Efficient Computation of Matched Solutions of the KV Envelope Equations for Periodic Focusing Lattices*

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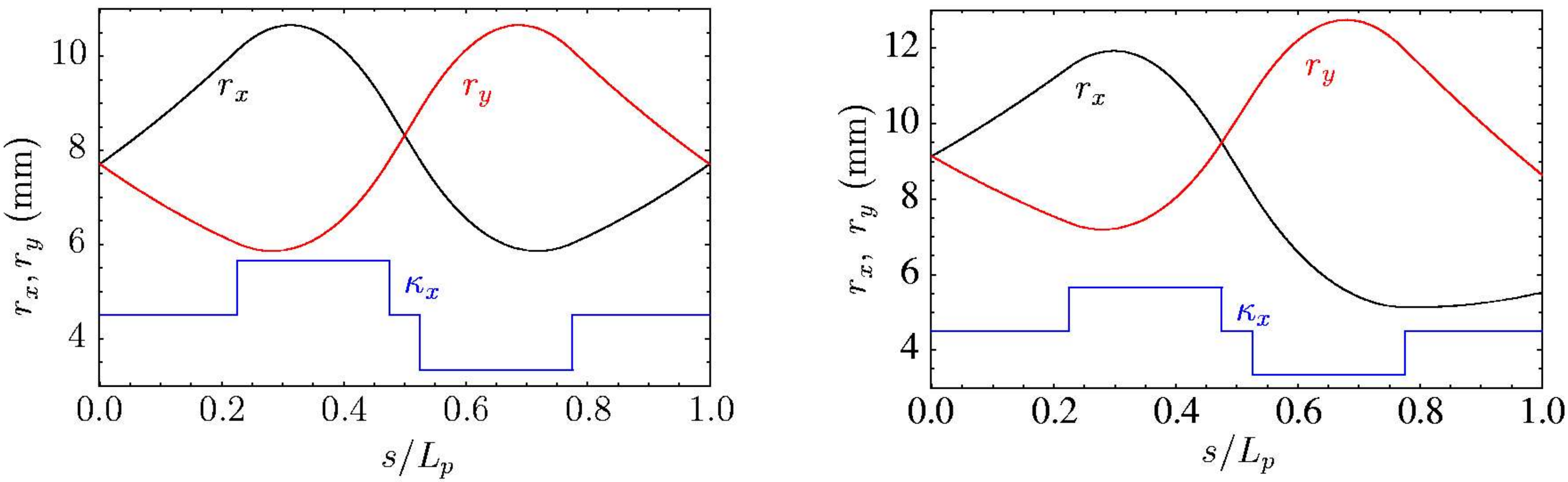
Conventional root-finding methods of solving the KV envelope equations often require *a priori* knowledge of initial conditions

Syncopated Quadrupole Lattice

$$L_p = 0.5 \text{ m}, \ \eta = 0.5, \ \alpha = 0.1, \ \sigma_0 = 80^\circ,$$
$$Q = 4 \times 10^{-4}, \ \varepsilon = 50 \text{ mm-mrad}$$



Actual initial conditions:

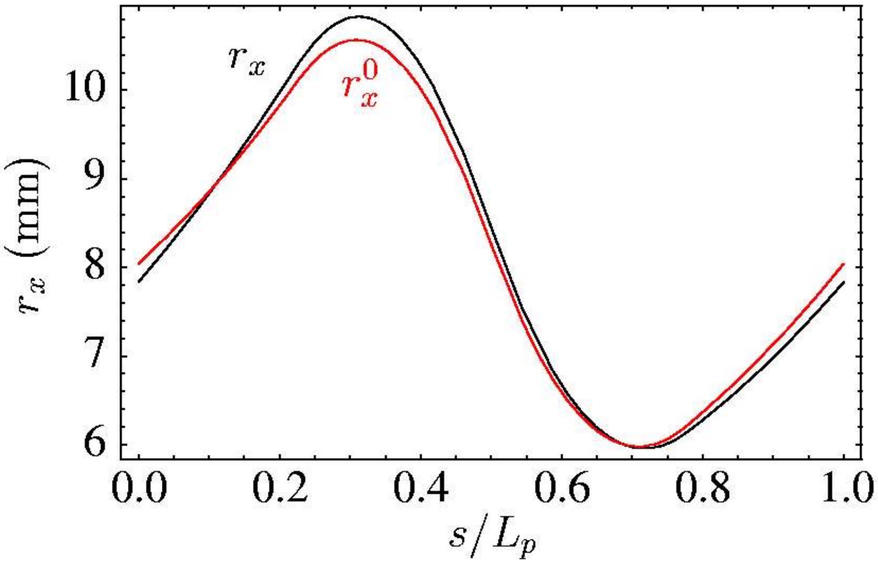
$$r_{xi} = r_{yi} = 7.71 \text{ mm}$$
$$r'_{xi} = -r'_{yi} = 0.0186$$

Incorrect IC's leading to non-matched solutions

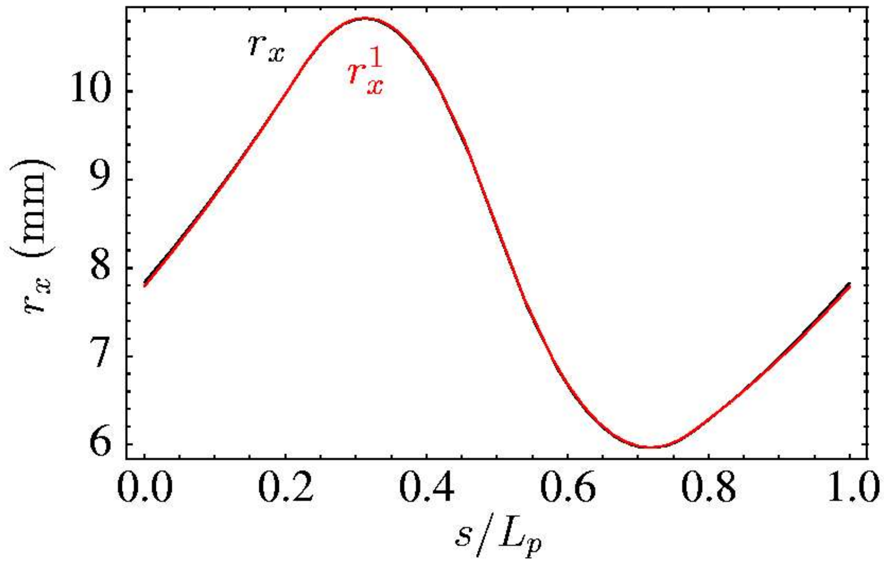
$$r_{xi} = 9 \text{ mm}, \ r_{yi} = 6 \text{ mm}$$
$$r'_{xi} = 0.016, \ r'_{yi} = -0.02$$

New iterative numerical method converges rapidly to matched solution without prior knowledge of initial conditions

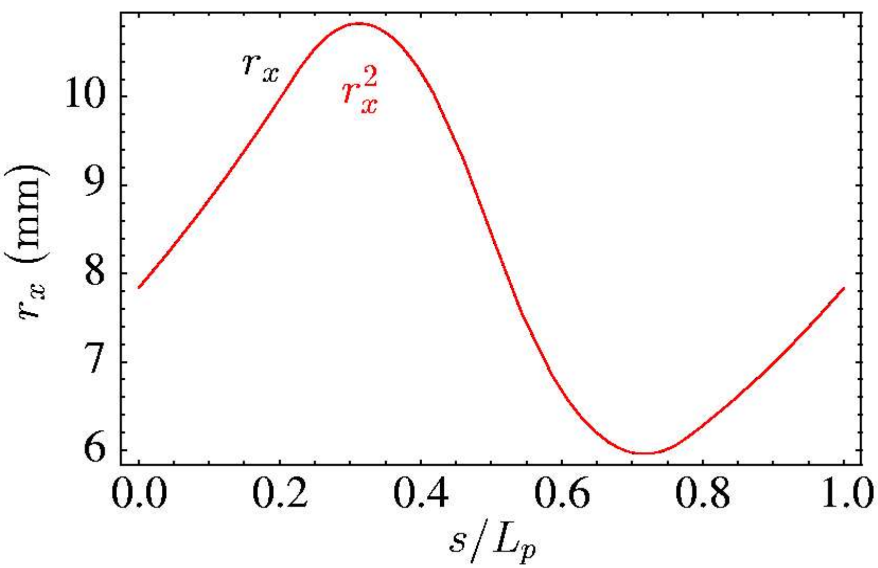
Iteration 0



Iteration 1



Iteration 2



$$
\begin{aligned}
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\end{aligned}
$$

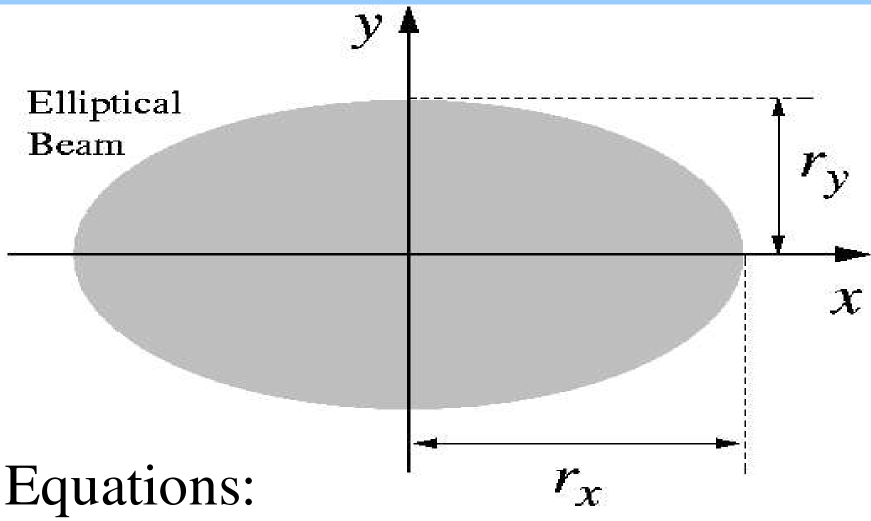
Introduction: New Iterative Numerical Method to construct matched solutions to the KV envelope equations

- Based on consistency between particle orbits and the matched beam envelope
  - Uses betatron formulation
- Method works over entire parameter space
- Works for all parameterizations of matched solutions
- Valid for all linear lattices without skew coupling
- Rapidly convergent and robust, even where envelope is unstable

Outline

- Introduction (Already Done)
- Theoretical Model
- Matched Envelope Properties
- Numerical Iterative Method
- Example Applications
- Conclusions

Theoretical Model: Definition of the KV Equations and Relevant Parameters



$$r_x = 2\langle x^2 \rangle^{1/2}$$
$$r_y = 2\langle y^2 \rangle^{1/2}$$

rms/KV envelope Equations:

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$
$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

Periodicity:

$$r_x(s + L_p) = r_x(s)$$
$$\kappa_x(s + L_p) = \kappa_x(s)$$

$Q = \frac{qI}{2\pi\epsilon_0 m c^3 \gamma_b^3 \beta_b^3}$ =const .... perveance

$\varepsilon_x = 4[\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2]^{1/2}$ .... rms edge emittance

$\kappa_x(s),\ \kappa_y(s)$ define applied focusing forces of the lattice

Undepressed particle phase advance σ_{0x} measures the strength of the applied focusing function $\kappa_x(s)$ of periodic lattices

Single-particle orbit without space-charge:

$$x'' + \kappa_x(s)x = 0$$

The same applies to y

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \mathbf{M}_{0x}(s \mid s_i) \cdot \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

$\mathbf{M}_{0x} =$ 2 x 2 Transfer Matrix from $s = s_i$ to s.

Undepressed particle phase advance:

$$\cos \sigma_{0x} = \frac{1}{2} \mathrm{Tr}\, \mathbf{M}_{0x}(s_i + L_p \mid s_i)$$

Undepressed Principal Orbit Equations

Transfer Matrix:

$$\mathbf{M}_{0x}(s|s_i) = \begin{pmatrix} C_{0x}(s|s_i) & S_{0x}(s|s_i) \\ C'_{0x}(s|s_i) & S'_{0x}(s|s_i) \end{pmatrix}$$

Cosine-like Principal Orbit Equation:

$$C''_{0x}(s|s_i) + \kappa_x(s)C_{0x}(s|s_i) = 0$$

Initial Conditions:

$$C_{0x}(s_i|s_i) = 1$$
$$C'_{0x}(s_i|s_i) = 0$$

Sine-like case analogous

y-plane analogous

Note that stability requires:

$$\frac{1}{2} \mid \mathrm{Tr}\, \mathbf{M}_{0x}(s_i + L_p \mid s_i) \mid < 1 \implies \sigma_{0x} < 180^\circ$$

[Courant and Snyder, Annals of Physics 3, 1 (1958)]

Depressed Principal Orbit Equations

Depression: $Q \neq 0$

Maintain same basic formulation as before except:

$$\kappa_x \rightarrow \boxed{\kappa_x} - \boxed{\frac{2Q}{(r_x + r_y) r_x}}$$

Applied focusing

Space-charge defocusing

Notation: drop 0 subscript to indicate depression

Cosine-like Principal Orbit Equation:

$$C_x''(s|s_i) + \left[\kappa_x(s) - \frac{2Q}{[r_x(s) + r_y(s)]}\right] C_x(s|s_i) = 0$$

The depressed particle phase advance provides a convenient measure of space-charge strength

Depressed single-particle phase advance in the presence of uniform space-charge for a particle moving in the matched beam envelope:

$$\cos\sigma_x = \frac{1}{2}[C_x(s_i + L_p|s_i) + S'_x(s_i + L_p|s_i)]$$

$$\sigma_x = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2}$$

$$\lim_{Q\to 0} \sigma_x = \sigma_{0x}$$

Normalized space charge strength or “depressed tune” :

$$0 \le \sigma_x/\sigma_{0x} \le 1$$

$\sigma_x/\sigma_{0x} \to 0$ Cold Beam (space-charge dominated) $\varepsilon_x \to 0$

$\sigma_x/\sigma_{0x} \to 1$ Warm Beam (kinetic dominated) $Q \to 0$

Parameterization Classes

Examples from here on assume a symmetric system:

$$\sigma_{0x} = \sigma_{0y} \equiv \sigma_0, \;\; \sigma_x = \sigma_y \equiv \sigma, \;\; \varepsilon_x = \varepsilon_y \equiv \varepsilon$$

Possible parameterizations of matched envelope solutions:

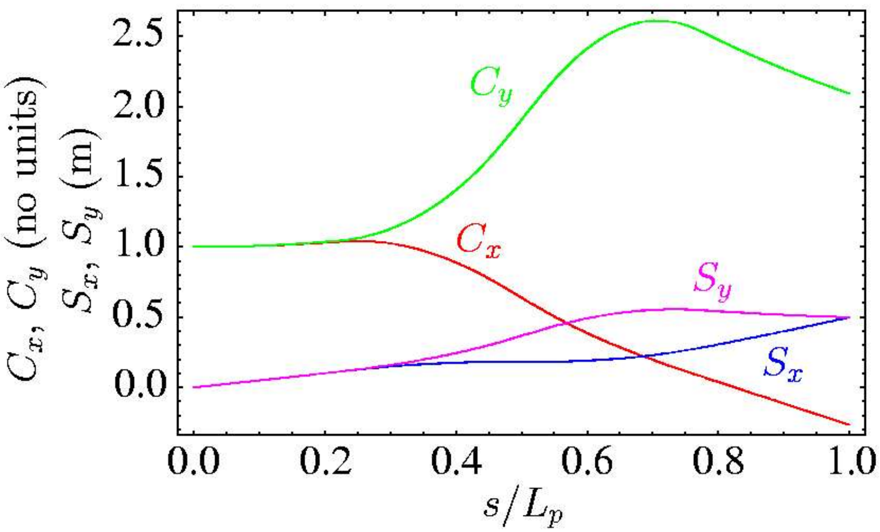
Case	Parameters
0	$\kappa_x \; (\sigma_0), \; Q, \; \varepsilon$
1	$\kappa_x \; (\sigma_0), \; Q, \; \sigma$
2	$\kappa_x \; (\sigma_0), \; \varepsilon, \; \sigma$

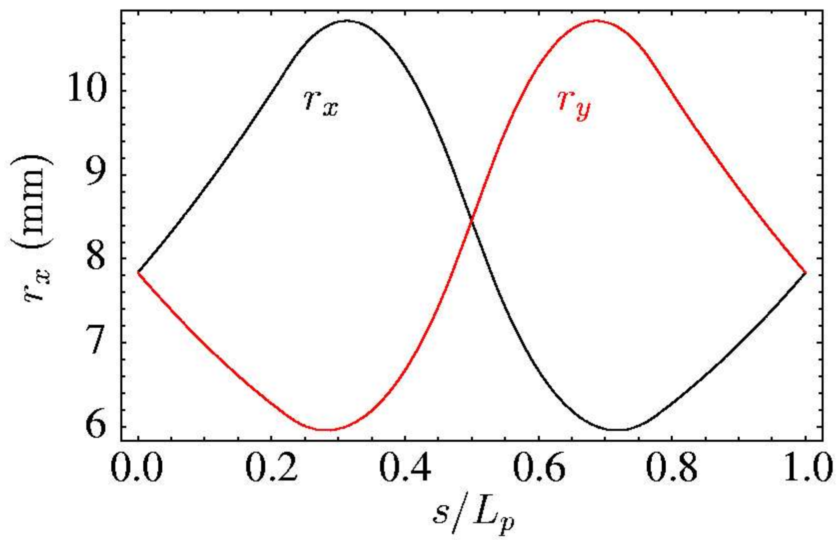
$\longleftarrow$ "Normal" parameterization (Case 0)

Typical principal orbit functions and corresponding matched envelope functions

Syncopated Quadrupole Lattice

$$L_p = 0.5 \text{ m},\ \eta = 0.5,\ \alpha = 0.1,\ \sigma_0 = 80^\circ,$$
$$\sigma/\sigma_0 = 0.3,\ \varepsilon = 50 \text{ mm-mrad}$$





The betatron consistency condition allows us to construct matched solutions of the KV equations

Consistency Condition:

$$\beta_x(s) = \frac{r_x^2(s)}{\varepsilon_x}$$
$$= \frac{S_x^2(s|s_i)}{S_x(s_i + L_p|s_i)/\sin\sigma}$$
$$+ \frac{S_x(s_i + L_p|s_i)}{\sin\sigma}\left[C_x(s|s_i) + \frac{\cos\sigma - C_x(s_i + L_p|s_i)}{S_x(s_i + L_p|s_i)} S_x(s|s_i)\right]^2$$

Used to formulate iterative numerical method for matched envelope soutions

The continuous limit is employed to seed the numerical method

Period Averages:

$$\overline{\zeta} \equiv \int_{s_i}^{s_i+L_p} \frac{ds}{L_p} \zeta(s)$$

Continuous Limit Replacements:

$$\kappa_x \to \left(\frac{\sigma_0}{L_p}\right)^2 \qquad r_x \to \overline{r_x} \qquad \overline{r_x} = \text{const}$$

Continuous Limit KV Envelope Equation (x-plane):

$$\left(\frac{\sigma_0}{L_p}\right)^2 \overline{r_x} - \frac{2Q}{\overline{r_x} + \overline{r_y}} - \frac{\varepsilon^2}{\overline{r_x}^3} = 0$$

Form of solution of continuous limit envelope equations depends on parameters specified

Q, ε parameterization:

Symmetric System: $\sigma_{0x} = \sigma_{0y} \equiv \sigma_0, \;\; \sigma_x = \sigma_y \equiv \sigma, \;\; \varepsilon_x = \varepsilon_y \equiv \varepsilon$

$$\overline{r_x} = \overline{r_y} = \frac{1}{(\sigma_0/L_p)} \left[\frac{Q}{2} + \frac{1}{2}\sqrt{Q^2 + 4\left(\frac{\sigma_0}{L_p}\right)^2 \varepsilon^2} \right]^{1/2}$$

Q, σ parameterization:

$$\overline{r_x} = \frac{\sqrt{2Q}L_p}{\sqrt{(\sigma_{0x}^2 - \sigma_x^2) + \frac{(\sigma_{0x}^2 - \sigma_x^2)^2}{(\sigma_{0y}^2 - \sigma_y^2)}}}$$

$$\overline{r_y} = \frac{\sqrt{2Q}L_p}{\sqrt{(\sigma_{0y}^2 - \sigma_y^2) + \frac{(\sigma_{0y}^2 - \sigma_y^2)^2}{(\sigma_{0x}^2 - \sigma_x^2)}}}$$

ε, σ parameterization:

$$\overline{r_x} = \overline{r_y} = \sqrt{\frac{\varepsilon}{(\sigma/L_p)}}$$

Numerical Iterative Method uses connection between principal orbits and envelope to generate a correction closer to actual matched solution

Notation: denote iteration order with superscript i $(i = 0,\ 1,\ 2,\ \cdots)$

For iterations $i \geq 1$, we calculate refinements of the principal orbit functions in terms of the envelope calculated at the previous iteration from

$$C_x^{i\,\prime\prime} + \kappa_x C_x^i - \frac{2Q^{i-1}}{(r_x^{i-1} + r_y^{i-1})r_x^{i-1}} C_x^i = 0$$

Space-charge defocusing from previous iteration

Cosine-like initial conditions: $C_x^i(s_i|s_i) = 1$

$C_x^{i\,\prime}(s_i|s_i) = 0$

Sine-like case analogous

β_x^i calculated from C_x^i and S_x^i using consistency condition

Unspecified parameters may be calculated with one or more of the constraint equations below

Depressed Phase Advance:

$$\cos\sigma_x^i = \frac{1}{2}[C_x^i(s_i + L_p|s_i) + S_x^{i\prime}(s_i + L_p|s_i)]$$

Period Averaged Envelope Equation:

$$\sqrt{\frac{\varepsilon_y^i}{\varepsilon_x^i}} = \frac{\overline{\kappa_x\sqrt{\beta_x^i}} - \overline{1/(\beta_x^i)^{3/2}}}{\overline{\kappa_y\sqrt{\beta_y^i}} - \overline{1/(\beta_y^i)^{3/2}}}$$

$$\frac{\varepsilon_x^i}{2Q^i} = \frac{\overline{\frac{1}{\sqrt{\beta_x^i}+\sqrt{\varepsilon_y^i/\varepsilon_x^i}\sqrt{\beta_y^i}}}}{\overline{\kappa_x\sqrt{\beta_x^i}} - \overline{1/(\beta_x^i)^{3/2}}} \qquad \frac{\varepsilon_y^i}{2Q^i} = \frac{\overline{\frac{1}{\sqrt{\varepsilon_x^i/\varepsilon_y^i}\sqrt{\beta_x^i}+\sqrt{\beta_y^i}}}}{\overline{\kappa_y\sqrt{\beta_y^i}} - \overline{1/(\beta_y^i)^{3/2}}}$$

Seed Iteration and Cutoff

Seed iteration:

$$C_x^{0\,\prime\prime} + \boxed{\kappa_x} C_x^0 - \boxed{\frac{2\overline{Q}}{(\overline{r_x} + \overline{r_y})\overline{r_x}}} C_x^0 = 0$$

Actual applied focus   Continuous focusing space-charge

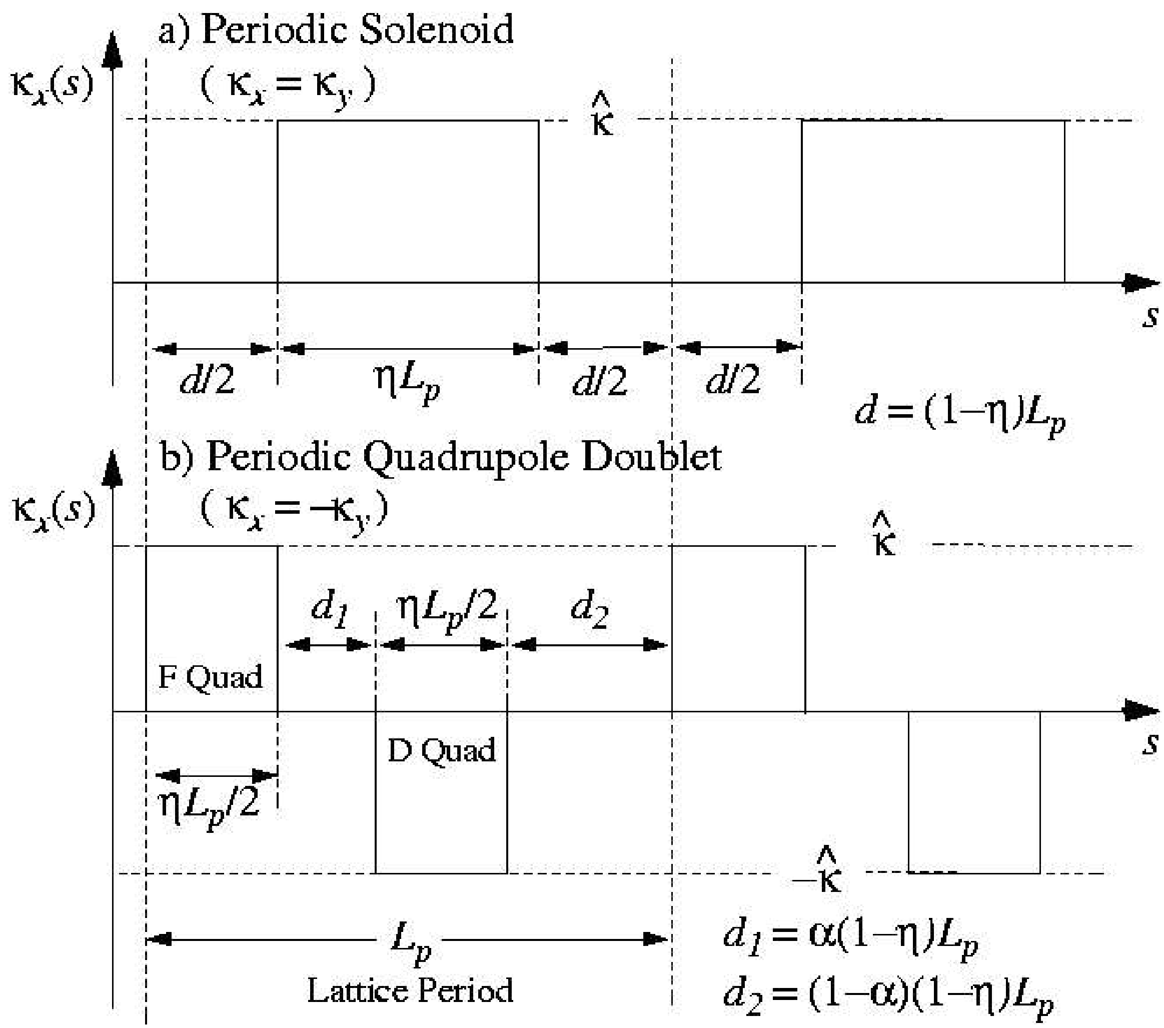
$\overline{Q}, \ \overline{r_x}$ calculated depending on parameterization case

Note: seed iteration is more accurate than continuous focusing limit

Cutoff: Terminate iterations when

$$\text{Max}\left|\frac{r_x^i - r_x^{i-1}}{r_x^i}\right| \leq \text{tol}$$

Example applications- solenoid and quadrupole lattices, treating the focusing functions as piecewise constant



Lattice Period L_p

Occupancy η

$\eta \in [0, 1]$

Solenoid description carried out implicitly in Larmor frame [see Lund and Bukh, PRST- AB7, 024801 (2004)]

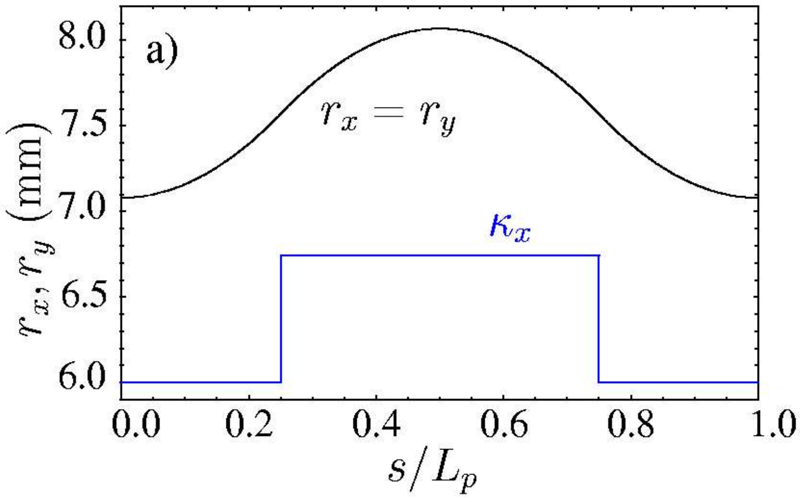
Syncopation Factor α

$\alpha \in [0, 1]$

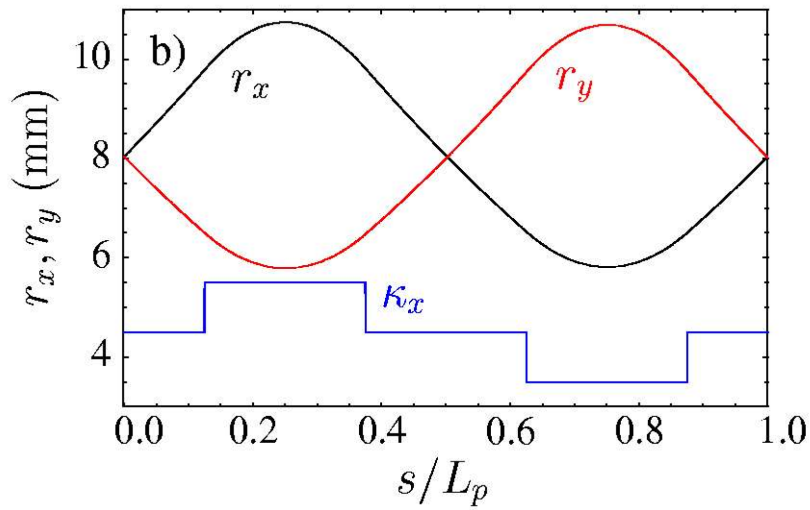
$$\alpha = \frac{1}{2} \implies FODO$$

Typical Matched Solutions

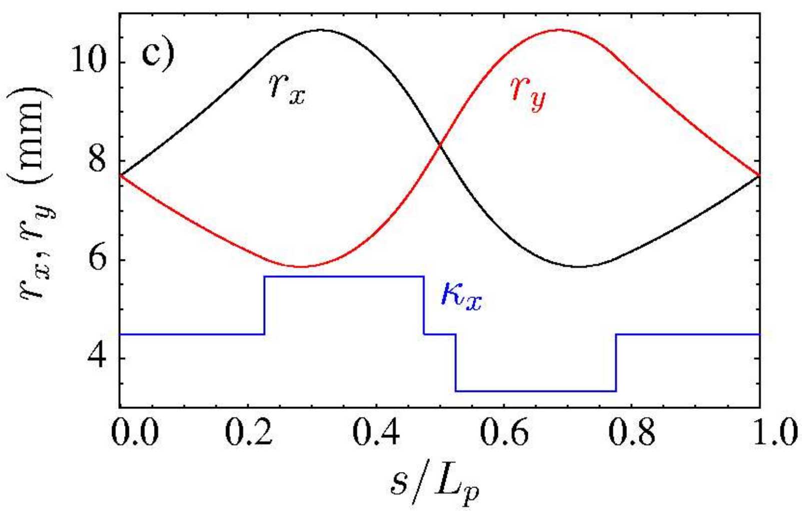
Solenoidal Lattice



FODO Quadrupole Lattice (α = 0.5)



Syncopated Quadrupole Lattice (α = 0.1)



$$
\begin{aligned}
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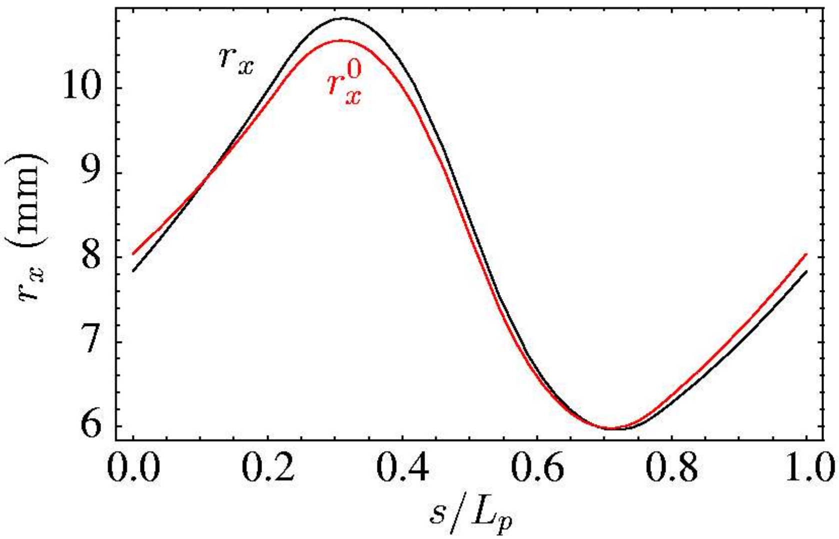
\eta &= 0.5 \\
\sigma_0 &= 80^\circ \\
Q &= 4 \times 10^{-4} \\
\varepsilon &= 50 \text{ mm-mrad}
\end{aligned}
$$

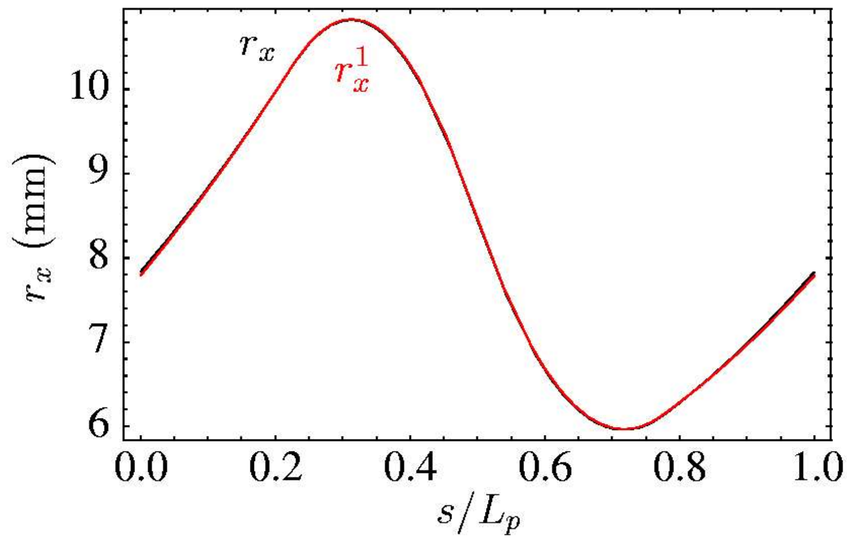
Iterative numerical method converges rapidly to matched solution for all parameterizations with specified σ

Syncopated Quadrupole Lattice

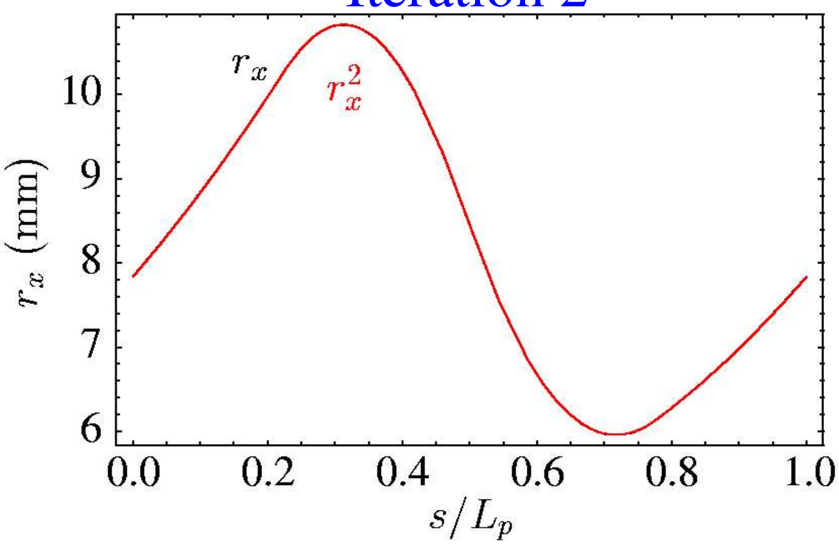
Iteration 0



Iteration 1



Iteration 2



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\end{aligned}
$$

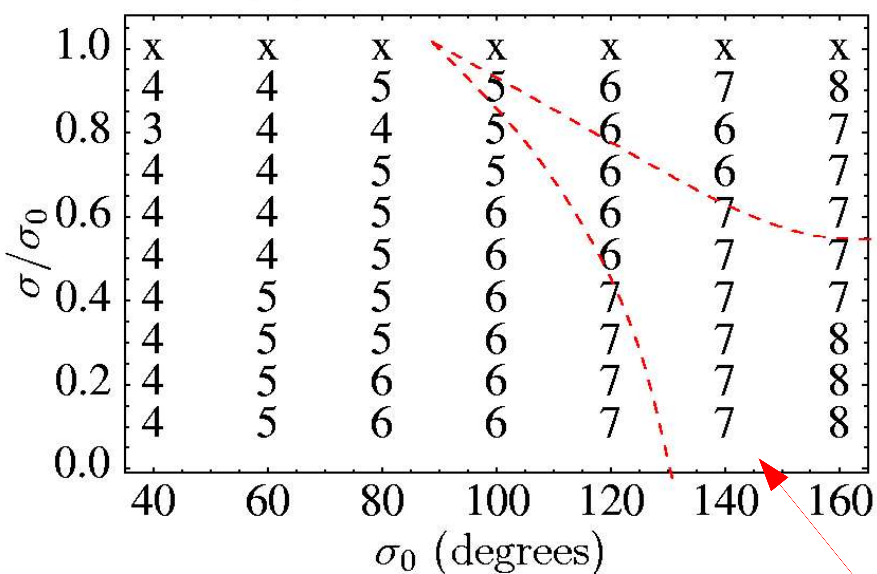
Parameter space plots illustrating the number of iterations necessary to achieve a fractional tolerance of 10^{-6}

Syncopated Quadrupole Lattice

$$L_p = 0.5 \text{ m}, \;\; \eta = 0.5, \;\; \alpha = 0.1$$

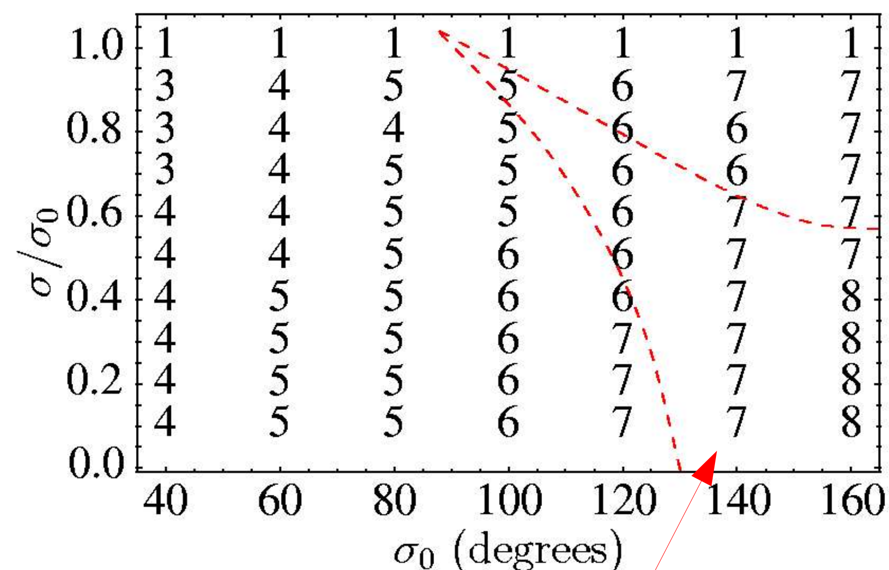
$Q, \;\sigma$ specified

c) Case 1, $Q = 10^{-4}$



$\varepsilon, \;\sigma$ specified

c) Case 2, $\varepsilon = 50$ mm-mrad

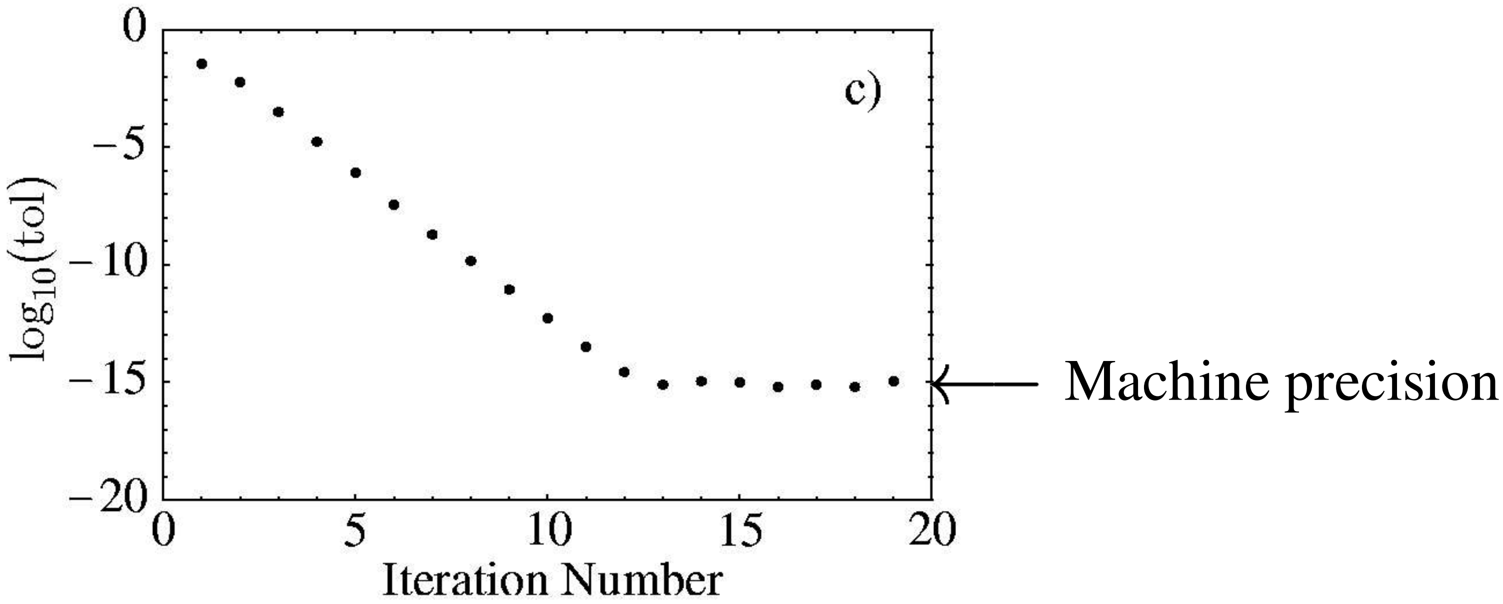


Envelope Instability Bands

Tolerance decreases rapidly toward numerical precision

Syncopated Quadrupole Lattice

$$L_p = 0.5 \text{ m}, \; \eta = 0.5, \; \alpha = 0.1, \; \sigma_0 = 80^\circ$$
$$\sigma/\sigma_0 = 0.2, \; \varepsilon = 50 \text{ mm-mrad}$$



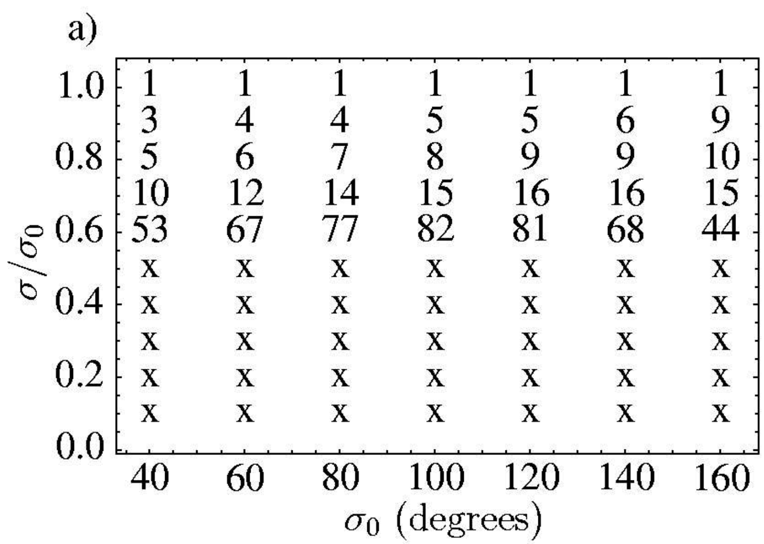
Tolerance decreases more slowly with:

- Increasing undepressed phase advance
- Increasing lattice complexity

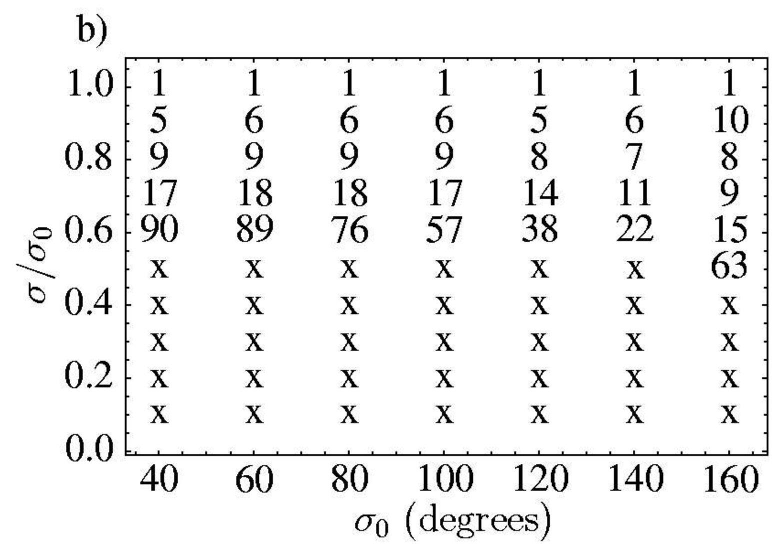
Problem: Simplest implementation of Q, ε parameterization fails over approximately half of the parameter space

$$L_p = 0.5 \text{ m}, \ \eta = 0.5, \ \varepsilon = 50 \text{ mm-mrad}$$

Solenoidal Lattice



FODO Quadrupole Lattice: $\alpha = 0.5$



x = Failure point due to complex σ^i in iterations:

Beam squeezed too hard for given Q; principal orbits overcompensate, grow too large, and yield complex phase advances

We attempted to implement the Q, ε parameterization in the entire parameter space through several methods

1) Calculate the depressed phase advances via previous iteration integral formula

$$\sigma^i = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{[r_x^{i-1}(s)]^2}$$

- Converges systematically to unphysical solutions

2) Vary perveance adaptively

- Raise Q until method fails, lower until method works, then increase adaptively
- Found this only works for very slow increases in Q, leading to many iterations

3) Hybrid Method

- Assume trial σ_x, σ_y values and find consistent values with specified Q and/or ε_x, ε_y using numerical root-finding

Fortunately, the Q, ε parameterization can be extended to the entire parameter space by employing hybrid methods

(Q, ε)/(Q, σ) Hybrid

Find $\sigma_x, \ \sigma_y$ satisfying

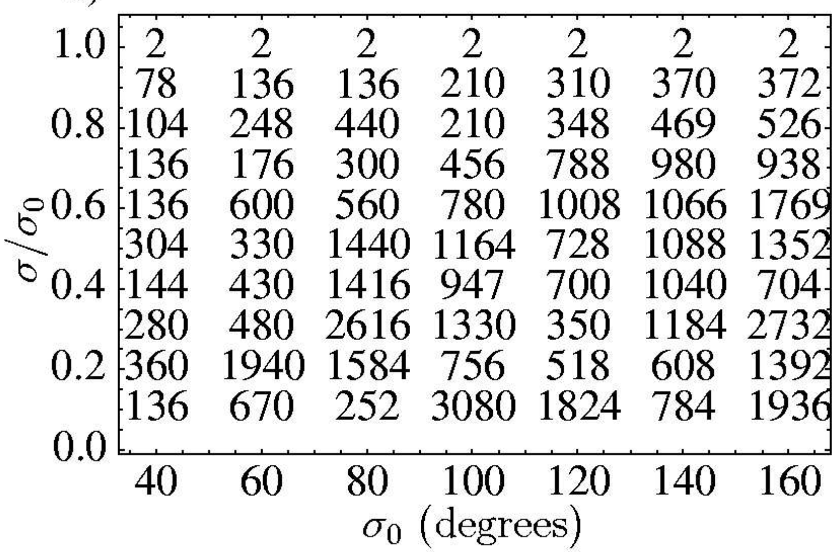
$$\varepsilon_j(\sigma_x, \ \sigma_y) = \varepsilon_j|_{\mathrm{specified}}$$

Then employ Q, σ method

$$Q = 10^{-4}$$

$$L_p = 0.5 \text{ m}, \ \eta = 0.5$$

b)



Conclusions

A new iterative method for generating matched envelope solutions to the KV equations has been developed

- Has a large basin of attraction
- Converges rapidly
- Works over entire parameter space, even in regions of strong instability
- Applicable to all linear lattices without skew coupling
- Straightforward to code

Downside: Direct application of Q, ε parameterization fails in about half of the parameter space

- However, the Q, ε method can be implemented with hybrids

Extra:

- Manuscript submitted to PRST-AB
- Programs and presentation slides (soon) available online