Problem 1 30 points

(assume non-relativistic equations throughout)
A mass 200 ion beam has an injection energy $qV = 1$ MeV, a pulse duration of 10 $\mu$s, a normalized transverse emittance of 1 mm mrad, and a fractional longitudinal momentum spread $\frac{\Delta p}{p} = 10^{-3}$.

Assume the transverse and longitudinal normalized emittance is conserved, and assume that in the final focus region the beam is neutralized, with a spot size determined by the emittance and chromatic effects only. (Note: the longitudinal normalized emittance $\epsilon_{n,l} \sim \Delta pl$, where $l$ is the length of the bunch).

$$r_{\text{spot}}^2 \approx \frac{\epsilon^2}{\theta^2} + \alpha^2 d^2 \theta^2 \left(\frac{\Delta p}{p}\right)^2$$

let $\alpha = 6$

here $\epsilon$ = the unnormalized emittance, $d$ is the distance between the end of the last magnet and the focal spot, and $\theta$ is the half angle of the convergent beam.
a) What is the optimum focusing angle which minimizes the spot radius (expressed in terms of $\epsilon$, $d$, and $\Delta p/p$)? What is the radius of the spot if the final ion energy were: (Assume $d = 6$ cm and final pulse duration 10 ns)

b) 10 GeV
c) 1 GeV
d) Under the assumption of this problem, show that $r_{\text{spot}} \sim \frac{1}{\beta^n}$ where $n$ is a positive real number, and find $n$.

**Problem 2 (TKS 01) - Moment equations and conservation constraints 35 points**

The non-relativistic Vlasov-equation is:

$$\left\{ \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} + \frac{q}{m} \left[ \vec{E} + \vec{v} \times \vec{B} \right] \cdot \frac{\partial}{\partial \vec{v}} \right\} f(\vec{x}, \vec{v}, t) = 0$$

Define a fluid density $n$ and a fluid flow velocity $\vec{V}$ by

$$n(\vec{x}, t) = \int d^3 v f(\vec{x}, \vec{v}, t)$$

$$n(\vec{x}, t)\vec{V}(\vec{x}, t) = \int d^3 v \vec{v} f(\vec{x}, \vec{v}, t)$$

a) Operate on the Vlasov equation with $\int d^3 v \ldots$ to derive the continuity equation:

$$\frac{\partial}{\partial t} n(\vec{x}, t) + \frac{\partial}{\partial \vec{x}} \cdot \left[ n(\vec{x}, t)\vec{V}(\vec{x}, t) \right] = 0$$

b) Can the continuity equation be solved by itself if you specify the initial density field $n(\vec{x}, t = 0)$? Why?

c) Operate on the Vlasov equation with $\int d^3 v \vec{v} \ldots$ to derive the fluid force equation.

$$\frac{\partial}{\partial t} \left( n\vec{V} \right) + \nabla \cdot \left( n\langle \vec{v}\vec{v} \rangle_v \right) = \frac{q}{m} n \left( \vec{E} + \vec{V} \times \vec{B} \right)$$

with $\langle \vec{v}\vec{v} \rangle_v \equiv \frac{\int d^3 v \vec{v}\vec{v} f}{\int d^3 v f}$

Defining a pressure tensor as

$$P_v = m \int d^3 v (\vec{v} - \vec{V})(\vec{v} - \vec{V}) f(\vec{x}, \vec{v}, t)$$

$$= mn\langle \vec{v}\vec{v} \rangle_v - mn\vec{V}\vec{V},$$
the fluid force equation can be expressed as

$$\frac{\partial}{\partial t} \vec{V} + \vec{V} \cdot \frac{\partial}{\partial \vec{x}} \vec{V} = \frac{q}{m} \left( \vec{E} + \vec{V} \times \vec{B} \right) - \frac{1}{mn} \frac{\partial}{\partial \vec{x}} \cdot \vec{P}.$$

This form is often used in fluid/plasma analysis.

d) If the continuity and force equation derived in parts a) and c) are analyzed, can they be solved in principle if you specify the initial density field $n(\vec{x}, t = 0)$ and the velocity field $\vec{V}(\vec{x}, t = 0)$? Why? Does the answer change if we assume a cold initial beam with $\vec{P} = 0$? Why?

e) Let $G(f)$ be some smooth, differentiable function of $f$, satisfying $G(f \to 0) = 0$. Show that

$$\int d^3x \int d^3v G(f) = \text{const.}$$

This so-called “generalized entropy” measure with $G$ specified can be used to check Vlasov simulations. For example:

- $G(f) = f : \int d^3x \int d^3v f = \text{const.} \rightarrow \text{charge conservation}$
- $G(f) = f^2 : \int d^3x \int d^3v f^2 = \text{const.} \rightarrow \text{“entropy” conservation}$
- $G(f) = f \ln f : \int d^3x \int d^3v f \ln f = \text{const.} \rightarrow \text{entropy conservation}$

**Problem 3 (TKS 02) - Gluckstern Modes on a KV Beam 35 points**

$n = 1$ Gluckstern mode and the KV envelope equation for the breathing mode. $r_b =$equilibrium matched beam radius.

a) The Gluckstern mode eigenfunction is given by

$$\delta \phi_n = \begin{cases} \frac{A_n}{2} \left[ P_{n-1} \left( 1 - \frac{r^2}{r_b^2} \right) + P_n \left( 1 - \frac{r^2}{r_p^2} \right) \right] & 0 \leq r \leq r_b \\ 0 & r_b < r \leq r_p \end{cases}$$

$n = 1, 2, 3, \ldots$

$P_n(x) = n^{th}$ order Legendre polynomial

write down the eigentfunction as an explicit polynomial in $r$ for $n = 1$ and plot this solution.

Legendre polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2} \left( 3x^2 - 1 \right)$
  
  \vdots
b) Apply the Poisson equation:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \delta \phi}{\partial r} \right) = -\frac{q}{\epsilon} \delta n_n(r)
\]

to calculate the perturbed mode density \(\delta n_n\) for \(\delta \phi_1\) as a function of \(r\) for \(0 \leq r \leq r_b\) (the “body-wave” component). Plot the result.

c) Use part b) to calculate the amount of charge introduced into the system by the “body-wave” perturbation \(\delta n_1(r)\) for \(0 \leq r \leq r_b\). How far would the beam edge radius \(r_e = r_b + \delta r_b\) need to change to conserve charge to linear order in \(A_1\)?

d) Obtain the \(n = 1\) Gluckstern mode dispersion relation from the general \(n\) formula presented in class:

\[
z_n + \frac{1 - (\sigma/\sigma_0)^2}{(\sigma/\sigma_0)^2} \left[ B_{n+1} \left( \frac{k/k_{30}}{\sigma/\sigma_0} \right) - B_n \left( \frac{k/k_{30}}{\sigma/\sigma_0} \right) \right] = 0
\]

From the definitions in the class notes for the \(B_n\) we have:

\[
B_0(\alpha) = 1
\]

\[
B_1(\alpha) = \frac{(\alpha/2)^2}{(\alpha/2)^2 - 1}
\]

Solve for the mode eigen”frequency” \(k\) as a function of \(k_{30}\) and \(\sigma/\sigma_0\).

\(k\) is a spatial wavenumber that we sometimes call a “frequency”.

e) Compare the wavenumber \(k\) calculated in part d) with the “breathing” envelope mode on a round KV equilibrium where we showed that the mode wavenumber is:

\[
k_{\text{envelope}} = \sqrt{2k_{30}^2 + 2k_{30}^2(\sigma/\sigma_0)^2}
\]

Are the wavenumbers the same? Is it reasonable to identify these as the same modes? (Explain why?) Would you expect the lowest order modes of a kinetic theory to always reproduce the KV envelope modes to lowest order? (Explain why?)