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I. Introduction

(related reading in parentheses)

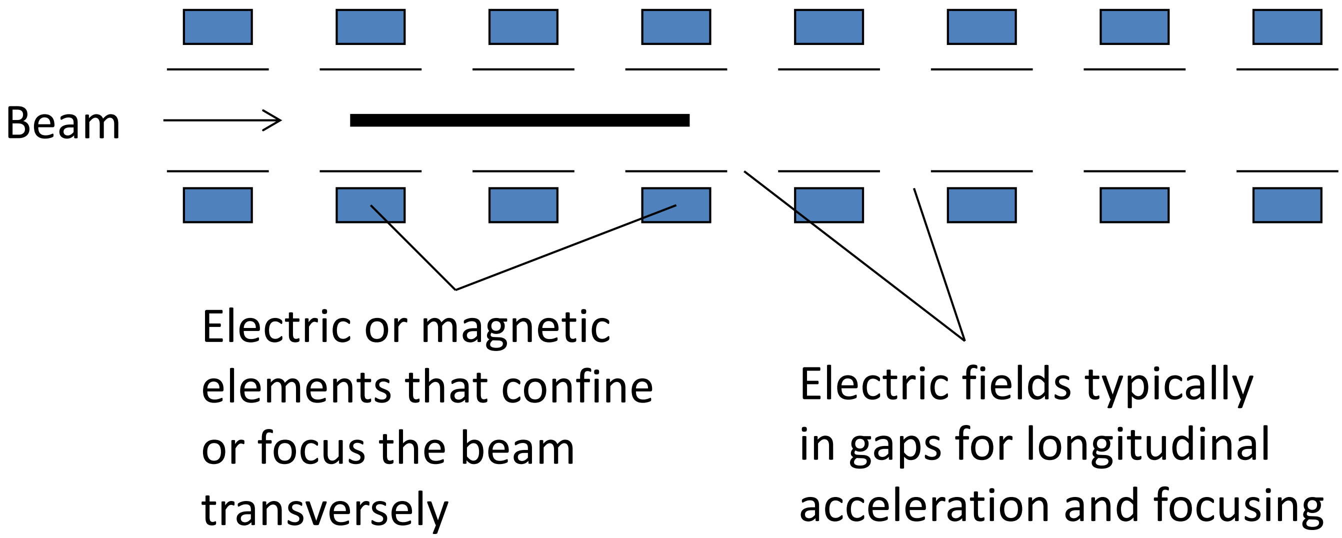
Particle motion (Reiser 2.1)
Equation of motion (Reiser 2.1)
Dimensionless quantities (Reiser 4.2)

Plasma physics of beams (Reiser 3.2, 4.1)

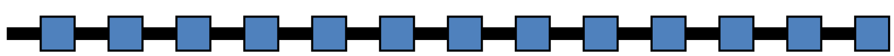
Emittance and brightness (Reiser 3.1 - 3.2)

LLNL-PRES-732305

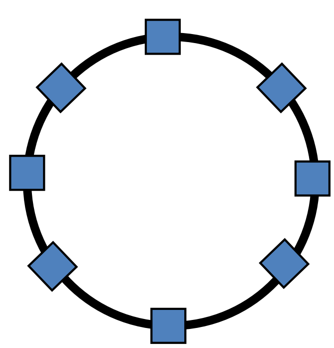
How do we describe and calculate the evolution of a collection of particles under the EM forces in an accelerator?



This array or "lattice" of focusing elements may be arranged in a linac or circular accelerator



or

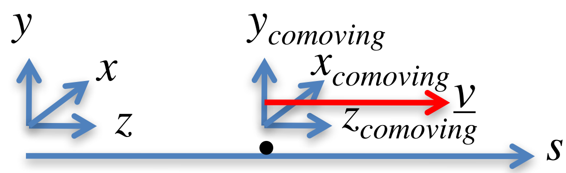


Particle equations of motion and dimensionless quantities

Consider the Lorentz force on a particle (mass m, charge q, momentum $\underline{p}$, velocity $\underline{v} = c\underline{\beta}$, Lorentz factor γ) under the influence of an electric ($\underline{E}$) and magnetic field ($\underline{B}$):

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B})$$ (SI units employed throughout)

$$\underline{p} = \gamma m \underline{v} \qquad \gamma^2 = \frac{1}{1-\beta^2} \qquad \underline{\beta} = \underline{v}/c$$



Beam center

Consider the x-component of the motion (transverse to the streaming direction). s is the coordinate of the "design" (ideal) orbit (equivalent to z for a linear accelerator) and subscripts "*comoving*" indicate coordinates comoving with the design particle.

We may transform to s as the independent variable:

$$dt = \frac{ds}{v_z}; \qquad v_x = \frac{dx}{dt} = v_z x'$$ where prime ' = $\frac{d}{ds}$

$$v_z \frac{d}{ds}\left(\gamma m v_z x'\right) = q(\underline{E} + \underline{v} \times \underline{B})_x$$

$$\gamma m v_z^2 x'' + x' m v_z \frac{d(\gamma v_z)}{ds} = q(\underline{E} + \underline{v} \times \underline{B})_x$$

$$\Rightarrow \quad x'' + \left[\frac{1}{\gamma v_z}\frac{d(\gamma v_z)}{ds}\right]x' = \frac{q}{\gamma m v_z^2}(\underline{E} + \underline{v} \times \underline{B})_x$$

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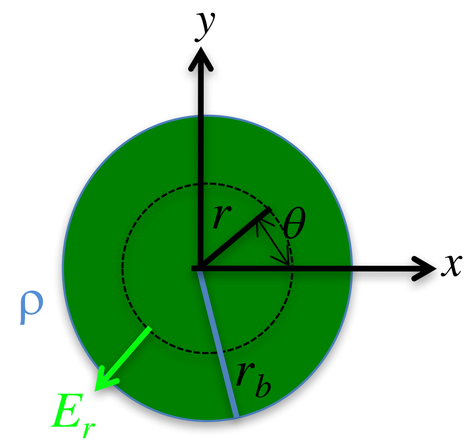
Now consider an unbunched beam of uniform charge density ρ and circular cross section, with radius r_b

Line charge density $\lambda = \pi r_b^2 \rho$

First calculate electric field:

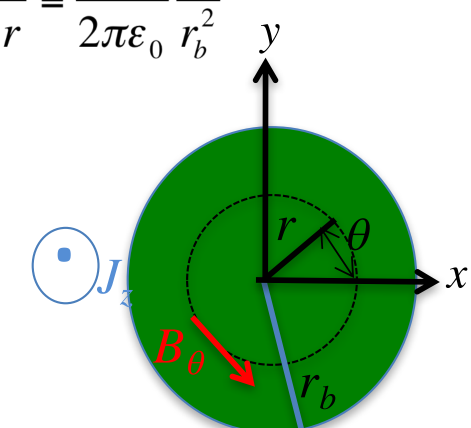


$$\nabla \bullet \underline{E} = \frac{\rho}{\varepsilon_0}$$

$$2\pi r E_r = \pi r^2 \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss theorem)}$$

$$\Rightarrow E_r = \frac{\rho}{2\varepsilon_0} r = \frac{\lambda}{2\pi\varepsilon_0}\frac{r}{r_b^2} \qquad E_x = E_r \cos\theta = E_r \frac{x}{r} = \frac{\lambda}{2\pi\varepsilon_0}\frac{x}{r_b^2}$$

Similarly, calculate the magnetic field:



$$\nabla \times \underline{B} = \mu_0 \underline{J}$$

$$2\pi r B_\theta = \mu_0 \pi r^2 \rho v_z \qquad \text{(Stokes theorem)}$$

$$\Rightarrow B_\theta = \mu_0 \frac{\lambda v_z}{2\pi\varepsilon_0}\frac{r}{r_b^2} \qquad B_y = B_\theta \cos\theta = B_\theta \frac{x}{r} = \mu_0 \frac{\lambda v_z}{2\pi}\frac{x}{r_b^2}$$

$$(B_z = 0)$$

Let $(\underline{E} + \underline{v} \times \underline{B})_x = (E_x - v_z B_y)^{self} + (E_x + v_y B_z - v_z B_y)^{ext}$

$$\Rightarrow \quad x'' + \left[\frac{1}{\gamma v_z}\frac{d(\gamma v_z)}{ds}\right] x' = \frac{q}{\gamma m v_z^2}\frac{\lambda}{2\pi\varepsilon_0}\frac{x}{r_b^2}\left[1 - \mu_0\varepsilon_0 v_z^2\right] + \frac{q}{\gamma m v_z^2}(E_x + v_y B_z - v_z B_y)^{ext}$$

Now $\mu_0\varepsilon_0 = \frac{1}{c^2}$; Assuming $\beta_x^2 + \beta_y^2 << \frac{1}{\gamma^2} \Rightarrow \gamma^2 \cong \frac{1}{1 - v_z^2/c^2}$ (Paraxial approximatior

$(\gamma^2 \cong 1/(1 - v_z^2/c^2)$ equivalent to assuming $\beta_x, \beta_y << 1$).

$$\Rightarrow x'' + \left[\frac{1}{\gamma v_z}\frac{d(\gamma v_z)}{ds}\right] x' = \frac{q}{\gamma^3 m v_z^2}\frac{\lambda}{2\pi\varepsilon_0}\frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2}(E_x + v_y B_z - v_z B_y)^{ext}$$

First consider the self-field.

$$x'' + \left[\frac{1}{\gamma v_z}\frac{d(\gamma v_z)}{ds}\right]x' = \frac{q}{\gamma^3 m v_z^2}\frac{\lambda}{2\pi\varepsilon_0}\frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2}(E_x + v_y B_z - v_z B_y)^{ext}$$

$$= Q\frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2}(E_x + v_y B_z - v_z B_y)^{ext}$$

$$Q \equiv \frac{q}{\gamma^3 m v_z^2}\frac{\lambda}{2\pi\varepsilon_0} \equiv \text{Generalized Perveance} \rightarrow \begin{cases} \dfrac{\lambda}{4\pi\varepsilon_0 V} & for\ \ \gamma^2 v_z^2 << c^2 \\ \dfrac{\lambda}{2\pi\varepsilon_0 V\left(\dfrac{qV}{mc^2}\right)^2} & for\ \ \gamma^2 v_z^2 >> c^2 \end{cases}$$

$$\equiv \frac{(q/e)}{m/m_{amu}}\frac{2I}{I_0}\frac{1}{\gamma^3\beta^3} \quad \text{where } I_0 \equiv \frac{4\pi\varepsilon_0 m_{amu} c^3}{e} \approx 31 \text{ MA}$$

Here $qV=(\gamma\text{-}1)mc^2$ = ion kinetic energy,
e is the proton charge, and m_{amu} is the atomic mass unit.
Also note in the non-relativistic limit:

$$Q \equiv \frac{1}{4\pi\varepsilon_0}\left(\frac{m}{2q}\right)^{1/2}\left(\frac{I}{V^{3/2}}\right) \quad \text{(non−relativistic)}$$

(same scaling as original term "perveance" characterizing injectors)

$$Q \equiv \frac{\phi_{self}}{V} = \frac{\int_0^{r_b}(E_r - v_z B_\theta)dr}{V} = \frac{\text{Potential energy of beam parti}}{\text{Kinetic energy of beam particle}}$$

Now consider the external field. We often try to create focusing forces that are linear in x (examples are: electric or magnetic quadrupoles, solenoids, Einzel lenses.) So let this focusing force be represented by $K(s)$.

$$\frac{q}{\gamma m v_z^2}(E_x + v_y B_z - v_z B_y)^{ext} = K(s)x$$

$$x'' + \left[\frac{1}{\gamma v_z}\frac{d(\gamma v_z)}{ds}\right]x' = Q\frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2}(E_x + v_y B_z - v_z B_y)^{ext}$$

$$= Q\frac{x}{r_b^2} + K(s)x$$

The focusing forces are often periodic:

$K(s)=K(s+L_p)$ where L_p=period of focusing element

(when dv_z/ds = 0, and Q is periodic with period L_p, then:

$x''=f(s)x$ where $f(s)$ is periodic. (Hill's equation).

For some purposes a suitable constant can be found which captures the "average" variation (over several periods) of the particle motion (continuous focusing approximation)

Then we replace the effects of the periodic lattice with a single focusing parameter $k_{\beta 0}^2$

$$x'' + \left[\frac{1}{\gamma v_z}\frac{d(\gamma v_z)}{ds}\right]x' = Q\frac{x}{r_b^2} - k_{\beta 0}^2 x$$

$k_{\beta 0}$ is defined as the "undepressed" betatron frequency

$$x'' + \left[\frac{1}{\gamma v_z}\frac{d(\gamma v_z)}{ds}\right]x' = Q\frac{x}{r_b^2} - k_{\beta 0}^2 x$$

Consider a drifting beam ($dv_z/ds = 0$). The particle equation becomes:

$$x'' = Q\frac{x}{r_b^2} - k_{\beta 0}^2 x$$

$$= -k_{\beta 0}^2\left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2}\right)x$$

This is simple harmonic oscillator equation.
Note some frequently encountered definitions:

$$k_{\beta 0}^2\left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2}\right) \equiv k_\beta^2 \equiv \text{ depressed betatron frequency}$$

Define also

$\sigma_0 \equiv k_{\beta 0} L_p \equiv$ undepressed phase advance (per period)

and

$\sigma \equiv k_\beta L_p \equiv$ depressed phase advance (per period) (includes space charge)

$$\frac{\sigma}{\sigma_0} \equiv \frac{k_\beta}{k_{\beta 0}} = \left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2}\right) = \text{tune depression}$$

Examples: $\frac{\sigma}{\sigma_0} = 0 \quad \Rightarrow$ Fully tune depressed

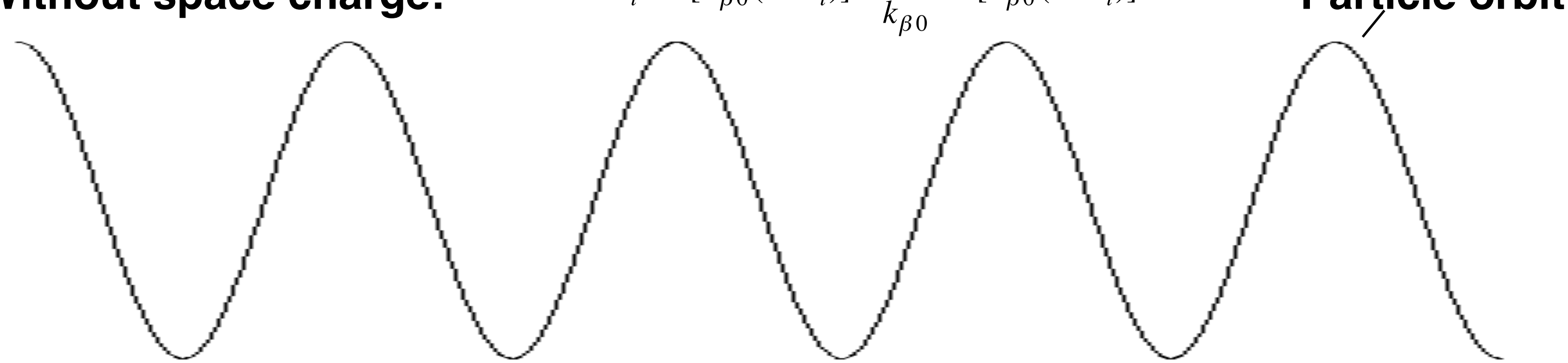
$\frac{\sigma}{\sigma_0} = 1 \quad \Rightarrow$ No space charge depressicn

(so two dimensionless parameters: Q characterizes space charge relative to ion kinetic energy, σ/σ_0 characterizes space charge force relative to focusing force)

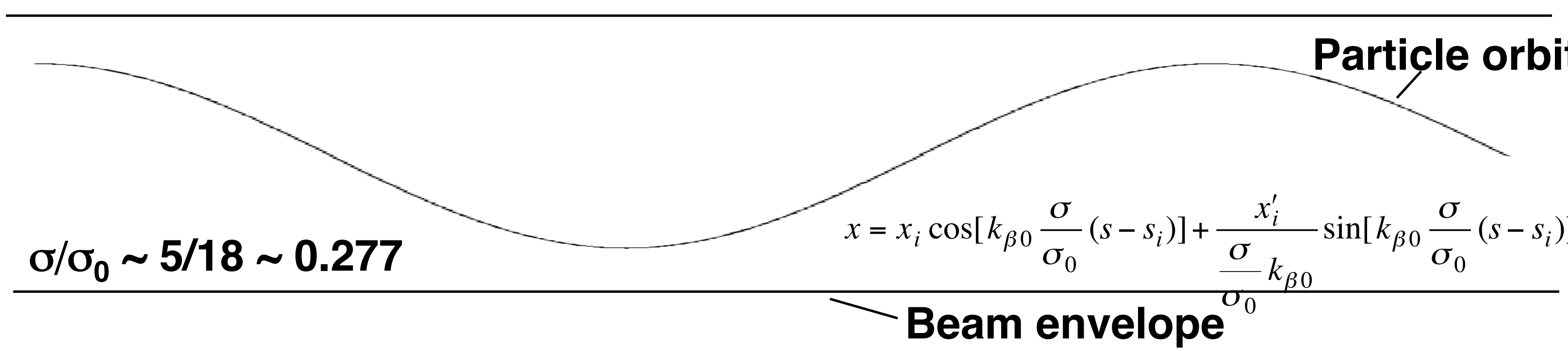
Space charge reduces betatron phase advance

Without space charge:

$$x = x_i \cos[k_{\beta 0}(s - s_i)] + \frac{x_i'}{k_{\beta 0}} \sin[k_{\beta 0}(s - s_i)]$$



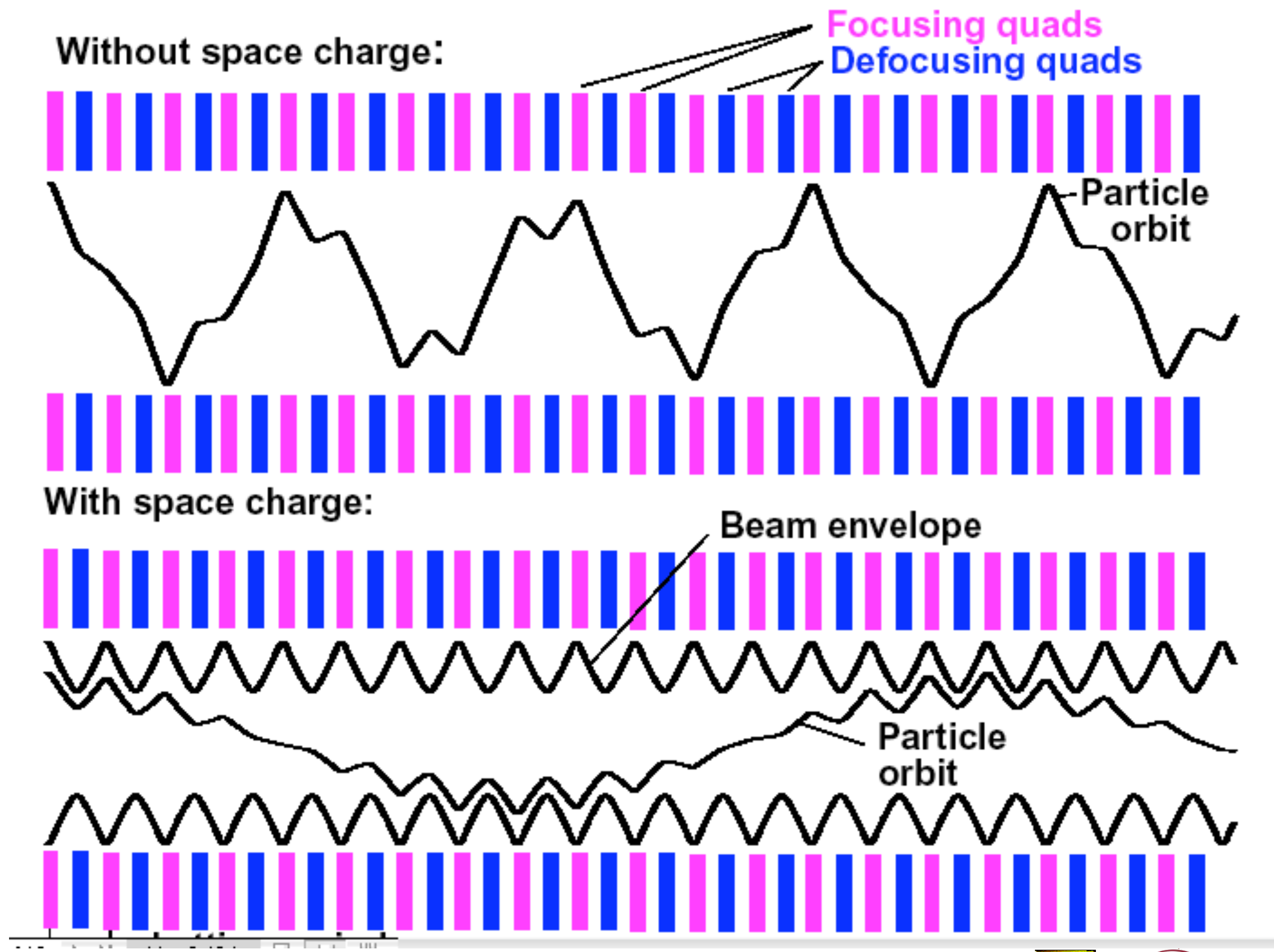
With space charge:



$$x = x_i \cos[k_{\beta 0} \frac{\sigma}{\sigma_0}(s - s_i)] + \frac{x_i'}{\frac{\sigma}{\sigma_0} k_{\beta 0}} \sin[k_{\beta 0} \frac{\sigma}{\sigma_0}(s - s_i)]$$

$\sigma/\sigma_0 \sim 5/18 \sim 0.277$

Space charge reduces betatron phase advance



BENDING BEAMS

RETURNING TO PARTICLE EQUATION WITH ARBITRARY E, & B:

$$x'' + \left[\frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\right] x' = \frac{q}{\gamma m v_z^2}\left(\underline{E} + \underline{v}\times\underline{B}\right)_x$$

IF EXTERNAL FORCE IS PROPORTIONAL TO $-x$

$\Rightarrow$ FOCUSING (HARMONIC OSCILLATIONS)

HOWEVER, IF $\underline{E} + \underline{v}\times\underline{B}$ = CONSTANT

$\Rightarrow$ BENDING

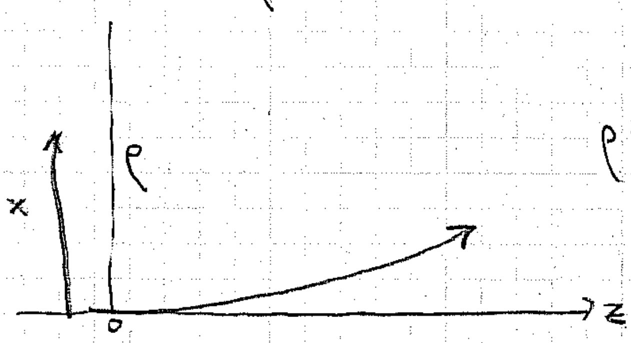
EXAMPLE: If $B = B_y \hat{e}_y$

$v = v_0 \hat{e}_z + v_x \hat{e}_x$ where $v_0 \gg v_x$

$$\Rightarrow \quad x'' = \frac{q B_y}{\gamma m v_z'} = \frac{B_y}{[B\rho]} \qquad [B\rho] \equiv \text{RIGIDITY} = \frac{\gamma m v_z}{q} = \frac{p}{q}$$

$$x' = \frac{B_y}{[B\rho]} z + x_0'$$

$$x = \frac{B_y}{[B\rho]}\frac{z^2}{2} + x_0' z + x_0$$



ρ = RADIUS OF CURVATURE OF ARC $= \dfrac{[B\rho]}{B_y}$

(BENDING CAN ALSO BE CARRIED OUT WITH ELECTRIC FIELDS E = const)

<u>Plasma physics of beams</u>

Plasma physics of beams = physics of space charge

Plasma parameter Λ:

$$q\phi_{IP} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r_{IP}} = \frac{1}{4\pi\varepsilon_0} q^2 n_0^{1/3}$$

Average potential energy $q\phi_{IP}$ of particle due to its nearest neighbor a distance $r_{IP} = n_0^{-1/3}$ (q = charge of particle; n_0 = number density)

If $q\phi_{IP} << k_B T$ $\Rightarrow$ "Weakly coupled plasma" or simply "plasma"

Define $\lambda_D \equiv \frac{(k_B T/m)^{1/2}}{(q^2 n_0/(\varepsilon_0 m))^{1/2}} \equiv \frac{v_t}{\omega_p} = \left(\frac{k_B T \varepsilon_0}{q^2 n_0}\right)^{1/2}$ = Debye Length

= characteristic distance whereby charges are shielded in plasma

Define Λ $\equiv \frac{4\pi}{3} n_0 \lambda_D^3$ ≡ Plasma Parameter

$$\sim \left(\frac{k_B T}{q\phi_{IP}}\right)^{1/2} >> 1 \qquad [\text{if } q\phi_{IP} << k_B T]$$

Klimontovich Equation

Ref.: "Introduction to Plasma Theory," D.R. Nicholson, [Wiley, 1983].

$$N(x,v,t) = \sum_{i=1}^{N_0} \delta(\underline{x} - \underline{X}_i(t))\delta(\underline{v} - \underline{V}_i(t))$$

$N(\underline{x},\underline{v},t)$ is the density of particles in phase space.
Note there are N_o particles: $\int N(\underline{x},\underline{v},t)\ \ d^3x\ \ d^3v = N_0$
$X_i(t)$ and $V_i(t)$ are position and velocity of the i^{th} particle.
The (non-relativistic) equations of motion are:

$$\dot{\underline{X}}_i = \underline{V}_i \qquad\qquad m\dot{\underline{V}}_i = q\underline{E}^m\left(\underline{X}_i(t),t\right) + q[\underline{V}_i \times \underline{B}^m\left(\underline{X}_i(t),t\right)]$$

Let $u = x - X_i(t) \ \Rightarrow\ \frac{\partial f(u)}{\partial x} = f'(u)$ and $\frac{\partial f(u)}{\partial t} = -\dot{X}(t) f'(u) = -\dot{X}(t)\frac{\partial f(u)}{\partial x}$

So taking the derivative of $N(x,v,t)$ with respect to t:

$$\frac{\partial N}{\partial t}(\underline{x},\underline{v},t) = -\sum_{i=1}^{N_0} \dot{\underline{X}}_i(t)\cdot \underline{\nabla}_x[\delta(\underline{x} - \underline{X}_i(t))\delta(\underline{v} - \underline{V}_i(t))]$$
$$- \sum_{i=1}^{N_0} \dot{\underline{V}}_i(t)\cdot \underline{\nabla}_v[\delta(\underline{x} - \underline{X}_i(t))\delta(\underline{v} - \underline{V}_i(t))]$$

Maxwell's equations:

$$\underline{\nabla}\cdot \underline{E}^m = \frac{\rho^m}{\varepsilon_0} \equiv \frac{1}{\varepsilon_0} q\int N(\underline{x},\underline{v},t) d^3v \qquad\qquad \underline{\nabla}\cdot \underline{B}^m = 0$$

$$\underline{\nabla} \times \underline{E}^m = -\frac{\partial \underline{B}^m}{\partial t} \qquad\qquad \underline{\nabla} \times \underline{B}^m = \mu_0 \underline{J}^m + \frac{\partial \underline{E}^m}{\partial t} \equiv \mu_0 q\int \underline{v} N(\underline{x},\underline{v},t) d^3v + \frac{\partial \underline{E}^m}{\partial t}$$

(Here superscript "m" denotes "microscopic" quantity, not averaged locally over a small volume).

$$\Rightarrow \frac{\partial N}{\partial t}(\underline{x},\underline{v},t) = -\sum_{i=1}^{N_0} \underline{V}_i(t)\cdot \underline{\nabla}_x[\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))]$$

$$-\sum_{i=1}^{N_0}\left(\frac{q}{m}E^m(X_i(t),t)+\frac{q}{m}[V_i\times B^m(X_i(t),t)]\right)\cdot \underline{\nabla}_v[\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))]$$

Note that $\underline{V}_i(t)\delta(\underline{v}-\underline{V}_i(t)) = v\delta(\underline{v}-\underline{V}_i(t))$ so

$$\Rightarrow \frac{\partial N}{\partial t}(\underline{x},\underline{v},t) = -\underline{v}\cdot \underline{\nabla}_x \sum_{i=1}^{N_0} \delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$$

$$-\left(\frac{q}{m}E^m(\underline{x},t)+\frac{q}{m}[\underline{v}\times \underline{B}^m(\underline{x},t)]\right)\cdot \underline{\nabla}_v \sum_{i=1}^{N_0} \delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$$

$$\Rightarrow \boxed{\frac{\partial N}{\partial t}(\underline{x},\underline{v},t) = -\underline{v}\cdot \underline{\nabla}_x N(\underline{x},\underline{v},t) - \left(\frac{q}{m}E^m(\underline{x},t)+\frac{q}{m}[\underline{v}\times \underline{B}^m(\underline{x},t)]\right)\cdot \underline{\nabla}_v N(\underline{x},\underline{v},t)}$$

Klimontivich Equation

Note that the total derivative of
a quantity along an orbit in phase space:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left.\frac{d\underline{x}}{dt}\right|_{orbit}\cdot \underline{\nabla}_x + \left.\frac{d\underline{v}}{dt}\right|_{orbit}\cdot \underline{\nabla}_v$$

$$\Rightarrow \boxed{\left.\frac{d}{dt}N(\underline{x},\underline{v},t)\right|_{orbit} = 0}$$

Note that N=0 or infinity,
nothing in between!

Average N over some box in phase space. Δx,and Δy are the dimensions of the box. Assume $n^{-1/3} << \Delta x << \lambda_D$
so that $f(\underline{x},\underline{v},t)$ is a smoothly varying function.

Now let $$f(\underline{x},\underline{v},t) = \frac{1}{\Delta x^3 \Delta v^3}\int^{\Delta x^3,\, \Delta v^3} N(\underline{x},\underline{v},t)d^3x d^3v \equiv \langle N(\underline{x},\underline{v},t)\rangle$$

Then $N = f + \delta f \qquad f \equiv \langle N\rangle \qquad \langle \delta f\rangle = 0$

$$\underline{E}^m = \underline{E} + \delta\underline{E} \qquad \underline{E} = \langle \underline{E}^m\rangle \qquad \langle \delta\underline{E}\rangle = 0$$

$$\underline{B}^m = \underline{B} + \delta\underline{B} \qquad \underline{B} = \langle \underline{B}^m\rangle \qquad \langle \delta\underline{B}\rangle = 0$$

$$\frac{\partial f}{\partial t} + \underline{v}\cdot \underline{\nabla}_x f + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{B})\cdot \underline{\nabla}_v f = -\frac{q}{m}\left\langle \delta \underline{E} + \underline{v} \times \delta \underline{B})\cdot \underline{\nabla}_v \delta f \right\rangle$$

LHS: Smoothly varying part

RHS: Average over "rapidly fluctuating quantities", includes "discrete particle effects" or "collisions"

If collisions are neglected (so set RHS to zero): we have the "Vlasov Equation":

$$\frac{\partial f}{\partial t} + \underline{v}\cdot \underline{\nabla}_x f + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{B})\cdot \underline{\nabla}_v f = 0$$

$$\Rightarrow \frac{d}{dt} f(\underline{x},\underline{v},t)\Big|_{orbit} = 0$$

Phase space density on trajectories is constant. (Liouville's theorem).

The RHS represents the effects of collisions (i.e. interactions with non-smoothly varying fields).
Very heuristically:

$$-\frac{q}{m}\left\langle \delta \underline{E} + \underline{v} \times \delta \underline{B}) \cdot \underline{\nabla}_v \delta f \right\rangle \sim \nu_c f$$

$$\nu_c \sim \sigma n v$$

$$\sigma \sim \pi r_c^2$$

where σ is the collision cross section.
For a large angle scattering the kinetic energy of particle will be of order the potential energy at cl approach, defining a collision radius by

$$k_B T \sim \frac{q^2}{4\pi\varepsilon_0 r_c} \quad \Rightarrow \quad r_c \sim \frac{q^2}{4\pi\varepsilon_0 k_B T}$$

$$\Rightarrow \nu_c \sim \pi\left(\frac{q^2}{4\pi\varepsilon_0 k_B T}\right)^2 n_0 \left(\frac{k_B T}{m}\right)^{1/2}$$

$$\sim \frac{1}{16\pi}\frac{v_{th}}{\lambda_D^4 n_0}$$

Recall the smoothed equation with the heuristic collision term:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla}_x f + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{B}) \cdot \underline{\nabla}_v f = \nu_c f$$

Consider the third term on the LHS of the equation:

We approximate $\nabla_v \sim 1/v_t$

and $\nabla_x \sim 1/\lambda_D$

and use $\underline{\nabla}_x \cdot \underline{E} = \rho/\varepsilon_0$ yielding:

$$\frac{q}{m}\underline{E} \cdot \underline{\nabla}_v f \sim \frac{q}{m}(\lambda_D \underline{\nabla} \cdot \underline{E})\underline{\nabla}_v f \sim \frac{q}{m}\left(\frac{q\lambda_D n_0}{\varepsilon_0}\right)\frac{f}{v_{th}} \sim \frac{\omega_p^2 \lambda_D}{v_{th}} f$$

$$\sim \omega_p f \qquad \text{where } v_t \sim \left(\frac{k_B T}{m}\right)^{1/2}$$

Similarly, the second term on the LHS of the equation is approximately:

$$\underline{v} \cdot \underline{\nabla}_x f \sim \frac{v_t}{\lambda_D} f \sim \omega_p f$$

The first term can be argued a priori to be no greater than

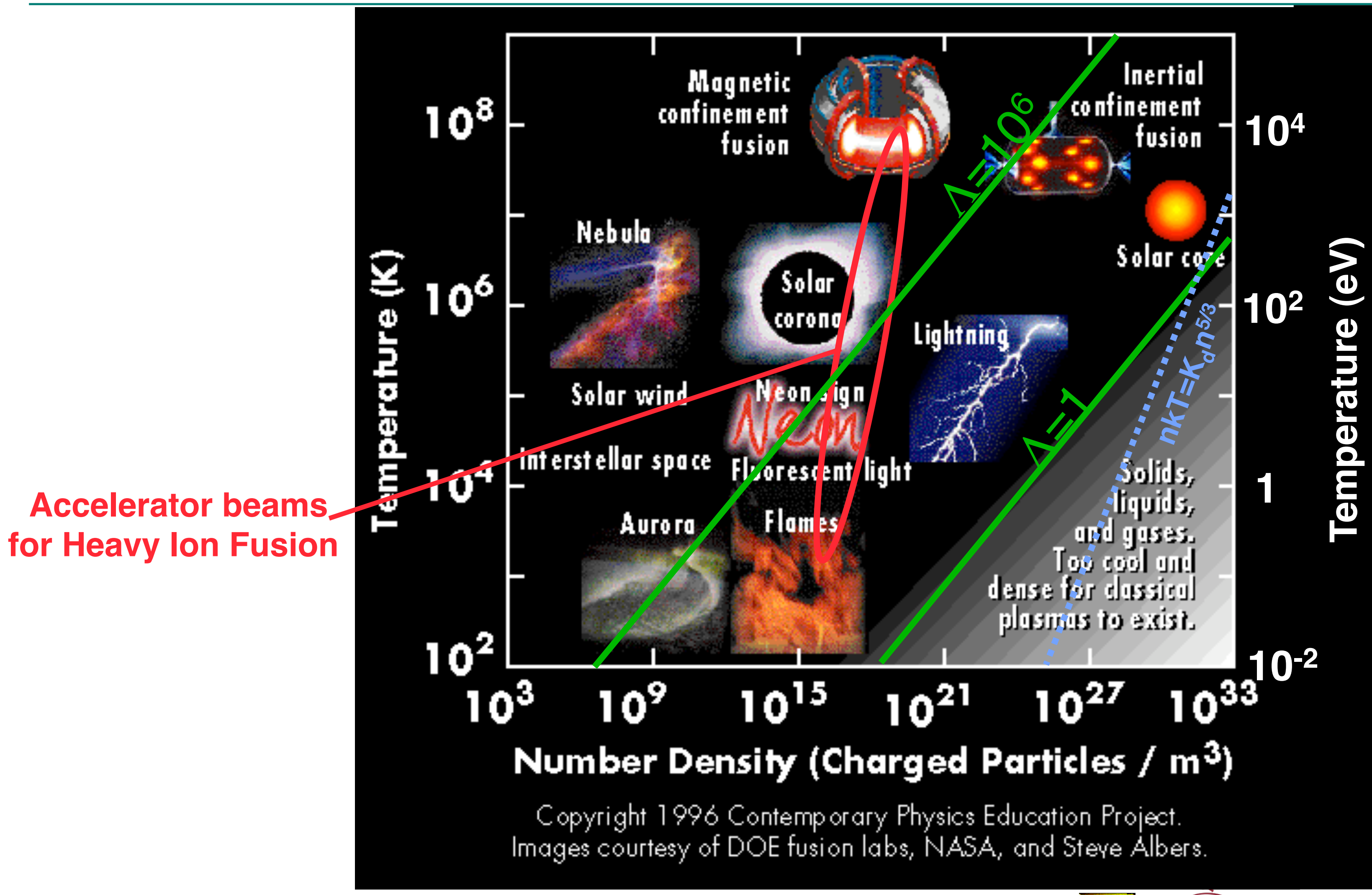
$$\frac{\partial f}{\partial t} <\sim \omega_p f$$

The fourth term can be of order the third term if it includes external focusing or is of order v^2/c^2 if it includes only the self magnetic field.

So the LHS ~ $\omega_p f$. Pulling it all together then:

$$\frac{\text{Collision term}}{\text{LHS}} \sim \frac{1}{16\pi\lambda_D^3 n_0} \sim \frac{1}{12\Lambda} << 1 \quad \text{when } \Lambda >> 1$$

Accelerator beams are non-neutral plasmas



Phase space density conservation

Liouville's theorem: $\frac{df}{dt} = 0$ along a trajectory in phase space.

Let $dN = f \quad dx \quad dy \quad dz \quad dp_x \quad dp_y \quad dp_z$

The continuity equation in phase space is

$$\frac{\partial f}{\partial t} + \underline{\nabla}_6 \cdot (f\, \underline{v}_6) = 0$$

where $\underline{v}_6 = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$ and $\underline{\nabla}_6 \cdot \underline{a}_6 = \frac{\partial a_1}{\partial q_1} + \frac{\partial a_2}{\partial q_2} + \frac{\partial a_3}{\partial q_3} + \frac{\partial a_4}{\partial p_1} + \frac{\partial a_5}{\partial p_2} + \frac{\partial a_6}{\partial p_3}$

If the system is governed by a Hamiltonian $H(\underline{q},\underline{p},t)$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Now $\underline{\nabla}_6 \cdot \underline{v}_6 = \sum_{i=1}^{3} \left(\frac{\partial}{\partial q_i}\left(\frac{dq_i}{dt}\right) + \frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt}\right) \right) = \sum_{i=1}^{3} \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0$

$$\Rightarrow \frac{\partial f}{\partial t} + \underline{\nabla}_6 \cdot (f\, \underline{v}_6) = \frac{\partial f}{\partial t} + \underbrace{f\, \underline{\nabla}_6 \cdot \underline{v}_6}_{0} + \underline{v}_6 \cdot \underline{\nabla}_6 f = 0$$

$$\Rightarrow \frac{df}{dt} = 0 \quad \text{along a 6D trajectory}$$

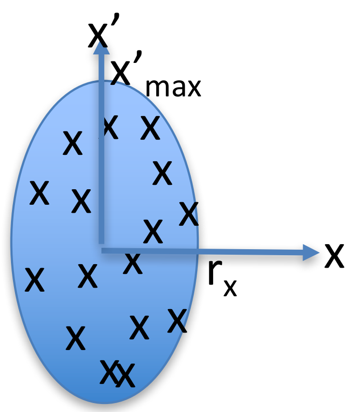
Emittance and Brightness:

Liouvilles equation or Vlasov equation $\Longrightarrow \dfrac{dN}{dx\, dy\, dz\, dp_x p_y p_z}$ = constant

If x'' = f(x) and not functions of y or z
y'' = f(y) and not functions of x or z
z'' = f(z) and not functions of x or y

$\Longrightarrow \dfrac{dN}{dx\ \ dp_x}$=constant; $\dfrac{dN}{dy\ \ dp_y}$=constant; and $\dfrac{dN}{dz\ \ dp_z}$=constant separately.

Definitions of emittance:
Trace space emittance: area/π of smallest ellipse that encloses all particles



For non-accelerating paraxial beam x' proportional to p_x, etc.

Statistical definition:
Involves statistical averages of 2nd order quantitites such $\langle x^2\rangle$,$\langle x'^2\rangle$, and $\langle xx'\rangle$

$\varepsilon_x = 4\,(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2)^{1/2}$

For an upright, unform density beam in phase space $\langle x^2\rangle = r_x^2/4$, $\langle x'^2\rangle = x'^2_{max}/4$, and $\langle xx'\rangle=0$, so $\varepsilon_x = x'_{max}\, r_x$ = Area/π

Normalized Emittance:

For a beam that is accelerating, return to x, p_x as appropriated definition of phase space area
$p_x = \gamma\beta m v_x = \gamma\beta m v_z x'$

normalized emittance can be defined:

=> $\varepsilon_{Nx} = 4\,\gamma\beta(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2)^{1/2} = \gamma\beta\varepsilon_x$

Here v_z is approximately equal to v.

Since emittance is related to the average phase space area (averaging over empty space) the emittance generally grows as a beam filaments (engulfing empty space).

Brightness:

The microscopic density f of particles in 6 D space is

$$f = \frac{dN}{dx\, dy\, dz\, dp_x p_y p_z}$$

A quantity that characterizes the average 6D phase space density is the 6 D brightness:

$$B_6 = \frac{I\Delta t/q}{\pi^3\ \varepsilon_x\varepsilon_y\varepsilon_z}$$

Note that f is normally constant along a trajectory whereas the 6D brightness can decrease.
Lower dimensional versions of the brightness are often used such as normalized brightness:

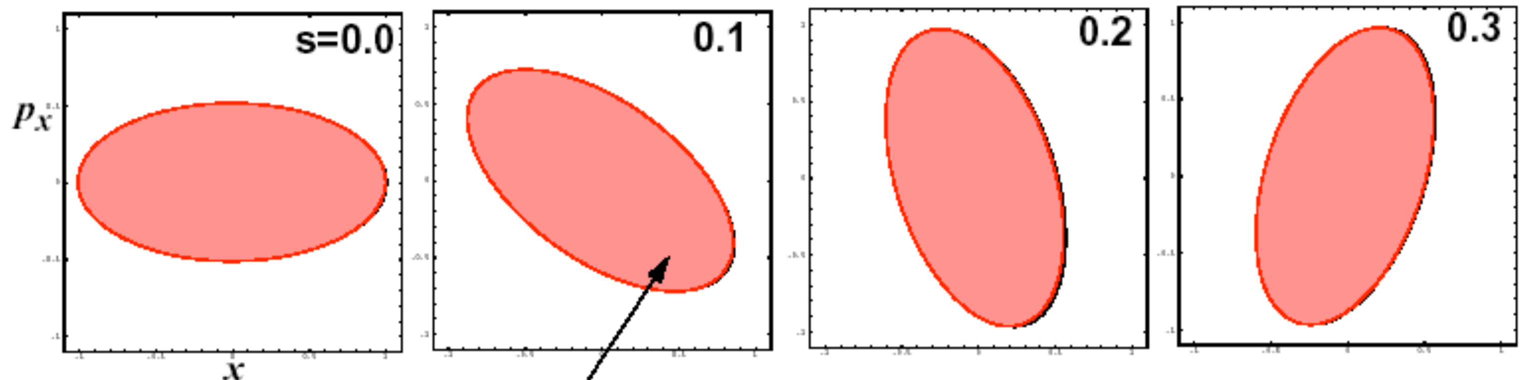
$$B_N = I/(\varepsilon_{Nx}\varepsilon_{Ny})$$

and unnormalized brightness:

$$B = I/(\varepsilon_x\varepsilon_y)$$

Emittance is constant for linear force profiles and matched beams

Linear force profile ($x'' = - k^2 x$) => Phase space area preserved, ellipse stays elliptical.



Emittance = phase space area
Emittance constant if forces linear

Here, width of beam is oscillating or "mismatched.'

Non-linear forces (e.g. $x'' = - k^2 x + \varepsilon x^3$) $\Rightarrow$ position-dependent frequency

$\Rightarrow$ phase mixing, increasing effective area $\Rightarrow$ Emittance increases if forces non-linear