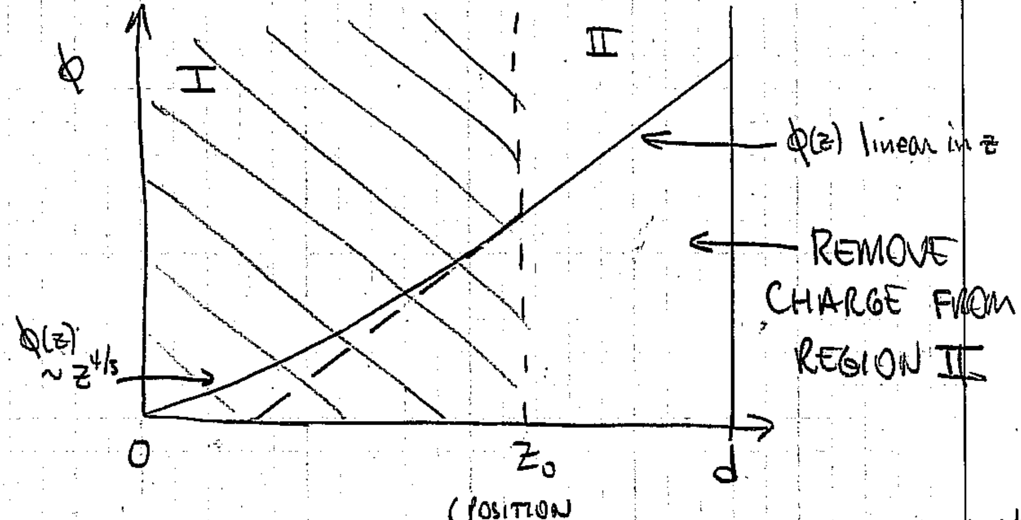
FIGURE FROM A.T. FORRESTER, "LARGE ION BEAMS, ..." Wiley, 1988

TRANSIENTS IN INJECTORS

(LAMPEL & TIEFENBACK, Appl. Phys. Lett, 43, 57, 1983)

DURING TURN-ON THERE IS NO SPACE CHARGE IN FRONT OF BEAM, SO FIELDS MAY NOT BE GIVEN BY CHILD-LANGMUIR LAW. ⇒ CURRENT SPIKES POSSIBLE ⇒ ADVERSE TRANSVERSE COUPLING.

SOLUTION: ADJUST VOLTAGE ON DIODE SUCH THAT C-L FIELD OCCURS EVERYWHERE THERE IS BEAM.



$$E(z_0) = \frac{4}{3}\frac{V_0}{d}\left(\frac{z_0}{d}\right)^{1/3}$$

$$\Phi(d) = \int_0^{z_0} E(z)\,dz + \int_{z_0}^{d} E(z)\,dz$$

$$= V_0\left(\frac{z_0}{d}\right)^{4/3} + \frac{4}{3}\frac{V_0}{d}\left(\frac{z_0}{d}\right)^{1/3}(d - z_0)$$

$$= V_0\left[\frac{4}{3}\left(\frac{z_0}{d}\right)^{1/3} - \frac{1}{3}\left(\frac{z_0}{d}\right)^{4/3}\right]$$

(NOTE V_0 IS THE DESIRED STEADY STATE VOLTAGE ACROSS DIODE).

SO IF WE KNOW $z_0(t)$ WE CAN DETERMINE $\Phi(t)$.

$$\frac{1}{2} m \dot{z}_0^2 = qV_0 \left(\frac{z_0}{d}\right)^{4/3}$$

(SINCE BY CONSTRUCTION, HEAD OF BEAM TRAVELS AT CHILD-LANGMUIR VELOCITY LIKE ALL PARTICLES).

$$\dot{z}_0 = \left(\frac{2qV_0}{m}\right)^{1/2} \left(\frac{z_0}{d}\right)^{2/3}$$

$$\frac{dz_0}{z_0^{2/3}} = \left(\frac{2qV_0}{m}\right)^{1/2} \frac{dt}{d^{2/3}}$$

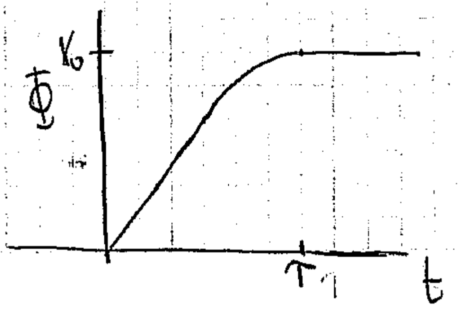
$$\Rightarrow 3 z_0^{1/3} = \left(\frac{2qV_0}{m}\right)^{1/2} \frac{t}{d^{2/3}} \Rightarrow t = \frac{3(z_0 d^2)^{1/3}}{\left(\frac{2qV_0}{m}\right)^{1/2}}$$

Let $\tau = \frac{3d}{\left(\frac{2qV_0}{m}\right)^{1/2}}$ = transit time across diode

$$\Rightarrow \frac{t}{\tau} = \left(\frac{z_0}{d}\right)^{1/3}$$

$$\Phi(d, z_0) = V_0 \left[\frac{4}{3}\left(\frac{z_0}{d}\right)^{1/3} - \frac{1}{3}\left(\frac{z_0}{d}\right)^{4/3}\right]$$

$$\Rightarrow \Phi(d,t) = \begin{cases} V_0 \left[\frac{4}{3}\left(\frac{t}{\tau}\right) - \frac{1}{3}\left(\frac{t}{\tau}\right)^4\right] & \text{for } 0 < t < \tau \\ V_0 & \text{for } t > \tau \end{cases}$$



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INJECTOR CHOICES

(cf. Kwan et al, NIMPRA, 464, 379 (2001).

CHILD - LANGMUIR $\Rightarrow J = \chi \dfrac{V^{3/2}}{d^2}$

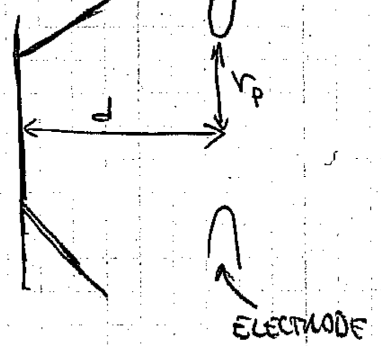
where $\chi = \frac{4}{9}\varepsilon_0\left(\frac{2q}{m}\right)^{1/2}$

SOME CONSTRAINTS:

(1) VOLTAGE BREAKDOWN

$$\text{EMPIRICALLY} \quad V \lesssim 100\,\text{kV} \begin{cases} \left(\dfrac{d}{1\,\text{cm}}\right) & \text{for } d \leq 1\,\text{cm} \\ \left(\dfrac{d}{1\,\text{cm}}\right)^{1/2} & \text{for } d \geq 1\,\text{cm} \end{cases}$$

(2) BEAM OPTICS ABERRATIONS: $d \gtrsim 3 \text{ to } 4\; r_p$ (TYPICALLY)



NOTE THAT

$$J \sim \frac{V^{3/2}}{d^2} \sim \begin{cases} V^{-1/2} \sim d^{-1/2} & d < 1 \text{ cm} \\ V^{-5/2} \sim d^{-5/4} & d > 1 \text{ cm} \end{cases} \qquad I \sim \pi r_p^2 J \sim \left\{ V^{3/2} \right.$$

Thus current density decreases with size and voltage, but I increases.

Air
$V = 24 S + 7\sqrt{S}$

from A. Faltens:

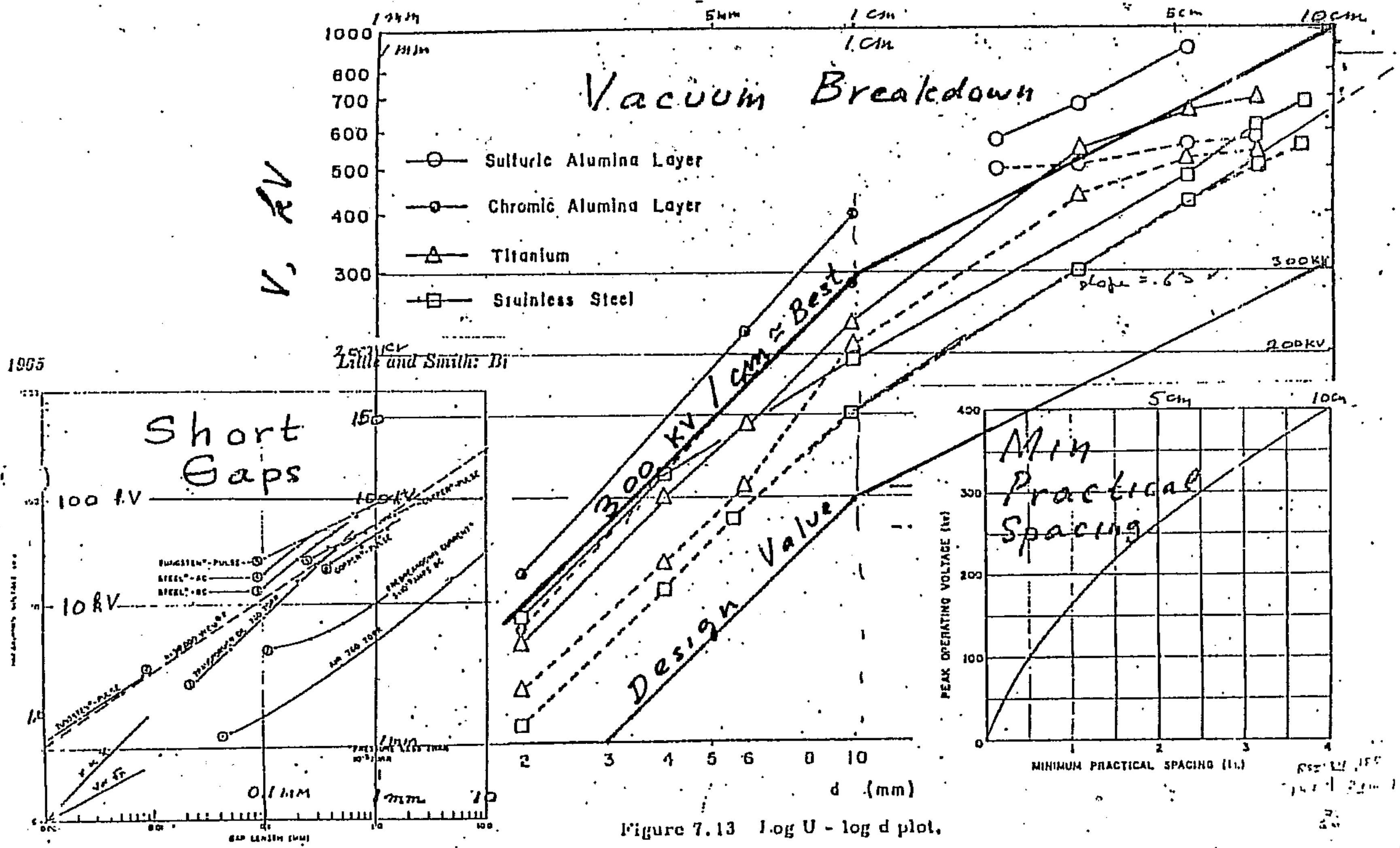


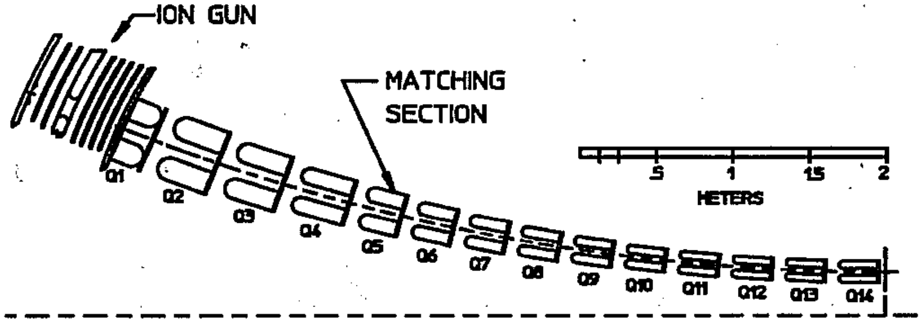
Figure 7.13 Log U - log d plot.

Fig. 1. Breakdown voltage-vs.-gap length for uniform-field and near-uniform-field geometry. Numbers on curves indicate the

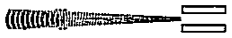
MULTIPLE BEAMLET INJECTORS CAN HAVE HIGHER CURRENT DENSITY

DECREASING SIZE OF INJECTOR

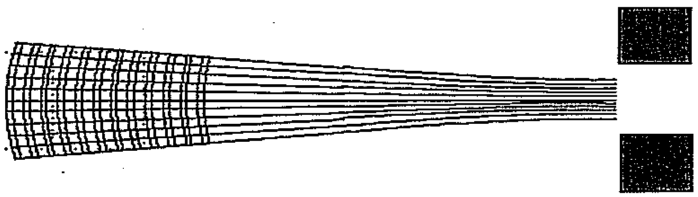
traditional design using single large diameter source →



advanced design using multiple beamlets →



Each beamlet carries higher current density; But merging beamlets increases thermal spread.



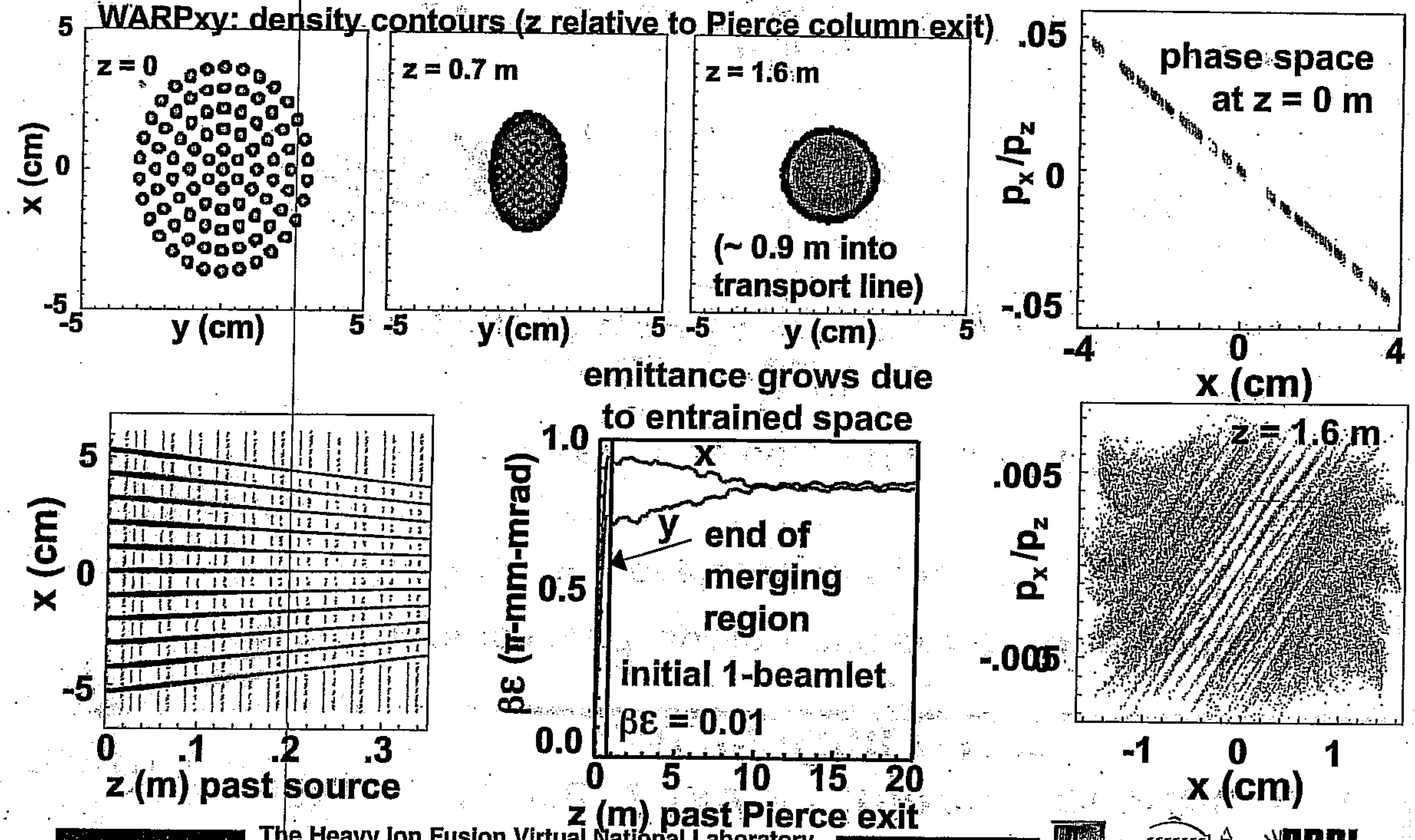
Merge and match beamlets into an ESQ channel

Child-Langmuir $J_{CL} \propto \frac{V^{3/2}}{d^2}$

Breakdown limit $V \propto d^{1.0\ to\ 0.5}$

$$\left.\begin{array}{l} J_{CL} \propto \frac{V^{3/2}}{d^2} \\ V \propto d^{1.0\ to\ 0.5} \end{array}\right\} J \propto V^{-1/2\ \ to\ \ -5/2} \propto d^{-1/2\ \ to\ \ -5/4}$$

Simulations of merging-beamlet injector

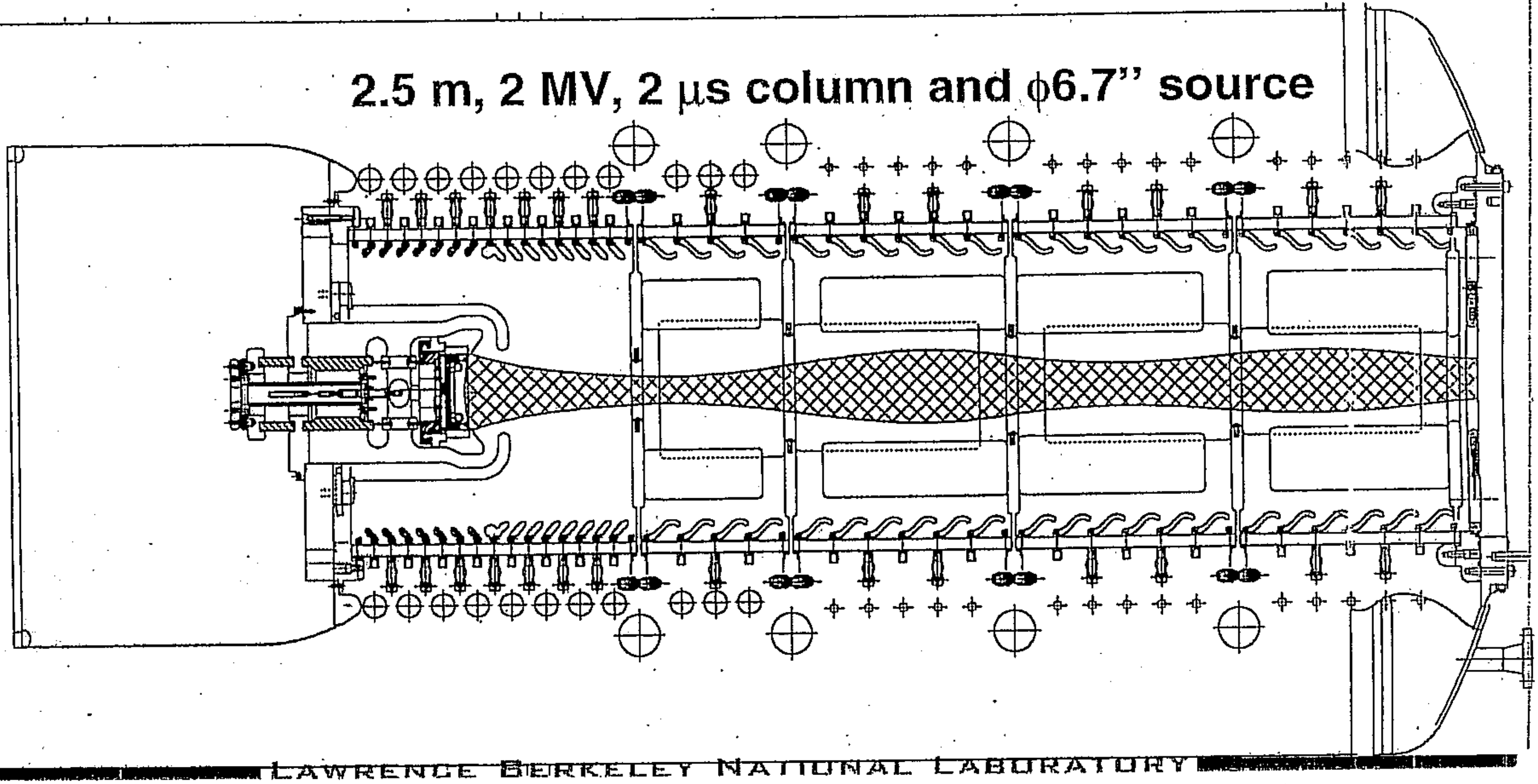


from D.P. Grote, E. Henestroza, J.W. Kwan, "Design & Simulation of the Multibeamlet Injector for a High Current Accelerator" submitted to [illegible] (200[illegible])

0.8 Ampere, 2 MV K^+ Injector produced a λ=0.25μC/m beam

Electrostatic Quadrupole Accelerator for simultaneous focusing and acceleration of ion beams to 2 MV.

2.5 m, 2 MV, 2 μs column and ϕ6.7" source



Filename for ease of reference

SCALING OF BRIGHTNESS IN INJECTORS

$$\epsilon_N = 4\beta \langle x^2 \rangle^{1/2} \langle x'^2 \rangle^{1/2} = \frac{4}{c}\left(\frac{r_b}{2}\right)\langle v_x^2 \rangle^{1/2}$$

$$\frac{1}{2} m v_x^2 = \frac{1}{2} kT$$

$$\epsilon_N = 2 r_b \sqrt{\frac{kT}{mc^2}}$$

$$\Rightarrow B = \frac{I}{\epsilon_N^2} = \frac{\pi J}{4(kT/mc^2)} \sim \frac{J}{T}$$

⇒ FOR HIGH BRIGHTNESS & HIGH CURRENT MAY WISH TO ACCELERATE MANY BEAMLETS AND THEN MERGE TO FORM SINGLE BEAM.

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MANY ISSUES NOT DISCUSSED HERE!

- SOURCES
- ELECTRON TRAPPING
- CONVERGING BEAMS
- MATCHING TO AN ESQ (e.g.)
- rf
- ...