Consider a diode of voltage $V_0$ and gap length $d$.

Let a current density $J$ be composed of two species such that $J_1 = \alpha J$ and $J_2 = (1-\alpha) J$ (so that $J = J_1 + J_2$).

Let the mass of ions in species 1 be $m_1$ and those of species 2 be $m_2$. What is the effective mass $m_{\text{eff}}$ that should be used in the resolting Child-Langmuir law: \[
J = \frac{4}{q} E_0 \left( \frac{2q}{m_{\text{eff}}} \right)^{1/2} \frac{V_0^{3/2}}{d^{1/2}} \]

(Both ion species have charge $q$).
Problem 7 Construct the following diode of voltage $V_0$ and length $d$.

Suppose at some time $t_f > t_0 \in \frac{3d}{(2qV_0)^{1/2}}$ the current is abruptly turned off. What voltage waveform is required to ensure that the electric field at the tail of the pulse is identical to the Child-Langmuir electric field?

\[ \varphi = V_0 \left( \frac{z}{d} \right)^{4/3} \]
Problem 3 (TED 1)

Consider a ⊥ unbunched ion beam described by

\[ f_\perp(\vec{x}_\perp, \vec{x}'_\perp, s) \sim \text{single particle distribution satisfying Vlasov’s equation.} \]

\[ H_\perp = \frac{1}{2} \dot{\vec{x}}_\perp^2 + \frac{\dot{\kappa}_x(s)x^2}{2} + \frac{\dot{\kappa}_y(s)y^2}{2} + \frac{q}{m\gamma^3_0\beta^2_0 c^2} \phi \]

\[ \nabla_\perp^2 \phi = -\frac{q}{\epsilon_0} \int d^2x' f_\perp(\vec{x}_\perp, \vec{x}'_\perp, s) \]

\[ \phi(r = r_p) = 0 \quad \text{Grounded pipe boundary condition.} \quad r_p = \text{pipe radius.} \]

(a) What are the first-order particle equations of motion for \( \frac{d}{ds} \vec{x}_\perp \) and \( \frac{d}{ds} \vec{x}'_\perp \) derived from \( H_\perp \)?

(b) Using the results of part a), what is the 2nd-order particle equation of motion for \( \frac{d^2}{ds^2} \vec{x}_\perp \)?

(c) Use the particle equations of motion to calculate \( \frac{d}{ds} \) of the single-particle Hamiltonian \( H_\perp \) and the “angular momentum”

\[ P_\theta \equiv xy' - yx' \]

I.e.,

\[ \frac{d}{ds} H_\perp = ? \]

\[ \frac{d}{ds} P_\theta = ? \]

d) Use the expression of part c) to show that for \( \dot{\kappa}_x = \text{const.}, \dot{\kappa}_y = \text{const.}, \) and \( f_\perp = f_\perp(H_\perp) \) that \( H_\perp = \text{const.} \)

Here \( f(H_\perp) \) can be any function of \( H_\perp \) with \( f(H_\perp) \geq 0 \).

e) Use the expressions of part c) to show that for axisymmetric beams \( \frac{\partial}{\partial \theta} = 0 \) (θ = azimuthal angle) with \( \dot{\kappa}_x = \dot{\kappa}_y = \dot{\kappa}(s) \) and \( f_\perp = f_\perp(H_\perp) \) that \( P_\theta = \text{const.} \).
Problem 4 (TED 2)

Consider a uniform density beam in free-space with circular cross-section, edge radius $r_b$, and uniform in $z$ ($\frac{\partial}{\partial z} = 0$).

- $r_b =$ beam edge radius
- $r = \sqrt{x^2 + y^2}$
- $\hat{n} =$ const.
- $\lambda = q\hat{n}\pi r_b^2 =$ line-charge

a) Directly construct the solution to Poisson’s equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = - \frac{q}{\epsilon_0} = \begin{cases} \hat{n}, & r < r_b \\ 0, & r > r_b. \end{cases}$$

satisfying

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \text{and} \quad \lim_{r \to \infty} - \frac{\partial \phi}{\partial r} = \frac{\lambda}{2\pi \epsilon_0 r}$$

b) Take derivatives of the interior solution ($r < r_b$) in part a) to obtain formulas for

$$E_x = - \frac{\partial \phi}{\partial x}$$
$$E_y = - \frac{\partial \phi}{\partial y}$$
c) Show that the ellipsoidal beam formulas

\[
E_x = -\frac{\partial \phi}{\partial x} = \frac{\lambda}{\pi \varepsilon_0 r_x + r_y} \frac{x}{x/r_x}
\]

\[
E_y = -\frac{\partial \phi}{\partial y} = \frac{\lambda}{\pi \varepsilon_0 r_x + r_y} \frac{y}{y/r_y}
\]

reduce to the results in part c) for a round beam with \(r_x = r_y = r_b\).

d) Would a grounded, conducting pipe of radius \(r = r_p > r_b\) change the answer in part b)?

\[
\Phi \left( \frac{r_p}{\hat{r}} \right) = 0
\]

\[
r_p > r_b
\]

e) Would a grounded conducting pipe of radius \(r = r_p > r_x, r_y\) change the fields calculated in class for the elliptical beam case with \(r_x \neq r_y\)? (no need to calculate any changes, just explain answer)
\[ \phi(\Gamma = \Gamma_P) = 0 \]
Problem 5 (TED 3)

For a KV-distribution:

\[ n(x, y) = \int dx' dy' f_\perp = \begin{cases} \hat{n}, & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1 \\ 0, & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{cases} \]

Use this result to verify the formulas

\[ r_x = 2\langle x^2 \rangle_\perp^{\frac{1}{2}} \]
\[ r_y = 2\langle y^2 \rangle_\perp^{\frac{1}{2}} \]

Hint:
Integrals may be more easily carried out if the elliptical integration domain is transformed to a circular domain.

\[ \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} = 1 \quad \text{Elliptical beam edge} \]
\[ x = r_x \rho \cos \Psi \rightarrow \rho^2 \cos^2 \Psi + \rho^2 \sin^2 \Psi = 1 \]
\[ y = r_y \rho \sin \Psi \quad \rho^2 = 1 \quad \text{beam edge} \]