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§9 Transport Limit Scaling Based on the Matched Beam Envelope Equations for Periodic Focusing Channels

The scaling of the maximum beam current, or equivalently, the maximum perveance Q that can be transported at a given energy with a specified focusing technology and lattice is of critical importance in designing optimal transport and acceleration channels. Needed equations can be derived from approximate analytical solutions to the matched beam envelope equations for a given lattice. Alternatively, numerical solutions of the envelope equations can be evaluated. But analytical solutions are preferable to understand scaling and enable rapid evaluation of design tradeoffs.

As a practical matter, equations derived must be applied to regimes where technology is feasible.

- Magnet Field Limits
- Electron breakdown
- Vacuum

⋮

Transport limits are inextricably linked to technology. Moreover, higher order stability constraints etc. must also be respected. Treatments of these topics are beyond the scope of this class. Here we present simplified treatments to highlight issues and methods.

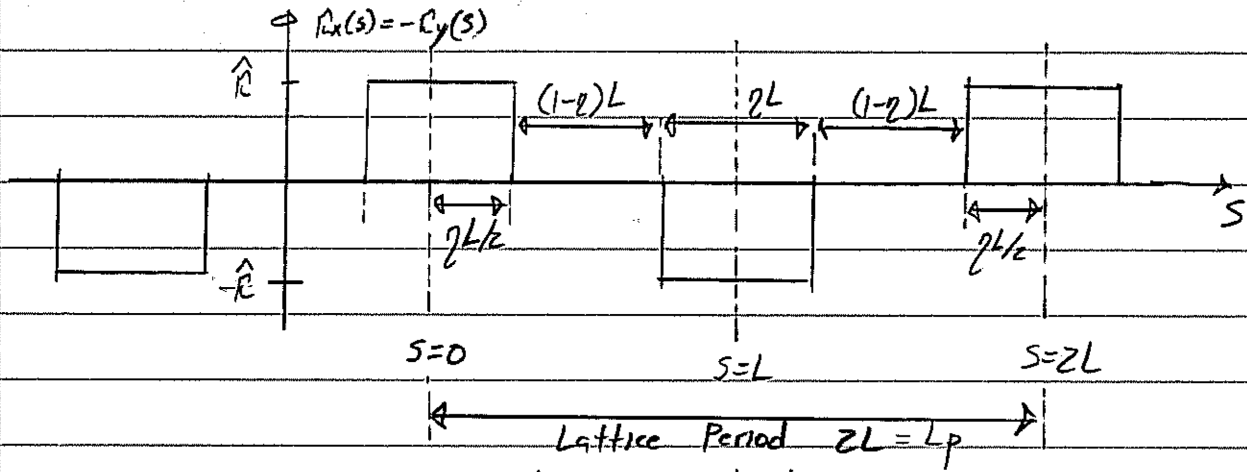
First review an example sketched by J. J. Barnard in the Intro. lectures.

Transport Limits of a Periodic FODO Quadrupole Transport Channel

$$r_{xm}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_{xm}' + \kappa_x r_{xm} - \frac{2Q}{r_{xm}+r_{ym}} - \frac{\varepsilon_x^2}{r_{xm}^3} = 0$$

$$r_{ym}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_{ym}' - \kappa_x r_{ym} - \frac{2Q}{r_{xm}+r_{ym}} - \frac{\varepsilon_y^2}{r_{ym}^3} = 0$$

$$r_{xm}(s+L_p) = r_{xm}(s) \quad ; \quad r_{ym}(s+L_p) = r_{ym}(s)$$



L = Half-Period $\quad L = L_p/2$

η = Quadrupole "occupancy" $\quad 0 < \eta \leq 1$

κ = Focus strength $\qquad \kappa = \begin{cases} \dfrac{qE_q'(s)}{m\gamma_b\beta_b^2c^2} \ ; & \text{Electric} \\ \dfrac{qB_q'(s)}{m\gamma_b\beta_b c} \ ; & \text{Magnetic} \end{cases}$

Expand $\kappa_x(s)$ as a Fourier Series:

$$\boxed{\begin{aligned} \kappa_x(s) &= \sum_{n=1}^{\infty} \kappa_n \cos\left(\frac{n\pi s}{L}\right) \\ \kappa_n &= \frac{1}{L}\int_0^{2L} \kappa_x(s)\cos\left(\frac{n\pi s}{L}\right) = \frac{2\hat{\kappa}}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi\eta}{2}\right) \end{aligned}}$$

And expand the periodic matched beam envelope by:

$$r_{xm} = r_b\left[1 + \Delta\cos\left(\frac{\pi s}{L}\right)\right] + \sum_{n=2}^{\infty}\Delta_{xn}\cos\left(\frac{n\pi s}{L}\right)$$

$$r_{ym} = r_b\left[1 - \Delta\cos\left(\frac{\pi s}{L}\right)\right] + \sum_{n=2}^{\infty}\Delta_{yn}\cos\left(\frac{n\pi s}{L}\right)$$

$r_b = \text{const} =$ avg. beam radius.

$|\Delta| = \text{const} \ll 1$

$\Delta_{\substack{x\\y}n}$ constants with $|\Delta_{\substack{x\\y}n}| \ll |\Delta|$

Take:

- $(\gamma_b\beta_b)' = 0 \Rightarrow$ coasting beam
- $\varepsilon_x = \varepsilon_y = \varepsilon \Rightarrow$ isotropic beam

and insert these expansions in the envelope equations.

Neglect:

- All terms $\mathcal{O}(\Delta^2)$ and higher
- Fast oscillation terms $\sim \cos\left(\frac{n\pi s}{L}\right)$ with $n \geq 2$.

to obtain two independent constraint equations:

Avg (const):
$$\frac{2\Delta\hat{\kappa}}{\pi} r_b \sin\left(\frac{\pi\eta}{2}\right) - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

Fundamental $\propto \cos\left(\frac{\pi s}{L}\right)$:
$$-\Delta\left(\frac{\pi}{L}\right)^2 r_b + \frac{4\hat{\kappa} r_b}{\pi}\sin\left(\frac{\pi\eta}{2}\right) + \frac{3\Delta\varepsilon^2}{r_b^3} = 0$$

These equations can be solved to express the maximum beam edge excursion as

$$\boxed{\mathrm{Max}[r_{xm}] = \mathrm{Max}[r_{ym}] \simeq r_b(1+|\Delta|) = r_b\left\{1 + \frac{4|\hat{\kappa}|L^2 \sin\left(\frac{\pi\eta}{2}\right)}{\pi^3\left(1 - 3\frac{L^2\varepsilon^2}{\pi^2 r_b^4}\right)}\right\}}$$

and the beam Perveance as:

$$\boxed{Q = \frac{2}{\pi^2}\left[\frac{\sin\left(\frac{\pi\eta}{2}\right)}{\left(\frac{\pi\eta}{2}\right)}\right]^2 \frac{\eta^2\hat{\kappa}^2L^2r_b^2}{\left(1 - \frac{3L^2\varepsilon^2}{\pi^2 r_b^4}\right)} - \frac{\varepsilon^2}{r_b^2}}$$

Design Strategy:

1) Choose a lattice period $2L$, quadrupole occupancy η, and clear machine "pipe" radius r_p consistent with focusing technology employed.
2) Choose the largest possible focus strength $\hat{\kappa}$ (quadrupole current or voltage excitation) for beam energy with undepressed particle phase advance:

$$\boxed{\sigma_0 \lesssim 80^\circ / \text{period}, \quad \text{"Tiefenback Limit"}}$$

- Larger phase advances correspond to stronger focus and smaller beam cross-sectional area for given values of Q, ε.

- Weaker phase advance suppresses various particle, envelope, and collective instabilities for reliable transport: [Ref: M.G. Tiefenback, "Space-Charge Limits on the Transport of Ion Beams," UC Berkeley Ph.d Thesis, 1986 LBL-22465

3) Choose a suitable beam-edge to aperture clearance factor:

$$\boxed{\begin{gathered} r_p = \text{Max}[r_{xm}] + \Delta_p \\ \Delta_p = \text{Clearance.} \end{gathered}}$$

to allow for misalignments, limit scraping of halo particles outside the beam core, reduce image charges, gas propagation times from the aperture to the beam, and other nonideal effects.

4) Evaluate choices made using higher-order theory, numerical simulations etc.. Iterate choices made to reoptimize when evaluating cost.

Effective application of this formulation requires extensive practical knowledge:

- Nonideal effects: collective instabilities, halo, electron and gas interactions (species contamination), ...
- Technology limits: Voltage breakdown, vacuum, superconducting magnets, ...

In practice, for intense beam transport, the emittance terms $\varepsilon_x, \varepsilon_y$ can often be neglected for the purpose of obtaining simpler scaling relations that are more easily understood.

$$\lim_{\varepsilon_x \to 0} \delta_x = 0$$

$$\lim_{\varepsilon_y \to 0} \delta_y = 0$$

$\Rightarrow$ Full space charge depression

In this limit $Q \to Q_{max}$, the maximum transportable perveance.

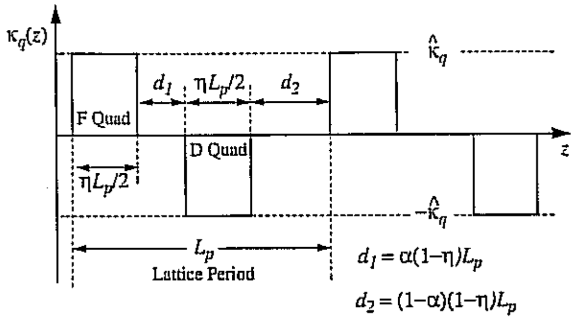
For our previous example for FODO quadrupoles, the $\varepsilon \to 0$ limit obtains:

$$\lim_{\varepsilon \to 0} \mathrm{Max}[r_{xm}] = r_b \left\{ 1 + \frac{4|\hat{\kappa}|L^2}{\pi^3} \sin\left(\frac{\pi}{2}\eta\right) \right\}$$

$$\lim_{\varepsilon \to 0} Q = Q_{max} = \frac{2}{\pi^2} \left[\frac{\sin\left(\frac{\pi}{2}\eta\right)^2}{\left(\frac{\pi}{2}\eta\right)} \right] \eta^2 \hat{\kappa}^2 L^2 r_b^2$$

Unfortunately, the method introduced before are inadequate for lattices with lesser degrees of symmetry such as syncopated quadrupole doublet lattices. However, methods introduced by Lee [E. P. Lee, Physics of Plasmas **9**, 4301 (2002)] can be applied in this situation and also obtain more accurate results. It is beyond the scope of this class to carry out derivations with these methods, but we summarize results derived.

Quadrupole Doublet Lattice



Denote:

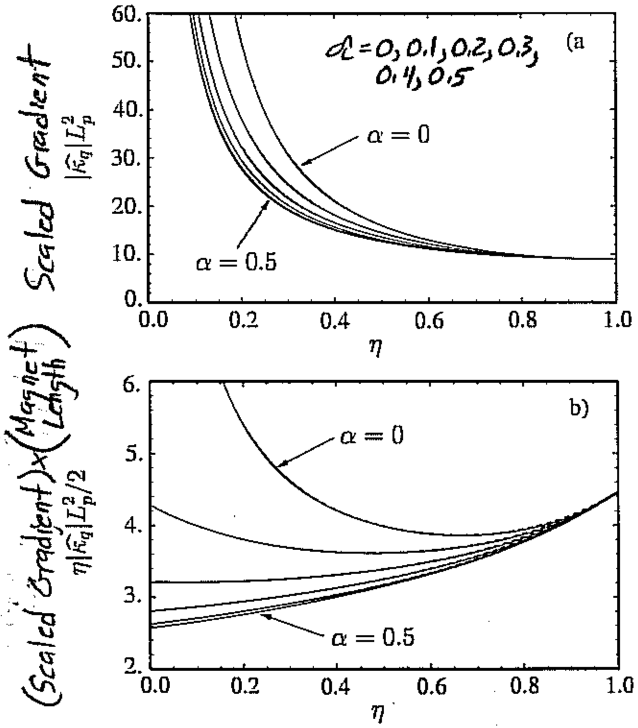
Avg Radius: $$\overline{r_m} = \int_0^{L_p} \frac{ds}{L_p}\, r_{xm}(s) = \int_0^{L_p} \frac{ds}{L_p}\, r_{ym}(s)$$

Max Excursion in period: $$\mathrm{Max}[r_m] \equiv \mathrm{Max}[r_{xm}, r_{ym}]$$

Phase Advance

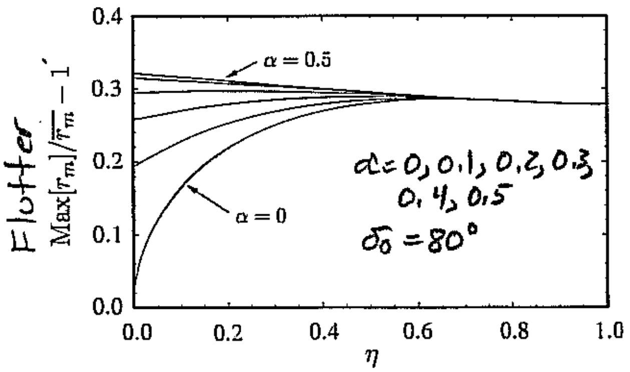
$$\cos\sigma_0 = 1 - \frac{(\eta\widehat{\kappa_q}L_p^2)^2}{32}\left[\left(1-\frac{2}{3}\eta\right) - 4\left(\alpha-\frac{1}{2}\right)^2(1-\eta)^2\right].$$

For $\sigma_0 = 80^\circ$



Envelope Flutter

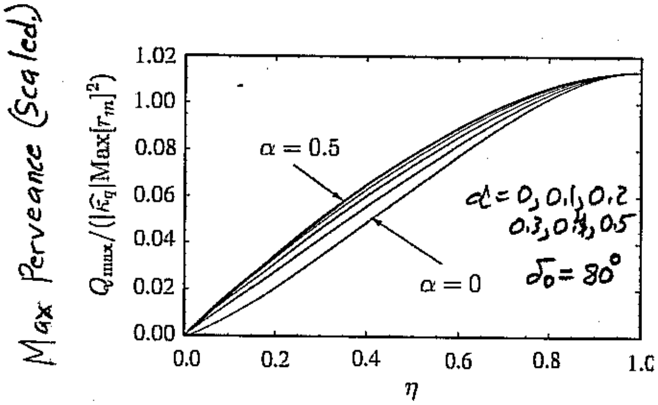
$$\frac{\mathrm{Max}[r_m]}{\overline{r_m}} - 1 = \frac{(1-\cos\sigma_0)^{1/2}(1-\eta/2)[1-4(\alpha-1/2)^2(1-\eta)^2]}{2^{3/2}\left[(1-2\eta/3)-4(\alpha-1/2)^2(1-\eta)^2\right]^{1/2}}.$$



Relations Connecting Max Transportable Perveance Q_{max} and Lattice Parameters

$$
\begin{aligned}
Q_{\max} &= \frac{(1-\cos\sigma_0)^{1/2}}{2^{3/2}} \frac{\eta[(1-2\eta/3) - 4(\alpha-1/2)^2(1-\eta)^2]^{1/2}}{(\mathrm{Max}[r_m]/\overline{r_m})^2} |\widehat{\kappa_q}| \,\mathrm{Max}[r_m]^2 \\
&= \frac{(1-\cos\sigma_0)^{1/2}}{2^{3/2}} \frac{\eta[(1-2\eta/3) - 4(\alpha-1/2)^2(1-\eta)^2]^{1/2}}{\left\{1 + \frac{(1-\cos\sigma_0)^{1/2}(1-\eta/2)[1-4(\alpha-1/2)^2(1-\eta)^2]}{2^{3/2}[(1-2\eta/3)-4(\alpha-1/2)^2(1-\eta)^2]^{1/2}}\right\}^2} |\widehat{\kappa_q}| \,\mathrm{Max}[r_m]^2.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{Max}[r_m]}{L_p} &= \sqrt{\frac{Q_{\max}}{2(1-\cos\sigma_0)}} \left(\frac{\mathrm{Max}[r_m]}{\overline{r_m}}\right) \\
&= \sqrt{\frac{Q_{\max}}{2(1-\cos\sigma_0)}} \left\{1 + \frac{(1-\cos\sigma_0)^{1/2}(1-\eta/2)[1-4(\alpha-1/2)^2(1-\eta)^2]}{2^{3/2}\,[(1-2\eta/3) - 4(\alpha-1/2)^2(1-\eta)^2]^{1/2}}\right\},
\end{aligned}
$$



Derivation and aplication of scaling relations can be complicated. They are often applied in systems codes to generate plots that can be interpreted more readily.