PROBLEM SET 3

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25 Points

1. CONSIDER A DIODE OF VOLTAGE V_0 AND gap length d.
Let a current density J be composed of two species such that $J_1 = \alpha J$ and $J_2 = (1-\alpha) J$ (so that $J = J_1 + J_2$).

Let the mass of ions in species 1 be m_1 and those of species 2 be m_2. What is the effective mass m_{eff} that should be used in the resulting Child Langmuir Law:

$$J = \frac{4}{9} \varepsilon_0 \left(\frac{2q}{m_{eff}}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$$

(Both ion species have charge q).

2. 25 Points

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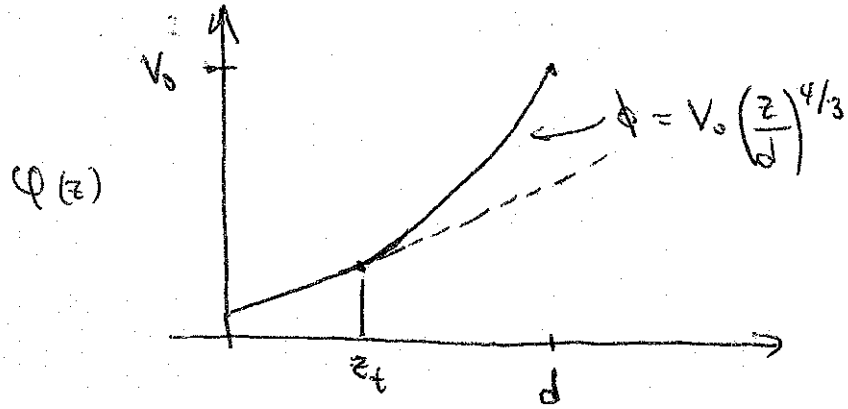
PROBLEM 2 CONSIDER THE FOLLOWING DIODE OF voltage V_0 AND LENGTH d.

SUPPOSE AT SOME TIME $t_p > \tau \equiv \frac{3d}{\left(\frac{2qV_0}{m}\right)^{1/2}}$ THE CURRENT IS ABRUPTLY TURNED OFF. WHAT VOLTAGE WAVEFORM IS REQUIRED TO ENSURE THAT THE ELECTRIC FIELD AT THE TAIL OF THE PULSE IS IDENTICAL TO THE CHILD-LANGMUIR ELECTRIC FIELD?



Problem 3 (TED 1)

Consider a $\perp$ unbunched ion beam described by

$$f_\perp(\vec{x}_\perp, \vec{x}'_\perp, s) \sim \text{single particle distribution satisfying Vlasov's equation.}$$

$$H_\perp = \frac{1}{2}\vec{x}'^2_\perp + \frac{\hat{\kappa}_x(s)x^2}{2} + \frac{\hat{\kappa}_y(s)y^2}{2} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

$$\nabla^2_\perp \phi = \frac{-q}{\epsilon_0}\int d^2x' f_\perp(\vec{x}_\perp, \vec{x}'_\perp, s)$$

$$\phi(r = r_p) = 0 \quad \text{Grounded pipe boundary condition. } r_p = \text{pipe radius.}$$

a) What are the first-order particle equations of motion for $\frac{d}{ds}\vec{x}_\perp$ and $\frac{d}{ds}\vec{x}'_\perp$ derived from $H_\perp$?

b) Using the results of part a), what is the 2nd-order particle equation of motion for $\frac{d^2}{ds^2}\vec{x}_\perp$?

c) Use the particle equations of motion to calculate $\frac{d}{ds}$ of the single-particle Hamiltonian $H_\perp$ and the "angular momentum"

$$P_\theta \equiv xy' - yx'$$

I.e.,

$$\frac{d}{ds}H_\perp = ?$$
$$\frac{d}{ds}P_\theta = ?$$

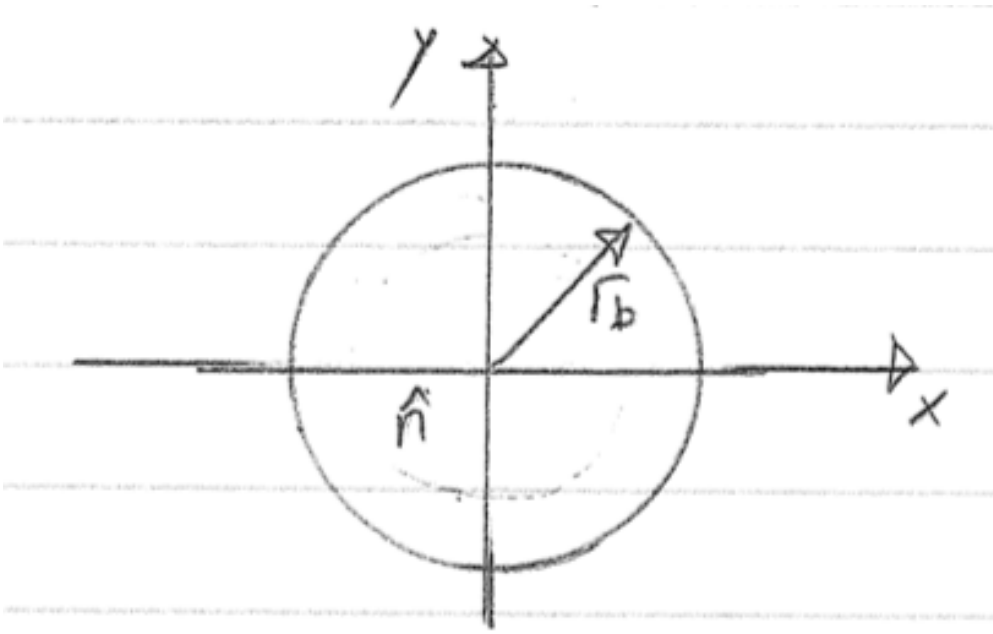
d) Use the expression of part c) to show that for $\hat{\kappa}_x = const.$, $\hat{\kappa}_y = const.$, and $f_\perp = f_\perp(H_\perp)$ that $H_\perp = const.$

Here $f(H_\perp)$ can be any function of $H_\perp$ with $f(H_\perp) \geq 0$.

e) Use the expresions of part c) to show that for axisymmetric beams $\frac{\partial}{\partial\theta} = 0$ (θ = azimuthal angle) with $\hat{\kappa}_x = \hat{\kappa}_y = \hat{\kappa}(s)$ and $f_\perp = f_\perp(H_\perp)$ that $P_\theta = const.$

Problem 4 (TED 2)

Consider a uniform density beam in free-space with circular cross-section, edge radius r_b, and uniform in z ($\frac{\partial}{\partial z} = 0$).



$$
\begin{aligned}
r_b &= \text{beam edge radius} \\
r &= \sqrt{x^2 + y^2} \\
\hat{n} &= const. \\
\lambda &= q\hat{n}\pi r_b^2 = \text{line-charge}
\end{aligned}
$$

a) *Directly* construct the solution to Poisson's equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{-q}{\epsilon_0} = \begin{cases} \hat{n}, & r < r_b \\ 0, & r > r_b. \end{cases}$$

satisfying

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \text{and} \quad \lim_{r\to\infty} -\frac{\partial \phi}{\partial r} = \frac{\lambda}{2\pi\epsilon_0 r}$$

b) Take derivatives of the interior solution ($r < r_b$) in part a) to obtain formulas for

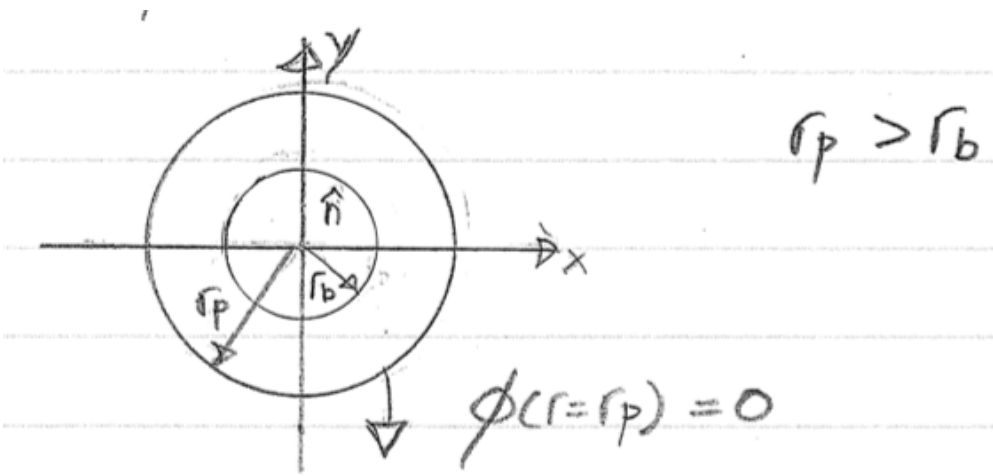
$$
\begin{aligned}
E_x &= -\frac{\partial \phi}{\partial x} \\
E_y &= -\frac{\partial \phi}{\partial y}
\end{aligned}
$$

c) Show that the ellipsoidal beam formulas

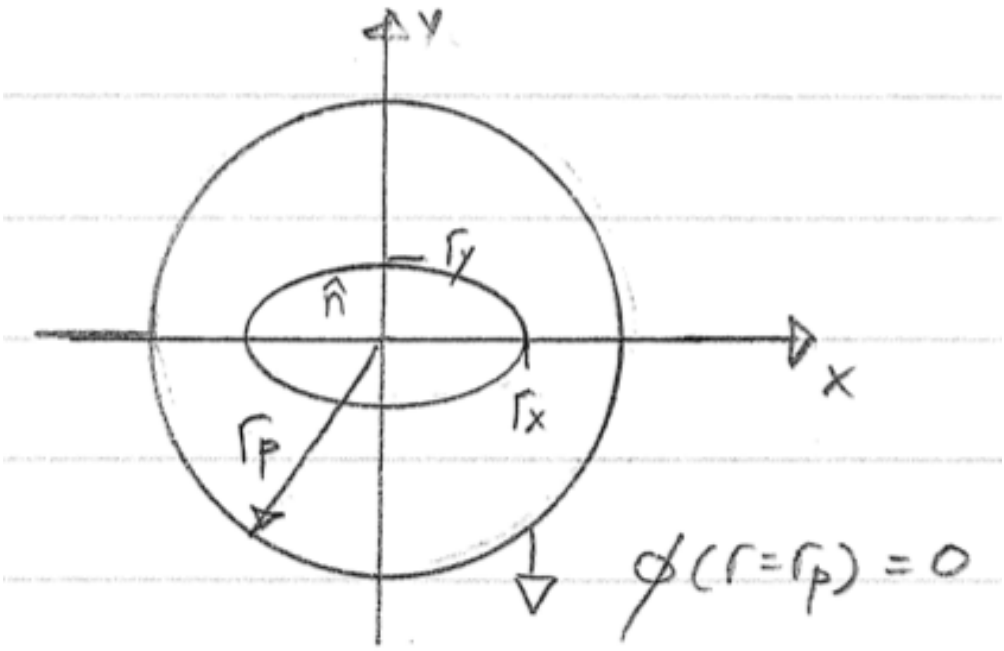
$$E_x = -\frac{\partial \phi}{\partial x} = \frac{\lambda}{\pi \epsilon_0} \frac{x/r_x}{r_x + r_y}$$
$$E_y = -\frac{\partial \phi}{\partial y} = \frac{\lambda}{\pi \epsilon_0} \frac{y/r_y}{r_x + r_y}$$

reduce to the results in part c) for a round beam with $r_x = r_y = r_b$.

d) Would a grounded, conducting pipe of radius $r = r_p > r_b$ change the answer in part b)?



e) Would a grounded conducting pipe of radius $r = r_p > r_x, r_y$ change the fields calculated in class for the elliptical beam case with $r_x \neq r_y$? (no need to calculate any changes, just explain answer)


y
x
r_y
$\hat{n}$
r_x
r_p
$\phi(r=r_p)=0$

Problem 5 (TED 3)

For a KV-distribution:

$$n(x,y) = \int dx'dy' f_\perp = \begin{cases} \hat{n}, & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1 \\ 0, & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{cases}$$

Use this result to verify the formulas

$$r_x = 2\langle x^2 \rangle_\perp^{\frac{1}{2}}$$
$$r_y = 2\langle y^2 \rangle_\perp^{\frac{1}{2}}$$

Hint:

Integrals may be more easily caried out if the elliptical integration domain is transformed to a circular domain.

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} = 1 \qquad \text{Elliptical beam edge}$$
$$x = r_x \rho \cos\Psi \quad \rightarrow \quad \rho^2 \cos^2\Psi + \rho^2 \sin^2\Psi = 1$$
$$y = r_y \rho \sin\Psi \qquad \rho^2 = 1 \quad \text{beam edge}$$