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USPAS
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Current limits

A. Axisymmetric
   1. Solenoids
   2. Einzel lens

B. Quadrupolar
   1. Derivation of envelope equations with elliptic symmetry
   2. Current limit using fourier transform method
   3. Alternative methods

Yesterday we derived the "Paraxial Ray Equation:"

$$r''+\frac{(\gamma\beta)'}{\gamma\beta}r'+\frac{\gamma''}{2\gamma\beta^2}r+\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r+\left(\frac{p_\theta}{\gamma\beta mc}\right)^2\frac{1}{r^3}-\frac{q}{\gamma^3 m\beta^2 c^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r}=0$$

Inertial

Accelerative damping (of angle r')

E_r from converging field lines

Solenoidal focusing ($v_\theta B_z$ – part of centrifugal term)

Part of centrifugal term

Self-field $(E_r^{self}-v_z B_\theta^{self})$

which together with the conservation of canonical angular momentum,

$$p_\theta \equiv \gamma\beta mcr^2\theta'+\frac{m\omega_c r^2}{2}$$

and initial conditions, specify the orbit of a particle in an axisymmetric field.

Taking statistical moments, we derived the radial envelope equation.

$$r_b''+\frac{(\gamma\beta)'}{\gamma\beta}r_b'+\frac{\gamma''}{2\gamma\beta^2}r_b+\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b-\left(\frac{2\langle p_\theta\rangle}{\gamma\beta mc}\right)^2\frac{1}{r_b^3}-\frac{\varepsilon_r^2}{r_b^3}-\frac{Q}{r_b}=0$$

where

$$\varepsilon_r^2=4\left(\langle r^2\rangle\langle r'^2\rangle-\langle rr'\rangle^2+\langle r^2\rangle\langle r^2\theta'^2\rangle-\langle r^2\theta'\rangle^2\right)$$

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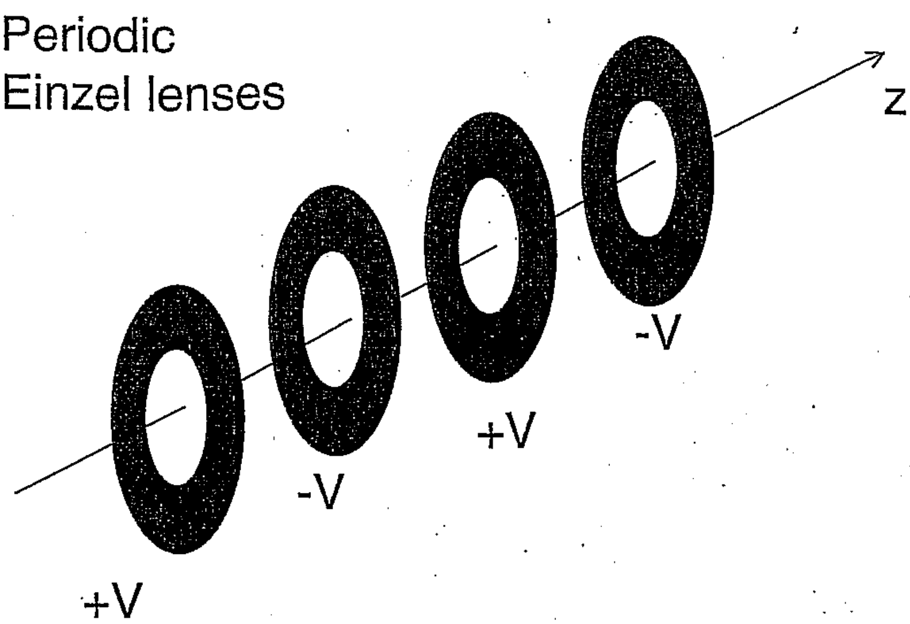

PERIODIC SOLENOIDS

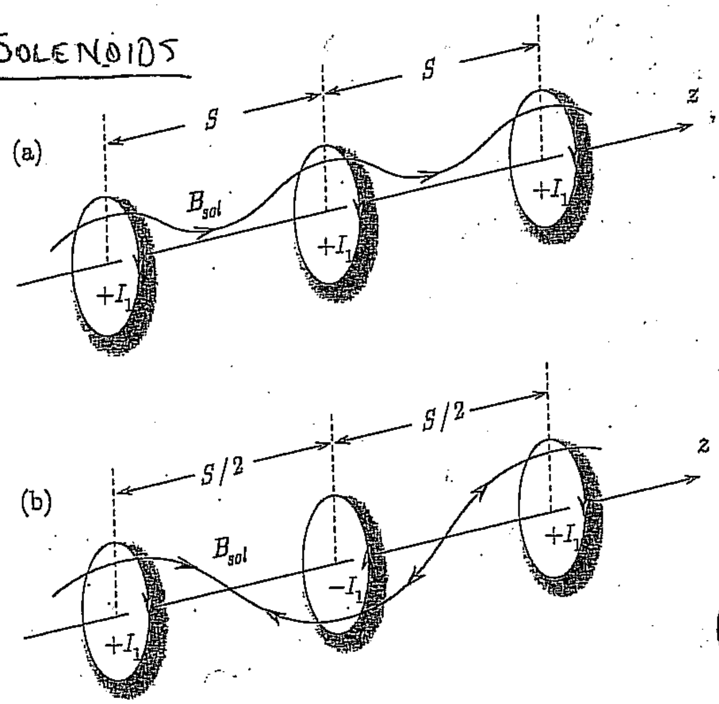

Figure 3.2. Schematic of magnet sets producing a periodic focusing solenoidal field with axial periodicity length S. In Fig. 3.2 (a), successive coils are spaced by S and have the same current polarity $+I_1, +I_1, \cdots$. In Fig. 3.2 (b), successive coils are spaced by $S/2$ and have alternating current polarities $+I_1, -I_1, +I_1, \cdots$.

(FIGURE FROM DAVIDSON & QIN, 2003) P. 55 "PHYSICS OF INTENSE CHARGED PARTICLE BEAMS IN HIGH ENERGY ACCELERATORS"

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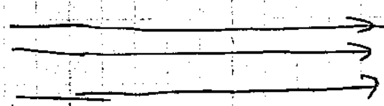
SOLENOIDAL FOCUSING



Let $\gamma' = \gamma'' = 0$

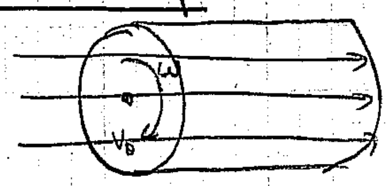
FOR MAXIMUM TRANSPORT $P_\theta = 0$ & $\epsilon_r^2 = 0$

$$\Rightarrow \quad r_b'' + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b = \frac{Q}{r_b}$$

FOR A MATCHED BEAM:

$$Q_{max} = \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b^2$$

HEURISTICALLY:



$$v_\theta = \omega r \qquad q\,\omega r\, B$$

$$m\omega^2 r + Q m v_z^2 \left(\frac{r}{r_b^2}\right) = q\, v_\theta\, B_z$$

centrifugal force — SPACE CHARGE FORCE — MAGNETIC FORCE INWARD

$$\Rightarrow \quad \omega^2 + \frac{QV^2}{r_b^2} = \omega\omega_c$$

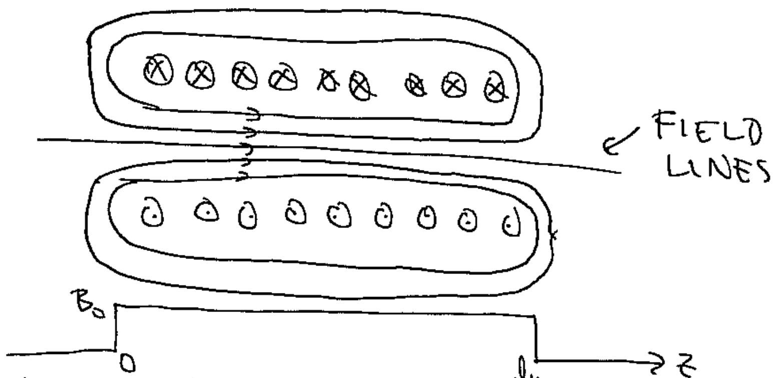
$\omega\omega_c - \omega^2$ = MAXIMUM WHEN $\omega = \frac{\omega_c}{2}$

$$\Rightarrow \quad Q_{max} = \left(\frac{\omega_c^2}{4}\right)\left(\frac{r_b^2}{V^2}\right)$$

SOLENOIDAL FOCUSING – CONTINUED

IN REALITY BEAM ACQUIRES V_θ AS BEAM ENTERS SOLENOID:

CONSIDER SIMPLE STEP FUNCTION APPROXIMATION TO SOLENOID FIELD:



LET $B_z = B_0\left[\Theta(z) + \Theta(l_m - z) - 1\right] = \begin{cases} 0 & z < 0 \\ B_0 & 0 < z < l_m \\ 0 & z > l_m \end{cases}$

$$\frac{\partial B_z}{\partial z} = B_0\left[\delta(z) - \delta(l_m - z)\right]$$

Here $\Theta(z) = \begin{cases} 1 & z > 0 \\ 0 & z < 0 \end{cases}$

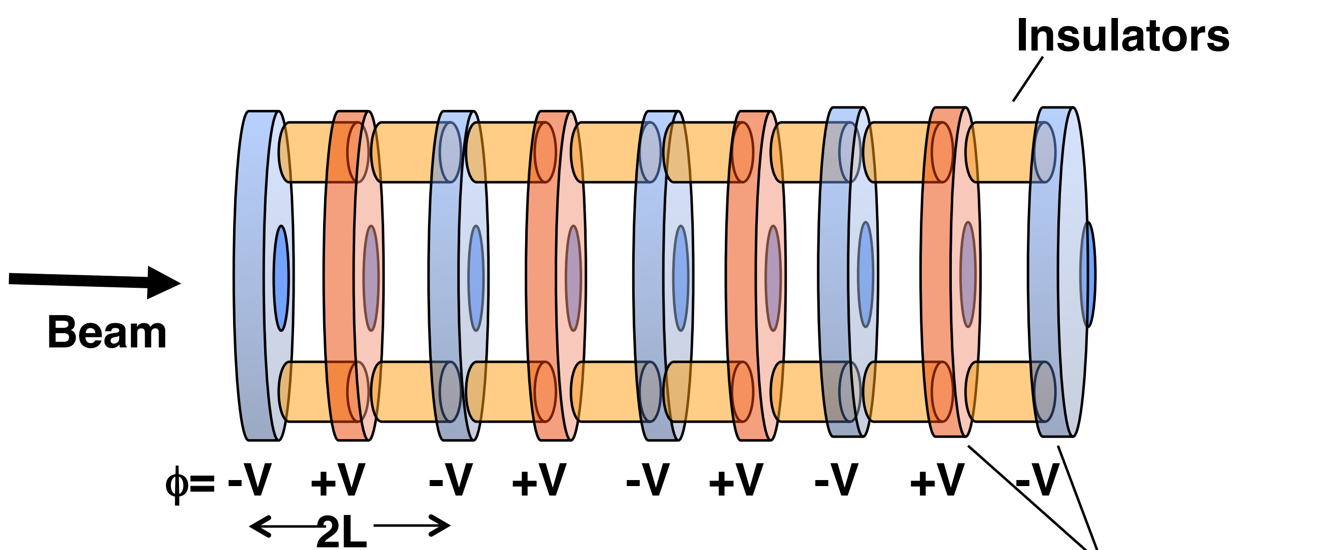
As we found earlier $\nabla \cdot B = 0 \Rightarrow$

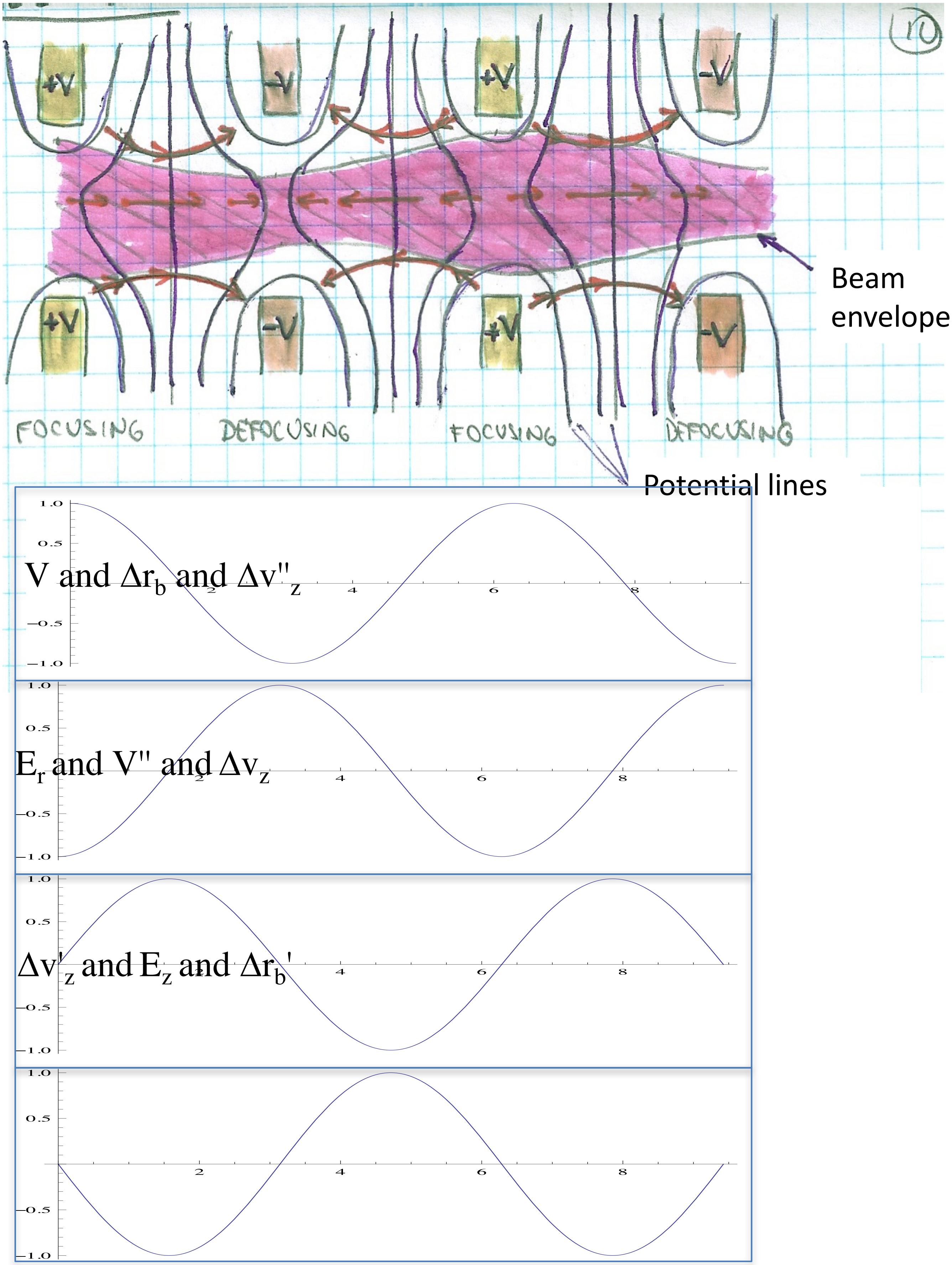
$$B_r(r,z) \simeq -\frac{r}{2}\frac{\partial B_z}{\partial z} + \cdots = -\frac{r}{2} B_0\left[\delta(z) + \delta(l_m - z)\right]$$

$$\Delta p_\theta^* = q\int v_z B_r\, dt = \int_{-\infty}^{0+\epsilon} q B_r\, dz = -\frac{r q B_0}{2}$$

$$\Rightarrow v_\theta = r\frac{qB_0}{2m} = \frac{r\omega_c}{2}$$

Schematic of Einzel lens





$$V \Rightarrow v_z,\ V'' \Rightarrow E_r \Rightarrow \Delta r$$

EINZEL LENS – ANALYSIS (DERIVATION FROM ED LEE)

NOW, LET $w_c = \langle P_\theta \rangle = \varepsilon_r^2 = 0$

$$\Rightarrow \quad r_b'' + \frac{\gamma'}{\beta^2 \gamma} r_b' + \frac{\gamma''}{2\beta^2\gamma} r_b - \frac{Q}{r_b} = 0$$

ALSO ASSUME $\beta \ll 1$, NON-RELATIVISTIC BEAM $\rightarrow$ $\gamma' \simeq \beta\beta'$, $\gamma'' \simeq \beta'^2 + \beta''\beta$

$$r_b'' + \frac{\beta'}{\beta} r_b' + \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b - \frac{Q}{r_b} = 0$$

To eliminate r_b' term try substitution

$$r_b = \left(\frac{\beta_0}{\beta}\right)^{1/2} R$$

$$r_b' = \left(\frac{\beta_0}{\beta}\right)^{1/2} R' - \frac{1}{2}\left(\frac{\beta}{\beta_0}\right)^{-3/2} \frac{R}{\beta_0}\beta'$$

$$r_b'' = \left(\frac{\beta_0}{\beta}\right)^{1/2} R'' - \left(\frac{\beta}{\beta_0}\right)^{-3/2} \frac{R'}{\beta_0}\beta' + \frac{3}{4}\left(\frac{\beta}{\beta_0}\right)^{-5/2} \frac{R}{\beta_0^2}\beta'^2 - \frac{1}{2}\left(\frac{\beta}{\beta_0}\right)^{-3/2} \frac{R}{\beta_0}\beta''$$

$$\Rightarrow \quad \left(\frac{\beta_0}{\beta}\right)^{1/2} R'' + \frac{3}{4}\left(\frac{\beta}{\beta_0}\right)^{-5/2} \frac{\beta'^2}{\beta_0^2} R = \frac{Q}{R}\left(\frac{\beta}{\beta_0}\right)^{1/2}$$

$$\Rightarrow \quad \boxed{R'' = \frac{Q}{R}\left(\frac{\beta}{\beta_0}\right) - \frac{3}{4}\left(\frac{\beta'}{\beta}\right)^2 R}$$

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EINZEL LENS – CONTINUED

MODEL: LET $\phi = \phi_0 \cos\left(\frac{\pi z}{L}\right)$

$$\frac{1}{2} m v^2 + q\phi = \text{constant}$$

$$\Rightarrow \quad v^2 = v_0^2 - \frac{2q\phi}{m}\cos\left(\frac{\pi z}{L}\right)$$

$$v' = \frac{q\phi_0}{mv}\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$

If $\left(\frac{2q\phi_0}{m}\right) << v_0^2$: $\left(\frac{\beta'}{\beta}\right)^2 \cong \left(\frac{q\phi_0}{mv_0^2}\right)^2\left(\frac{\pi}{L}\right)^2 \sin^2\left(\frac{\pi z}{L}\right)$

$$R'' = \frac{Q}{R}\left(\frac{\beta}{\beta_0}\right) - \frac{3}{4}\left(\frac{\beta'}{\beta}\right)^2 R$$

FOR EQUILIBRIUM LOOK AT D.C. COMPONENT: $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$\overline{R''} = 0 \quad \Rightarrow \quad \frac{Q}{\overline{R}} = \frac{3}{4}\overline{\left(\frac{\beta'}{\beta}\right)^2}\,\overline{R}$$

$$R \equiv \left(\frac{\beta}{\beta_0}\right)^{1/2} r_b \quad \Rightarrow \quad \overline{R} = r_b$$

$$\overline{\left(\frac{\beta'}{\beta}\right)^2} = \frac{1}{2}\left(\frac{q\phi_0}{mv_0^2}\right)^2\left(\frac{\pi}{L}\right)^2$$

$$\Rightarrow \quad Q_{max} = \frac{3\pi^2}{8}\left(\frac{q\phi_0}{mv_0^2}\right)^2\left(\frac{r_b}{L}\right)^2$$

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From Reiser, p.112

$E_x = -E'x$
$E_y = E'y$

$F_x = -qE'x$
$F_y = qE'y$

Electrostatic Quads

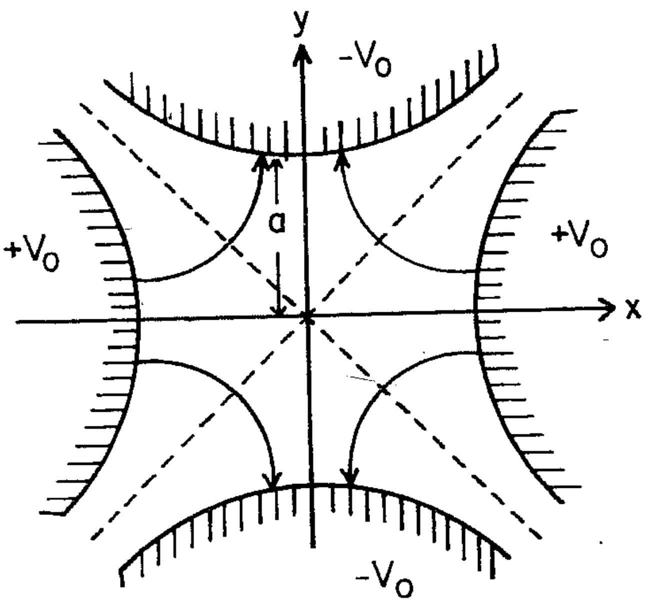
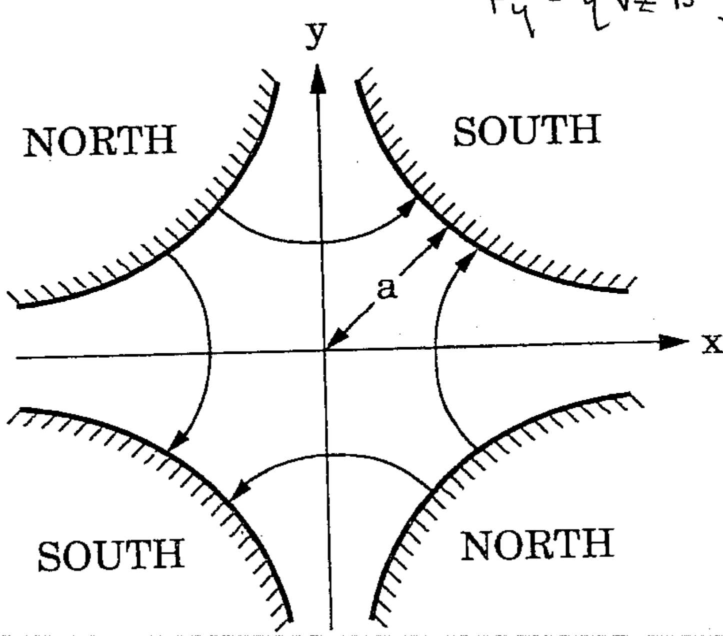


Figure 3.15. Electrodes and force lines in an electrostatic quadrupole.

$B_x = B'y$
$B_y = B'x$

$F_x = -qV_z B'x$
$F_y = qV_z B'y$

Magnetic Quads



ENVELOPE EQUATIONS FOR NON-AXISYMMETRIC SYSTEMS

$$r_x^2 \equiv 4\langle x^2\rangle \qquad r_y^2 \equiv 4\langle y^2\rangle$$

$$2 r_x r_x' = 8\langle x x'\rangle$$

$$r_x' = \frac{4\langle x x'\rangle}{r_x}$$

$$r_x'' = \frac{4\langle x x''\rangle}{r_x} + \frac{4\langle x'^2\rangle}{r_x} - \frac{4\langle x x'\rangle}{r_x^2} r_x'$$

$$= \frac{4\langle x x''\rangle}{r_x} + \frac{16\langle x'^2\rangle\langle x^2\rangle}{r_x^3} - \frac{16\langle x x'\rangle^2}{r_x^3}$$

DEFINE $\varepsilon_x^2 = 16(\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2)$

$$\Rightarrow \boxed{r_x'' = \frac{4\langle x x''\rangle}{r_x} + \frac{\varepsilon_x^2}{r_x^3}}$$

SO HOW DO WE CALCULATE $\langle x x''\rangle$?

RETURN TO SINGLE PARTICLE EQUATION (IN CARTESIAN COORDINATES)

$$\frac{d}{dt}(\gamma m \dot{x}) = \dot{\gamma} m \dot{x} + \gamma m \ddot{x} = q(E_x + \dot{y} B_z - \dot{z} B_y)$$

$\downarrow$ x'' & similarly y''

$\downarrow$ QUADRUPOLE FOCUSING, SPACE-CHARGE OF ELLIPTICAL BEAMS

EQUATION OF MOTION

RETURN TO X, Y COORDINATES

$$x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\,x' = \frac{-q}{\gamma^3 m v_z^2}\frac{\partial \phi}{\partial x} \pm \begin{cases} \dfrac{qB'}{\gamma m v_z}\,x & \text{for magnetic quads} \\[2ex] \dfrac{qE'}{\gamma m v_z^2}\,x & \text{for electric quads} \end{cases}$$

Let $\dfrac{\gamma m v_z}{q} = \dfrac{p}{q} \equiv [B\rho] \equiv$ RIGIDITY

$$y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\,y' = \frac{-q}{\gamma^3 m v_z^2}\frac{\partial \phi}{\partial y} \mp \begin{cases} \dfrac{B'}{[B\rho]}\,y & \text{magnetic} \\[2ex] \dfrac{qE'}{\gamma m v_z^2}\,y & \text{electric} \end{cases}$$

ENVELOPE EQUATION

$$r_x^2 = 4\langle x^2\rangle ; \qquad r_y^2 = 4\langle y^2\rangle$$

$$r_x' = \frac{4\langle x x'\rangle}{r_x}$$

$$r_x'' = \frac{4\langle x x''\rangle}{r_x} + \frac{\varepsilon_x^2}{r_x^3} ; \qquad \varepsilon_x^2 = 16\left(\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2\right)$$

$$r_y'' = \frac{4\langle y y''\rangle}{r_y} + \frac{\varepsilon_y^2}{r_y^3} \qquad \varepsilon_y^2 = 16\left(\langle y^2\rangle\langle y'^2\rangle - \langle y y'\rangle^2\right)$$

for magnetic focusing:

$$r_x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\,r_x' + \frac{4q}{\gamma^3 m v_z^2}\frac{\langle x \frac{\partial \phi}{\partial x}\rangle}{r_x} \mp \frac{B'}{[B\rho]}\,r_x - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\,r_y' + \frac{4q}{\gamma^3 m v_z^2}\frac{\langle y \frac{\partial \phi}{\partial y}\rangle}{r_y} \pm \frac{B'}{[B\rho]}\,r_x - \frac{\varepsilon_y^2}{r_y^3} = 0$$

(for electric focusing $\dfrac{B'}{[B\rho]} \to \dfrac{qE'}{\gamma m v_z^2}$)

SPACE CHARGE TERM WITH ELLIPTICAL SYMMETRY

#4: ELLIPTICAL SYMMETRY: $\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$

CAN BE SHOWN THAT $\left\langle x \frac{\partial\phi}{\partial x} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_x}{r_x + r_y}$

$$\left\langle y \frac{\partial\phi}{\partial y} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_y}{r_x + r_y}$$

DEFINING $Q = \frac{2q\lambda}{4\pi\varepsilon_0 \gamma^3 m v_z^2}$

$$\Rightarrow \boxed{\begin{aligned} & r_x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z) r_x' - \frac{2Q}{r_x + r_y} \mp \frac{B'}{[B\rho]} r_x - \frac{\varepsilon_x^2}{r_x^3} = 0 \\ & r_y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z) r_y' - \frac{2Q}{r_x + r_y} \pm \frac{B'}{[B\rho]} r_y - \frac{\varepsilon_y^2}{r_y^3} = 0 \end{aligned}}$$

(for Electric Focusing $\frac{B'}{[B\rho]} \to \frac{qE'}{[illegible]}$).

SPACE CHARGE TERM WITH ELLIPTICAL SYMMETRY II

ELLIPTICAL SYMMETRY: $\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$

CAN BE SHOWN THAT $\left\langle x \frac{\partial \varphi}{\partial x} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_x}{r_x + r_y}$

(Sacherer, 1971)

& $\left\langle y \frac{\partial \varphi}{\partial y} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_y}{r_x + r_y}$

OUTLINE OF PROOF: (from R. Ryne)

Let $\chi = \frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}$

DEFINE $\eta(\chi)$ such that $\rho(x,y) = \left.\frac{d\eta(\chi)}{d\chi}\right|_{s=0} = \left.\hat{\rho}(\chi)\right|_{s=0}$

So $\rho = \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right) = \left.\hat{\rho}(\chi)\right|_{s=0}$

DEFINE $\Psi(x,y) = \frac{-r_x r_y}{4\varepsilon_0} \int_0^\infty \frac{\eta(\chi)\, ds}{\sqrt{r_x^2+s}\sqrt{r_y^2+s}}$

It follows that $\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\frac{\rho}{\varepsilon_0}$ AND SO IS A SOLUTION OF POISSON'S EQUATION (since $\Psi \to 0$ as $x,y \to \infty$)

WHAT IS $\left\langle x \frac{\partial \Psi}{\partial x} \right\rangle$?

$$\left\langle x \frac{\partial \Psi}{\partial x} \right\rangle = \frac{-r_x r_y}{4\lambda\varepsilon_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, x\, \rho(x,y) \int_0^\infty \frac{\eta' \frac{\partial \chi}{\partial x}\, ds}{\sqrt{r_x^2+s}\sqrt{r_y^2+s}}$$

where $\lambda = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \rho(x,y)$

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So
$$\left\langle x \frac{\partial\phi}{\partial x} \right\rangle = \frac{-2 r_x r_y}{4\lambda\varepsilon_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; x^2 \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right) \int_0^{\infty} \frac{\hat{\rho}\left(\frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}\right) ds}{(r_x^2+s)^{3/2}(r_y^2+s)^{1/2}}$$

$$\boxed{\text{Let} \quad r\cos\theta = \frac{x}{\sqrt{r_x^2+s}} \qquad r\sin\theta = \frac{y}{\sqrt{r_y^2+s}}}$$

$$\det J = \sqrt{r_x^2+s}\,\sqrt{r_y^2+s}\; r$$
where J is the Jacobian
$dx\,dy = \det J \cdot dr\,d\theta$

$$\Rightarrow \left\langle x \frac{\partial\psi}{\partial x} \right\rangle = \frac{-r_x r_y}{\lambda 2\varepsilon_0} \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_0^{\infty} ds \; r^3 \hat{\rho}(r^2)\, \rho\left(\frac{r_x^2+s}{r_x^2} r^2\cos^2\theta + \frac{r_y^2+s}{r_y^2} r^2\sin^2\theta\right) \cdot \cos^2\theta$$

$$\boxed{\text{Let} \quad r'^2 = \frac{r_x^2+s}{r_x^2} r^2\cos^2\theta + \frac{r_y^2+s}{r_y^2} r^2\sin^2\theta}$$
$$= r^2\left[1 + s\left(\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}\right)\right]$$

with r fixed $\quad 2r'dr' = r^2\left(\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}\right) ds$

$$\Rightarrow \left\langle x \frac{\partial\psi}{\partial x} \right\rangle = \frac{-r_x r_y}{2\lambda\varepsilon_0} \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_r^{\infty} dr' \; \frac{2r'dr'\, r^3 \hat{\rho}(r^2) \hat{\rho}(r'^2)\cos^2\theta}{r^2\left(\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}\right)}$$

$$\int_0^{2\pi} \frac{\cos^2\theta \, d\theta}{\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}} = \frac{2\pi\, r_x^2 r_y}{r_x + r_y}$$

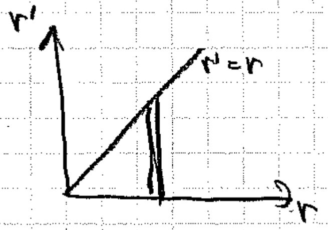
$$\Rightarrow \left\langle x \frac{\partial\psi}{\partial x} \right\rangle = \frac{-r_x^3 r_y^2}{\lambda 2\pi\varepsilon_0 (r_x+r_y)} \int_0^{\infty} dr\, 2\pi r\, \hat{\rho}(r^2) \int_r^{\infty} dr'\, 2\pi r'\, \hat{\rho}(r'^2)$$

Recall $\lambda = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, \rho(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$

$$\boxed{\text{Let} \quad \frac{x}{r_x} = r\cos\theta \qquad \frac{y}{r_y} = r\sin\theta} \qquad \det J = r_x r_y r$$

$$\Rightarrow \lambda = \int_0^{\infty} dr \int_0^{2\pi} d\theta\, \hat{\rho}(r^2)\, r_x r_y r = 2\pi r_x r_y \int_0^{\infty} dr\, r\, \hat{\rho}(r^2)$$

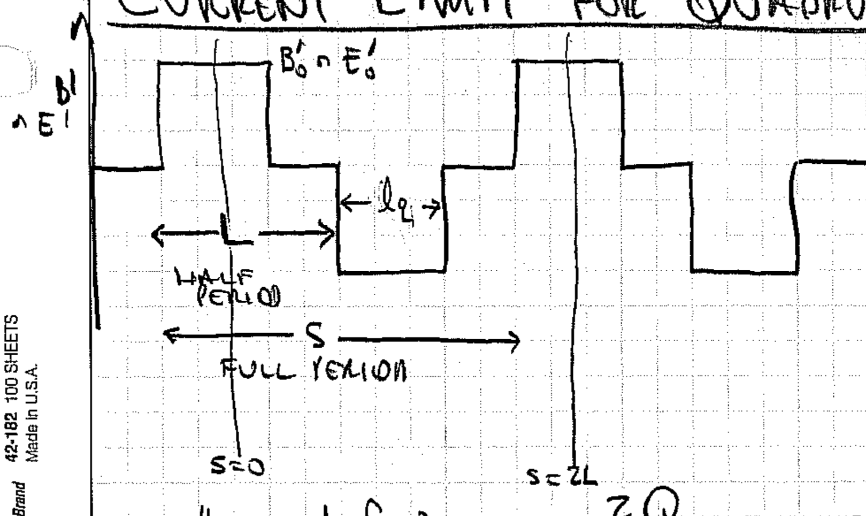
$$\text{Now} \quad \int_0^\infty dr\, r\, \hat{\rho}(r^2) \int_0^r dr'\, r'\, \hat{\rho}(r'^2) = \frac{1}{2} \overbrace{\int_0^\infty dr\, r\, \hat{\rho}(r^2)}^{\lambda/2\pi r_x r_y} \overbrace{\int_0^\infty dr'\, r'\, \hat{\rho}(r'^2)}^{\lambda/2\pi r_x r_y}$$



(by symmetry & consideration of diagram at left.)

$$\Rightarrow \quad \left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-\lambda}{4\pi\epsilon_0} \frac{r_x}{r_x + r_y}$$

CURRENT LIMIT FOR QUADRUPOLES



$$k = \begin{cases} \dfrac{B_0'}{[B\rho]} & \text{MAGNETIC} \\ \dfrac{qE_0'}{\gamma m v_z^2} & \text{ELECTRIC} \end{cases}$$

$$r_x'' + k f(s) r_x - \frac{2Q}{r_x + r_y} = 0$$

(NOTE WE HAVE SET $\epsilon = 0$).

$$r_y'' - k f(s) r_y - \frac{2Q}{r_x + r_y} = 0$$

$$f(s) = \begin{cases} 1 & 0 < s < \eta L/2 \\ -1 & L - \frac{\eta L}{2} < s < L + \eta L/2 \\ 1 & 2L - \eta L/2 < s < 2L \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{2L}\int_0^{2L} f(s)\cos\left(\frac{\pi s}{L}\right)ds = \left(\frac{4}{\pi}\right)\sin\left(\frac{\pi\eta}{2}\right)$$ = fourier amplitude at fundamental lattice frequency

Let $r_x = r_b\left(1 + \delta\cos\left(\frac{\pi s}{L}\right)\right)$

$r_y = r_b\left(1 - \delta\cos\left(\frac{\pi s}{L}\right)\right)$

Let $k f(s) \to k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right)\cos\left(\frac{\pi s}{L}\right)$

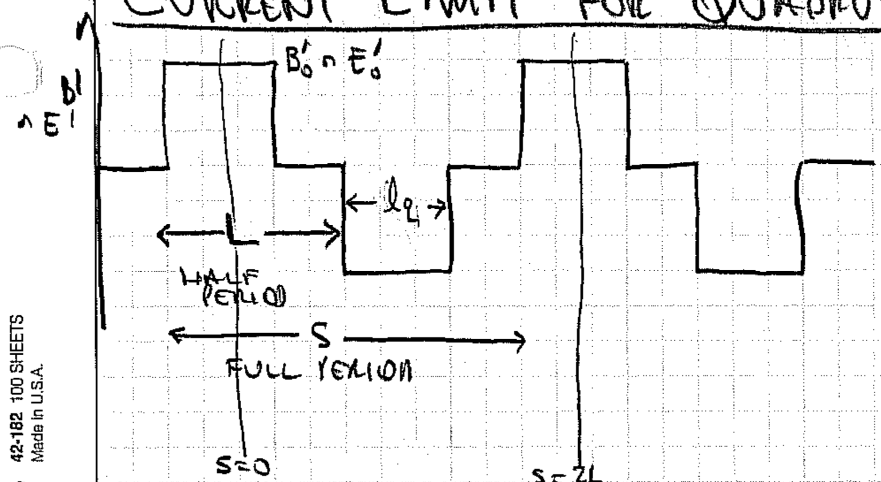
COLLECTING "FAST" TERMS AND "SLOW" TERMS

$$\left[-\frac{\pi^2}{L^2} r_b \delta + k r_b\left(\frac{4\sin\left(\frac{\eta\pi}{2}\right)}{\pi}\right)\right]\cos\left(\frac{\pi s}{L}\right) = 0 \qquad \text{(fast)}$$

$$\delta k r_b\left(\frac{2\sin\left(\frac{\eta\pi}{2}\right)}{\pi}\right) = \frac{Q}{r_b} \qquad \text{(slow)}$$

Fast ⇒ $\delta = \frac{4kL^2}{\pi^3}\sin\left(\frac{\eta\pi}{2}\right)$ & $Q_{max} \cong \frac{2\eta^2 k^2 L^2}{\pi^2}\left(\frac{\sin\left(\frac{\eta\pi}{2}\right)}{\left(\frac{\eta\pi}{2}\right)}\right)^2 r_b^2$

CURRENT LIMIT FOR QUADRUPOLES



$$k = \begin{cases} \dfrac{B_0'}{[B\rho]} & \text{MAGNETIC} \\[2ex] \dfrac{q E_0'}{\gamma m v_z^2} & \text{ELECTRIC} \end{cases}$$

$$r_x'' + k f(s) r_x - \frac{2Q}{r_x + r_y} = 0$$

(NOTE WE HAVE SET $\epsilon = 0$).

$$r_y'' - k f(s) r_y - \frac{2Q}{r_x + r_y} = 0$$

$$f(s) = \begin{cases} 1 & 0 < s < \eta L/2 \\ -1 & L - \frac{\eta L}{2} < s < L + \eta L/2 \\ 1 & 2L - \eta L/2 < s < 2L \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{L}\int_0^{2L} f(s) \cos\left(\frac{\pi s}{L}\right) ds = \left(\frac{4}{\pi}\right) \sin\left(\frac{\pi \eta}{2}\right)$$

= fourier amplitude at fundamental lattice frequency

Let $r_x = r_b\left(1 + \delta \cos\left(\frac{\pi s}{L}\right)\right)$

$r_y = r_b\left(1 - \delta \cos\left(\frac{\pi s}{L}\right)\right)$

Let $k f(s) \rightarrow k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta \pi}{2}\right) \cos\left(\frac{\pi s}{L}\right)$

COLLECTING "FAST" TERMS AND "SLOW" TERMS

$$\left[-\frac{\pi^2}{L^2} r_b \delta + k r_b \left(\frac{4 \sin\left(\frac{\eta\pi}{2}\right)}{\pi}\right)\right] \cos\left(\frac{\pi s}{L}\right) = 0 \qquad \text{(fast)}$$

$$\delta k r_b \left(\frac{2 \sin\left(\frac{\eta\pi}{2}\right)}{\pi}\right) = \frac{Q}{r_b} \qquad \text{(slow)}$$

Fast $\Rightarrow$ $\delta = \dfrac{4 k L^2}{\pi^3} \sin\left(\dfrac{\eta\pi}{2}\right)$ & $Q_{max} \cong \dfrac{2\eta^2 k^2 L^2}{\pi^2} \left(\dfrac{\sin\left(\frac{\eta\pi}{2}\right)}{\left(\frac{\eta\pi}{2}\right)}\right)^2 r_b^2$

Focusing term has both a fast and slow component:

$$kf(s)r_x \rightarrow k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right)\cos\left(\frac{\pi s}{L}\right)r_b\left(1+\delta\cos\left(\frac{\pi s}{L}\right)\right)$$

$$= r_b k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right)\cos\left(\frac{\pi s}{L}\right) + \delta r_b k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right)\cos\left(\frac{\pi s}{L}\right)^2$$

$$= r_b k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right)\cos\left(\frac{\pi s}{L}\right) + \delta r_b k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right)\left(\frac{1}{2}+\cancel{\frac{1}{2}\cos\left(\frac{2\pi s}{L}\right)}\right)$$

$$\cong r_b k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right)\cos\left(\frac{\pi s}{L}\right) + \delta r_b k\left(\frac{4}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right)\left(\frac{1}{2}\right)$$

$$r_x'' = -r_b\left(\frac{\pi^2}{L^2}\delta\right)\cos\left(\frac{\pi s}{L}\right)$$

$$r_y'' = r_b\left(\frac{\pi^2}{L^2}\delta\right)\cos\left(\frac{\pi s}{L}\right)$$

$$\frac{Q}{r_x+r_y}=\frac{Q}{2r_b}$$

CONTINUOUS FOCUSING

$$r_x'' = -k_{\beta 0}^2 r_x + \frac{2Q}{r_x + r_y} - \frac{\epsilon^2}{r_x^3}$$

$$r_y'' = -k_{\beta 0}^2 r_y + \frac{2Q}{r_x + r_y} - \frac{\epsilon^2}{r_y^3}$$

CURRENT LIMIT BALANCES PERVEANCE & EXTERNAL FOCUSING ($r_x = r_y = r_b$):

$$k_{\beta 0}^2 r_b = \frac{Q_{max}}{r_b}$$

Effective $k_{\beta 0}^2$ FOR QUADRUPOLES FOUND FROM DOMINANT FOURIER COMPONENT

$$k_{\beta 0}^2 = \frac{2\eta^2 k^2 L^2}{\pi^2} \left(\frac{\sin\left(\frac{\eta\pi}{2}\right)}{\frac{\eta\pi}{2}} \right)^2 \qquad \text{where } k = \frac{B'}{[B\rho]}$$

FOR CONTINUOUS FOCUSING: $k_{\beta 0}^2 = \dfrac{\sigma_0^2}{4L^2}$

ELIMINATING L:

$$Q_{max} = \frac{\eta k \sigma_0}{\sqrt{2}\pi} \left(\frac{\sin\frac{\eta\pi}{2}}{\frac{\eta\pi}{2}} \right) r_b^2 \longleftarrow \text{PERVEANCE LIMIT FOR FODO QUADRUPOLES}$$

Envelope instabilities set upper limit on "single particle" phase advance σ_0

Experimental data (Tiefenback, 1986) from LBL's Single Beam Transport Experiment (SBTE)

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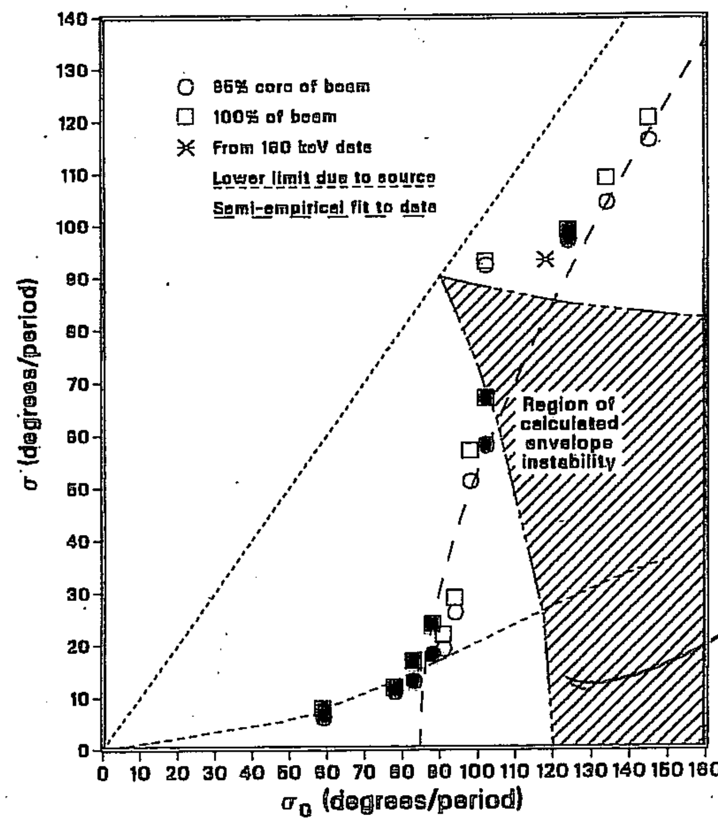
$\sigma_0 < 85^\circ$

Experimental limits on beam stability in terms of σ and σ_0



SEE LUND & CHAWLA 2006, NIMPR-A, FOR HIGHER ORDER PARTICLE–LATTICE RESONANCES WHICH CLARIFIES $\sigma_0 = 85^\circ$ LIMIT

SEE STRUCKMEIR & REISER, PARTICLE ACCELERATORS 14, 227, (1984) & LUND & BUKH, PRSTAB, 7 024801 (2004)

J. BARNARD

Quadrupole Current Limit — continued

$$Q_{max} \cong \frac{\eta k \Theta_0}{\sqrt{2\pi}} \left(\frac{\sin \frac{\eta\pi}{2}}{\frac{\eta\pi}{2}} \right) r_b^2$$

here $k = \begin{cases} \dfrac{dB/dx}{[B\ell]} \sim \dfrac{B}{[B\ell]\, r_p} & \text{(MAGNETIC QUAD FODO)} \\[2ex] \dfrac{q\, dE/dx}{\gamma m v_z^2} \sim \dfrac{2 q V_q}{\gamma m v_z^2\, r_p^2} & \text{where } V_q = \frac{1}{2}\frac{dE}{dx} r_p^2 \text{ (ELECTRIC QUAD FODO)} \end{cases}$

So

$$Q_{max} \cong \frac{\eta \Theta_0}{\sqrt{2\pi}} \left(\frac{\sin \frac{\eta\pi}{2}}{\frac{\eta\pi}{2}} \right) \begin{cases} \dfrac{B\, r_b}{[B\ell]} \left[\dfrac{r_b}{r_p} \right] & \text{(MAGNETIC QUAD)} \\[2ex] \dfrac{2 q V_q}{\gamma m v_z^2} \left[\dfrac{r_b^2}{r_p^2} \right] & \text{(ELECTRIC QUAD)} \end{cases}$$

Summary of Current Limits From Different Focusing Methods

Einzel Lens

$$Q_{max} \cong \frac{3\pi^2}{8}\left(\frac{q\phi_0}{mv_0^2}\right)^2\left(\frac{r_b}{L}\right)^2$$

Solenoids

$$Q_{max} = \left(\frac{\omega_c r_b}{2\gamma\beta c}\right)^2$$

Quadrupole Focusing

$$Q_{max} \cong \frac{\eta\,\Phi_{0}}{\sqrt{2\pi}}\left(\frac{\sin\frac{\eta\pi}{2}}{\frac{\eta\pi}{2}}\right)\left[\begin{array}{ll}\frac{B r_b}{[B\rho]}\left[\frac{r_b}{r_p}\right] & \text{MAGNETIC} \\ \frac{2qV_q}{\gamma m v_z^2}\left[\frac{r_b^2}{r_p^2}\right] & \text{Electric}\end{array}\right.$$

For Non-Relativistic Beams

$$\lambda_{max} \propto \frac{\phi_0^2}{V} \qquad \lambda_{max} \propto \frac{q}{m}B^2 r_p^2 \qquad \lambda_{max} \propto \begin{cases} B_1 V^{1/2} r_p \\ V_q \end{cases}$$

Note
ϕ_0 = Voltage between Einzel lenses
V_q = Voltage on a quad relative to ground
V = particle energy / q