Problem 1

Consider a round uniform ion beam with a current of 1 ampere, composed of Hg^+ ions (atomic mass A =200), a kinetic energy of 2 MeV, a beam radius of 2 cm and normalized emittance of 1 mm-mrad.

Calculate for these beam parameters (to 1 or 2 significant figures):

a) $\beta = v_0/c$ (assume non-relativistic beam)
b) n = number density of ions in beam
c) kT = transverse temperature (express in eV)
d) λ_D = transverse Debye length
e) Q = generalized perveance
f) Λ = plasma parameter
g) $\Delta\phi$ = potential difference between center and edge of beam.

For reference:

e = 1.6 x 10^{-19} C [proton charge]
k_B= 1.38 x 10^{-23} J/K [Boltzmann's constant]
ε_0 = 8.854 x 10^{-12} F/m [permittivity of free space]
c = 3 x 10^8 m/s [speed of light in free space]
m_{amu} = 1.66 x 10^{-27} kg [atomic mass unit]
$m_{amu}c^2$ = 931.1 x 10^6 eV [atomic mass unit in eV]

Problem 2

Show that:

$$\left\langle r\frac{\partial\phi}{\partial r}\right\rangle = -\frac{\lambda}{4\pi\varepsilon_0}$$

for a charge distributionin which $\rho(r,\theta) = \rho(r)$ only.

Here λ = line charge density= $\int_0^\infty 2\pi r\rho(r)dr$

$$\langle g\rangle = \frac{1}{\lambda}\int_0^\infty g(r)2\pi r\rho(r)dr$$

where g is any beam quantity that is a function of r only.

Problem 3

Let the equation of motion for a single particle be:

$$x'' = -\alpha(s)x^n$$

Here x is the usual transverse coordinate and s is the longitudinal coordinate. $\alpha(s)$ is a coefficient that depends only on s.

Calculate the derivative with respect to s of the square of the emittance:

$$\varepsilon^2 = 16\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)$$

Express $\frac{d\varepsilon^2}{ds}$ in terms of $\langle x^2\rangle, \langle xx'\rangle, \langle x' x^n\rangle$, and $\langle x^{n+1}\rangle$.

For what value of n is $\frac{d\varepsilon^2}{ds}$ identically zero?

Problem 4 (TPD 1) - Lamor Frame

For a uniform solenoidal channel:

$$B_z^a(s) = B_0 = const.$$

with no acceleration

$$\gamma_b \beta_b = const.$$

and an axisymmetric ($\partial/\partial\theta = 0$) beam with

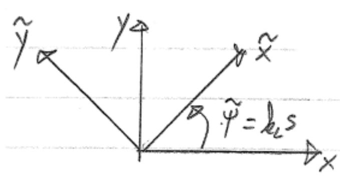
$$\frac{\partial \phi}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\vec{x}_\perp}{r} \qquad r = \sqrt{x^2 + y^2}$$

the particle equations of motion reduce to:

$$x'' = \frac{qB_0}{m\gamma_b\beta_b c}y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{x}{r}$$

$$y'' = -\frac{qB_0}{m\gamma_b\beta_b c}x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{y}{r}$$

a) Parralel steps taken in the class notes to transform the quations of motion to a co-rotating frame:



k_L = const = Larmor wavenumber

$$\tilde{x} = x\cos(k_l s) + y\sin(k_l s)$$

$$\tilde{y} = -x\sin(k_l s) + y\cos(k_l s)$$

Find an expression for k_l to reduce the equations of motion to the decoupled form:

$$\tilde{x}'' + \hat{k}\tilde{x} = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r}$$

$$\tilde{y}'' + \hat{k}\tilde{y} = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r}$$

and identify $\hat{k} = const.$

Hint:

The transformation can be carried out directly. But you may find the algreba simpler using complex coordinates as in the class notes:

$$z = x + iy \qquad i = \sqrt{-1} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$\tilde{z} = \tilde{x} + i\tilde{y}$$

b) If the direction of the magnetic field is reversed:

$$B_0 \to -B_0$$

how will the dynamics be influenced?

c) Neglect space-charge:

$$\phi = 0$$

and sketch a typical orbit in the rotating Lamor frame. Will this orbit appear more complicated in the Laboratory frame? Why?

Bonus:

Sketch the orbit taking advantage of simple choices of inital conditions that can always be made through choice of coordinates.

Problem 5 (TPD 14) - Lamor Frame

a) From the Lorentz Force equation, show that a static magnetic field $\vec{B}^a$ cannot change the kinetic energy of a particle; $E = (\gamma - 1)mc^2 = const.$

$$m\frac{d}{dt}\left(\gamma\vec{\beta}\right) = q\vec{\beta}\times\vec{B}^a \quad \text{Lorentz Force Equation}$$
$$\gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}} \qquad \vec{\beta} = \frac{1}{c}\frac{d\vec{x}}{dt}$$

Here γ and $\vec{\beta}$ are exact (unexpanded) forms that should not be confusd with γ_b and $\beta_b\hat{z}$.

b) In Class, it was shown for a solenoid magnet with azimuthal symmetry ($\frac{\partial}{\partial\theta} = 0$), that the magnetic field can be expanded in terms of the on-axis field as:

$$B_r^a(r,z) = \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu!(\nu-1)!}\frac{\partial^{2\nu-1}B_{z0}(z)}{\partial z^{2\nu-1}}\left(\frac{r}{2}\right)^{2\nu-1}$$
$$B_z^a(r,z) = B_{z0}(z) + \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{(\nu!)^2}\frac{\partial^{2\nu}B_{z0}(z)}{\partial z^{2\nu}}\left(\frac{r}{2}\right)^{2\nu}$$
$$B_{z0}(z) \equiv B_z^a(r=0,z) \quad \text{on-axis field}$$

Take $E \simeq E_b$ and apply the paraxial equations of motion (see Sec. S1G class notes) to show that if nonlinear applied force terms are dropped ($\propto x^2, xy, xy'$, etc.), the equations of motion are:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' - \frac{B'_{z0}}{2[B\rho]}y - \frac{B_{z0}}{[B\rho]}y' = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$
$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \frac{B'_{z0}}{2[B\rho]}x + \frac{B_{z0}}{[B\rho]}x' = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$
$$[B\rho] \equiv \frac{\gamma_b\beta_b mc}{q} \qquad B'_{z0} = \frac{\partial B_{z0}(z)}{\partial z}$$

c) *Qualitative only*: If there are no axial acceleration fields, we take $\gamma_b\beta_b = const.$, $[B\rho] = const.$, are the results of part b) inconsistent with part a)? If so, could they still be OK to use?

d) Show if we take $\vec{B}^a = \nabla\times\vec{A}$, we can generate the linear field components:

$$B_r^a = -\frac{1}{2}\frac{\partial B_{z0}}{\partial z}r$$
$$B_z^a = B_{z0}$$

from $\vec{A} = \frac{\hat{\theta}}{2}B_{z0}r$

$$\nabla\times\vec{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \frac{\hat{z}}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

e) Paraxial approximate the canonincal angular momentum

$$P_\theta = \left[\vec{x} \times \left(\vec{p} + q\vec{A}\right)\right] \cdot \vec{z}$$

as

$$P_\theta \equiv m\gamma_b\beta_b c(xy' - yx') + q\frac{B_{z0}}{2}\left(x^2 + y^2\right)$$

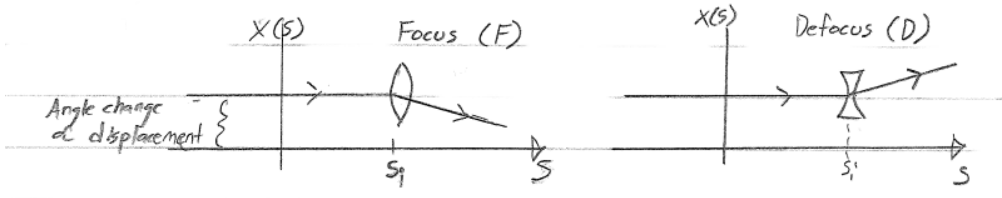
Use equation of motion in part b) to show that

$$\frac{d}{dz}P_\theta = 0 \;\Rightarrow\; P_\theta = const.$$

for a beam with $\phi = \phi(r)$.

Problem 6 (TPD 4)

A thin lense changes the angle of a particle trajectory but not the coordinate: This action can be



specified by transfer matrices applied at $s = s_i$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M(s|s_i) \begin{pmatrix} x_i \\ x_i' \end{pmatrix}$$

Focusing:

$$M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad f > 0$$

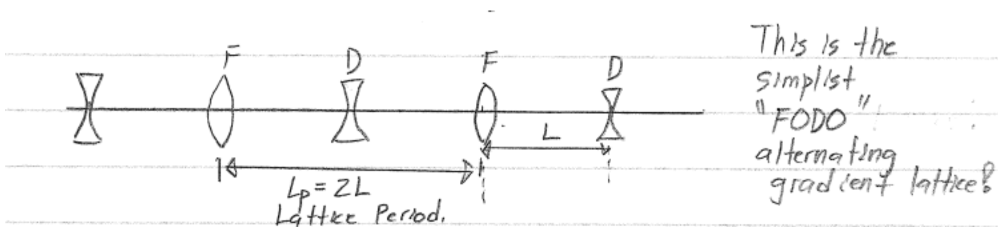
Defocusing:

$$M_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad f > 0$$

From TPD Problem 2 a), free space drift of length L has a transport matrix:

$$M_0 = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Consider a lattice of period $2L$ made up of equally spaced F and D lenses with eqaul values of f.



a) Use the transfer matrix analysis developed in class to find the range of f for which the particle orbit is stable.

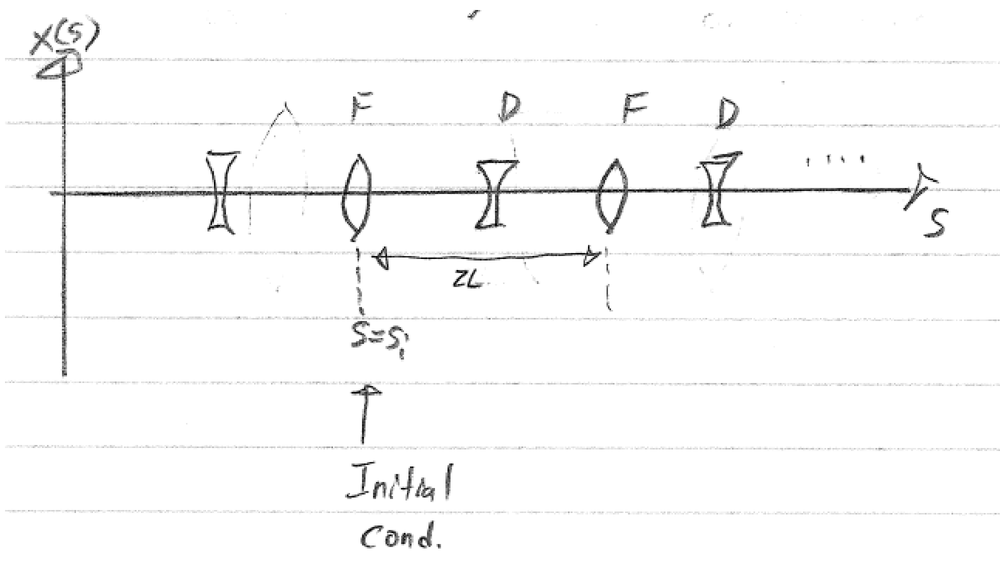
b) Calculate $\cos\sigma_0$ where σ_0 is the particle phase advance.

c) For the case of f chosen to correspond to the stability limit, sketch the motion of a particle initial conditions:

$$\lim_{s \to s_i} x(s) = x_0$$

$$\lim_{s \to s_i} x'(s) = x_o/L$$

where $s = s_i$ is the axial location of a focusing thin lens kick, and $s \to s_i$ is just before the kick. Sketch the particle orbit for focusing strength slightly larger than the stability limit. Superimpose the orbit sketch on a diagram of the lattice (see below):



X(S)
F
D
F
D
S
2L
S=S_i
Initial
Cond.