John Barnard
Steven Lund
USPAS
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Injectors and longitudinal physics -- II

1. Acceleration - introduction
2. Space charge of short bunches (rf)
3. Space charge of long bunches
4. Longitudinal space charge waves
5. Longitudinal rarefaction waves and bunch ends

Summary of fluid equations

Let $n(\underline{x},t) = \int d^3p \; f(\underline{x},\underline{p},t)$ PARTICLE DENSITY

$$\underline{\bar{v}}(\underline{x},t) \equiv \frac{1}{n(\underline{x},t)} \int d^3p \; \frac{\underline{p}}{\gamma m} \, f(\underline{x},\underline{p},t)$$ FLUID VELOCITY

$$\underline{\bar{p}}(\underline{x},t) \equiv \frac{1}{n(\underline{x},t)} \int d^3p \; \underline{p} \, f(\underline{x},\underline{p},t)$$ FLUID MOMENTUM

$$\underline{\underline{\mathbb{P}}}(\underline{x},t) \equiv \int d^3p \, (\underline{p} - \underline{\bar{p}})\left(\frac{\underline{p}}{\gamma m} - \underline{\bar{v}}\right) f(\underline{x},\underline{p},t)$$ PRESSURE TENSOR

$$\frac{d\underline{x}}{dt} = \frac{\underline{p}}{\gamma m} \qquad \frac{d\underline{p}}{dt} = q\left(\underline{E}(\underline{x},t) + \frac{\underline{p}}{\gamma m} \times \underline{B}(\underline{x},t)\right) \qquad \gamma^2 = \frac{\underline{p}\cdot\underline{p}}{(mc)^2} + 1$$

CONTINUITY EQUATION:
$$\frac{\partial n(\underline{x},t)}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot n(\underline{x},t)\, \underline{\bar{v}}(\underline{x},t) = 0$$

MOMENTUM EQUATION:
$$\frac{\partial \underline{\bar{p}}}{\partial t} + \underline{v} \cdot \frac{\partial \underline{\bar{p}}}{\partial \underline{x}} = q\left(\underline{E} + \underline{\bar{v}} \times \underline{B}\right) - \frac{1}{n(\underline{x},t)} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{\mathbb{P}}}$$

THE ABOVE EQUATIONS ARE RELATIVISTICALLY CORRECT. IN THE NON-RELATIVISTIC LIMIT THE CONTINUITY EQUATION REMAINS UNCHANGED & THE MOMENTUM EQUATION MAY BE WRITTEN:

NON RELATIVISTIC $\longrightarrow$
$$\frac{\partial \underline{\bar{v}}}{\partial t} + \underline{\bar{v}} \cdot \frac{\partial \underline{\bar{v}}}{\partial \underline{x}} = \frac{q}{m}\left(\underline{E} + \underline{\bar{v}} \times \underline{B}\right) - \frac{1}{mn} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{\mathbb{P}}}$$

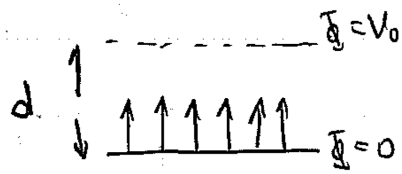
THESE EQUATIONS ARE SUPPLEMENTED WITH MAXWELL'S EQUATIONS: for $\underline{E}(\underline{x},t)$ & $\underline{B}(\underline{x},t)$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{E} = \frac{q\, n(\underline{x},t)}{\varepsilon_0} \qquad \frac{\partial}{\partial \underline{x}} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{B} = 0 \qquad \frac{\partial}{\partial \underline{x}} \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \qquad \underline{J}(\underline{x},t) = q\, n(\underline{x},t)\, \underline{\bar{v}}(\underline{x})$$

NEED ADDITIONAL EQUATIONS SUCH AS $\underline{\underline{\mathbb{P}}} = 0$ OR ENERGY EQUATION TO TERMINATE SET OF EQUATIONS.

Summary of Child Langmuir Law



1D $\qquad \Phi = -\phi$ (ACTUAL POTENTIAL)

CURRENT DENSITY: $$J = \frac{4}{9}\varepsilon_0 \left(\frac{2q}{m}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$$

$$\Phi(z) = V_0 \left(\frac{z}{d}\right)^{4/3}$$ – ELECTROSTATIC POTENTIAL

$$E(z) = \frac{4}{3}\frac{V_0}{d}\left(\frac{z}{d}\right)^{1/3}$$ ELECTRIC FIELD

$$v(z) = \left(\frac{2qV_0}{m}\right)^{1/2}\left(\frac{z}{d}\right)^{2/3}$$ LONGITUDINAL VELOCITY

$$\rho(z) = \frac{J}{v(z)} = \left(\frac{J^2 m}{2qV_0}\right)^{1/2}\left(\frac{z}{d}\right)^{-2/3}$$

IF WE MULTIPLY BY πr_b^2 (TO ACCOUNT FOR FINITE BEAM RADIUS):

$$I = \underbrace{\frac{4}{9}\varepsilon_0\left(\frac{2q}{m}\right)^{1/2}\left(\frac{r_b}{d}\right)^2}_{K \,\equiv\, \text{Gun perveance} \,\equiv\, \frac{I}{V_0^{3/2}}} V_0^{3/2}$$

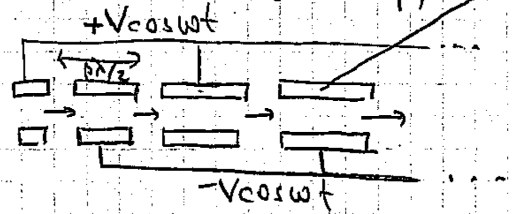
$\left[\text{DIMENSIONS: } \frac{\text{CURRENT}}{\text{VOLTAGE}^{3/2}}\right]$

GENERALIZED PERVEANCE: (DIMENSIONLESS)
$$Q(z) = \frac{\lambda}{4\pi\varepsilon_0 \Phi(z)} = \frac{\pi r_b^2 \rho(z)}{4\pi\varepsilon_0 \Phi(z)} = \frac{1}{9}\left(\frac{r_b^2}{z^2}\right)$$

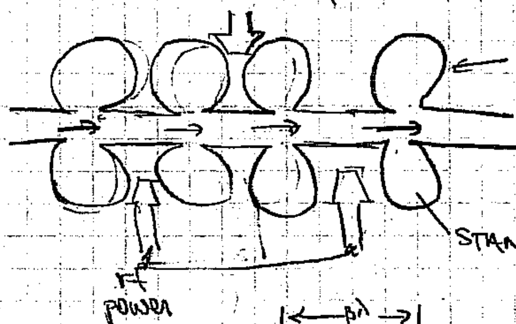
(NOTE THAT CHILD-LANGMUIR LAW ONLY VALID FOR $z \gg \lambda_D$, WHERE $\frac{1}{2}mv(z)^2 \gg kT$ & $\lambda_D = \frac{v_{th}}{\omega_p} = \frac{\sqrt{kT/m}}{\left(\frac{q n(z)}{\varepsilon_0 m}\right)^{1/2}}$)

ACCELERATION

rf (radio-frequency)

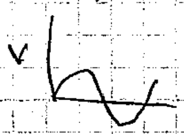


(Wideroe linac)
LOW FREQUENCIES (< 100 MHz)



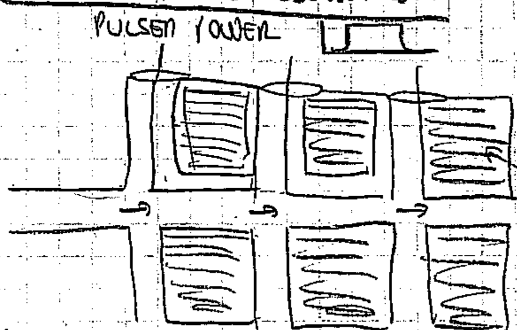
(COUPLED CAVITY Linac)
$0.4 < \beta < 1.0$

FREQUENCIES ~ 100's MHz – ~ 1 GHz



IN EACH GAP $E = E_{peak} \cos \omega t$

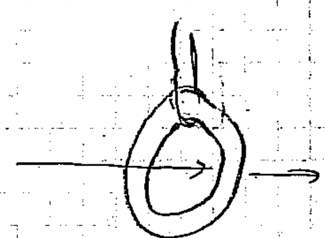
Induction acceleration



(INDUCTION LINAC)

$\nabla \times E = -\partial B/\partial t$

IN EACH GAP E = CONSTANT
(OR SOME PRESCRIBED FUNCTION)



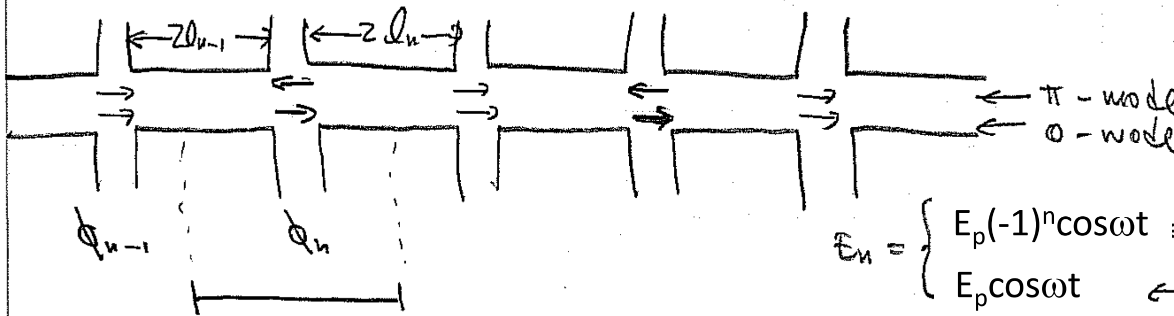
TRANSFORMER

I STRATEGY FOR CALCULATING LONGITUDINAL EQUATIONS OF MOTION

1. rf / short bunches
   - EXTERNAL FIELD: ACCELERATES AND FOCUSES
     - CALCULATE CHANGE IN PHASE AND ENERGY AS PARTICLE MOVES FROM ACCELERATING GAP TO ACCELERATING GAP
     - APPROXIMATE MOTION AS CONTINUOUS
   - SPACE CHARGE FIELD: UNIFORM DENSITY ELLIPSOID

2. INDUCTION / LONG BUNCHES
   - EXTERNAL FIELD: ACCELERATION, FOCUSING, AND COMPRESSION ACCOMPLISHED BY PRESCRIBING VOLTAGE WAVE FORM
   - FOCUSING (CONFINEMENT) IS DONE AT BEAM ENDS
   - SPACE CHARGE FIELD: $E_z \propto \frac{\partial \lambda}{\partial z}$

rf longitudinal equation of motion



$$E_n = \begin{cases} E_p(-1)^n \cos\omega t & \leftarrow \pi\text{-mode} \\ E_p \cos\omega t & \leftarrow 0\text{-mode} \end{cases}$$

L_n = center to center distance between drift tubes

$E_z = E_p \cos(\phi_s)$ ⟵ synchronous particle enters each gap at same phase

RESONANCE CONDITION ON SYNCHRONOUS PARTICLE:

$$\boxed{l_{n-1} = \frac{\beta_s \lambda}{2} \begin{Bmatrix} \frac{1}{2} & \pi\text{-mode} \\ 1 & 0\text{-mode} \end{Bmatrix} \qquad \lambda = \frac{2\pi}{\omega} c = c\Delta t_{rf}}$$

λ = light travel distance in one rf oscillation

(IT TAKES $\begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$ OSCILLATION PERIOD TO TRAVEL BETWEEN GAPS).

$\beta_s = \frac{V_s}{c}$ = velocity of synchronous particle

PARTICLE PHASE RELATIVE TO rf at the n^{th} gap:

$$\phi_n = \phi_{n-1} + \omega \frac{2l_{n-1}}{\beta_{n-1} c} + \begin{Bmatrix} \pi & \pi\text{-mode} \\ 0 & 0\text{-mode} \end{Bmatrix}$$

$$\Delta(\phi - \phi_s)_n = 2\pi \beta_{s,n-1} \left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right) \begin{Bmatrix} 1/2 & \pi\text{-mode} \\ 1 & 0\text{-mode} \end{Bmatrix}$$

$$\cong -2\pi \frac{\delta\beta}{\beta_{s,n-1}} \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$$

A VELOCITY DIFFERENCE LEADS TO A PHASE DIFFERENCE!

$$\boxed{\Delta(\phi - \phi_s)_n \cong -2\pi \frac{W_{n-1} - W_{s,n-1}}{mc^2 \gamma^3_{s,n-1} \beta^2_{s,n-1}} \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}}$$

$W = (\gamma - 1)mc^2$

$\frac{1}{\beta} - \frac{1}{\beta_s} \cong \frac{-\delta\beta}{\beta_s^2}$

$\delta W = \gamma_s^3 \beta_s mc^2 \, \delta\beta$

SIMILARLY, A PHASE DIFFERENCE PRODUCES AN ENERGY CHANGE (RELATIVE TO SYNCHRONOUS PARTICLE)

$$\boxed{\Delta (W - W_s)_n = q E_0 L_n (\cos\varphi_n - \cos\varphi_{s,n})}$$

$$L_n = \frac{(\beta_{s,n-1} + \beta_{s,n})\lambda}{2} \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix} =$$ CENTER-TO-CENTER DISTANCE BETWEEN DRIFT SECTIONS

$$(\Delta W_s = q E_0 L_n \cos\phi_s)$$

ENERGY (VELOCITY) DIFFERENCE $\Rightarrow$ ARRIVAL TIME DIFFERENCE (PHASE DIFFERENCE)

PHASE DIFFERENCE IN rf FIELD $\Rightarrow$ DIFFERENCE IN ENERGY GAIN

Note that E_o = the electric field averaged over the distance between accelerating gaps times the "transit time factor". The transit time factor accounts for the change of the field during the transit of the beam through the gap of width d.

$E_o \cong E_p (d/L_n)\, T$ where E_p is the peak field and

where $T \cong \int_{-d/2}^{d/2} E_{peak} \cos(2\pi z/\beta\lambda)\, dz \big/ \int_{-d/2}^{d/2} E_{peak}\, dz$

$= \sin(\pi d/\beta\lambda)/(\pi d/\beta\lambda)$ --> 1 if $\pi d/\beta\lambda << 1$

CONVERTING TO A CONTINUOUS VARIABLE:

$$\Delta(\phi - \phi_s) \rightarrow \frac{d\,\Delta\phi}{dn} \qquad \Delta(W - W_s) \rightarrow \frac{d\,\Delta W}{dn}$$

$$dn = \frac{ds}{\beta_s \lambda} \cdot \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \frac{ds}{2l_{n-1}}$$

$$\Rightarrow \boxed{\begin{aligned} \gamma_s^3 \beta_s^3 \frac{d\Delta\phi}{ds} &= -2\pi \frac{\Delta W}{mc^2 \lambda} \\ \frac{d\Delta W}{ds} &= qE_0 (\cos\phi - \cos\phi_s) \end{aligned}}$$

$$\frac{d}{ds}\left[\gamma_s^3 \beta_s^3 \frac{d\Delta\phi}{ds}\right] = -2\pi \frac{qE_0}{mc^2 \lambda}\left[\cos\phi - \cos\phi_s\right] \qquad (\mathrm{I})$$

NOW THE SPATIAL SEPARATION IS GIVEN BY:

$$\Delta z \equiv z - z_s = -\frac{\beta_s \lambda}{2\pi} \Delta\phi \quad ,$$

$\Rightarrow \frac{d}{ds}$ ALSO, LET $\cos\phi - \cos\phi_s \simeq -\sin\phi_s \Delta\phi \qquad \left[\text{for } \frac{2\pi \Delta z}{\beta\lambda} = \Delta\phi \ll 1\right]$

$$\Rightarrow \frac{d}{ds}\left[\gamma_s^3 \beta_s^3 \frac{d}{ds}\left(\frac{\Delta z}{\beta_s}\right)\right] \cong \frac{2\pi}{\lambda} \frac{qE_0}{mc^2} \sin\phi_s \frac{\Delta z}{\beta_s}$$

WHEN THE ACCELERATION RATE IS SMALL

$$\Rightarrow \frac{d^2}{ds^2}\Delta z \cong \frac{2\pi}{\lambda} \frac{qE_0 \sin\phi_s}{\gamma_s^3 \beta_s^3 mc^2} \Delta z$$

$$\equiv -k_{s0}^2 \Delta z \qquad \text{(synchrotron oscillations)}$$

RETURNING TO $\Delta W - \phi$ NOTATION

Let $w = \frac{\Delta W}{mc^2}$ $\quad A = \frac{2\pi}{\beta_s^3 \gamma_s^3 \lambda}$ $\quad B = \frac{qE_0}{mc^2}$

$$\boxed{\begin{aligned} \Rightarrow \quad w' &= B(\cos\phi - \cos\phi_s) \\ \phi' &= -Aw \\ \phi'' &= -AB(\cos\phi - \cos\phi_s) \end{aligned}}$$

MULTIPLYING BY ϕ' AND INTEGRATING:

$$\frac{\phi'^2}{2} = -AB(\sin\phi - \phi\cos\phi_s) + \text{const}$$

USING $\phi' = -Aw$ (& DIVIDING BY A)

$$\Rightarrow \quad \underbrace{\frac{Aw^2}{2}}_{\text{kinetic energy}} + \underbrace{B(\sin\phi - \phi\cos\phi_s)}_{\text{potential energy}} = \text{CONST.}$$

$$\frac{dW_s}{ds} \sim qE_0\cos\phi_s$$

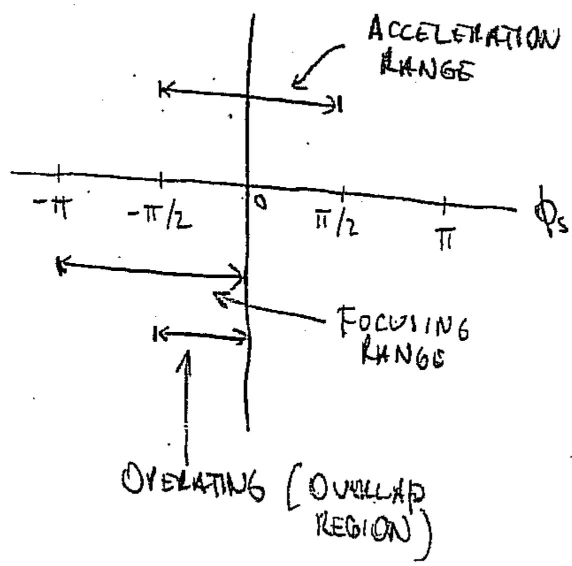
$$V(\phi) = B(\sin\phi - \phi\cos\phi_s)$$

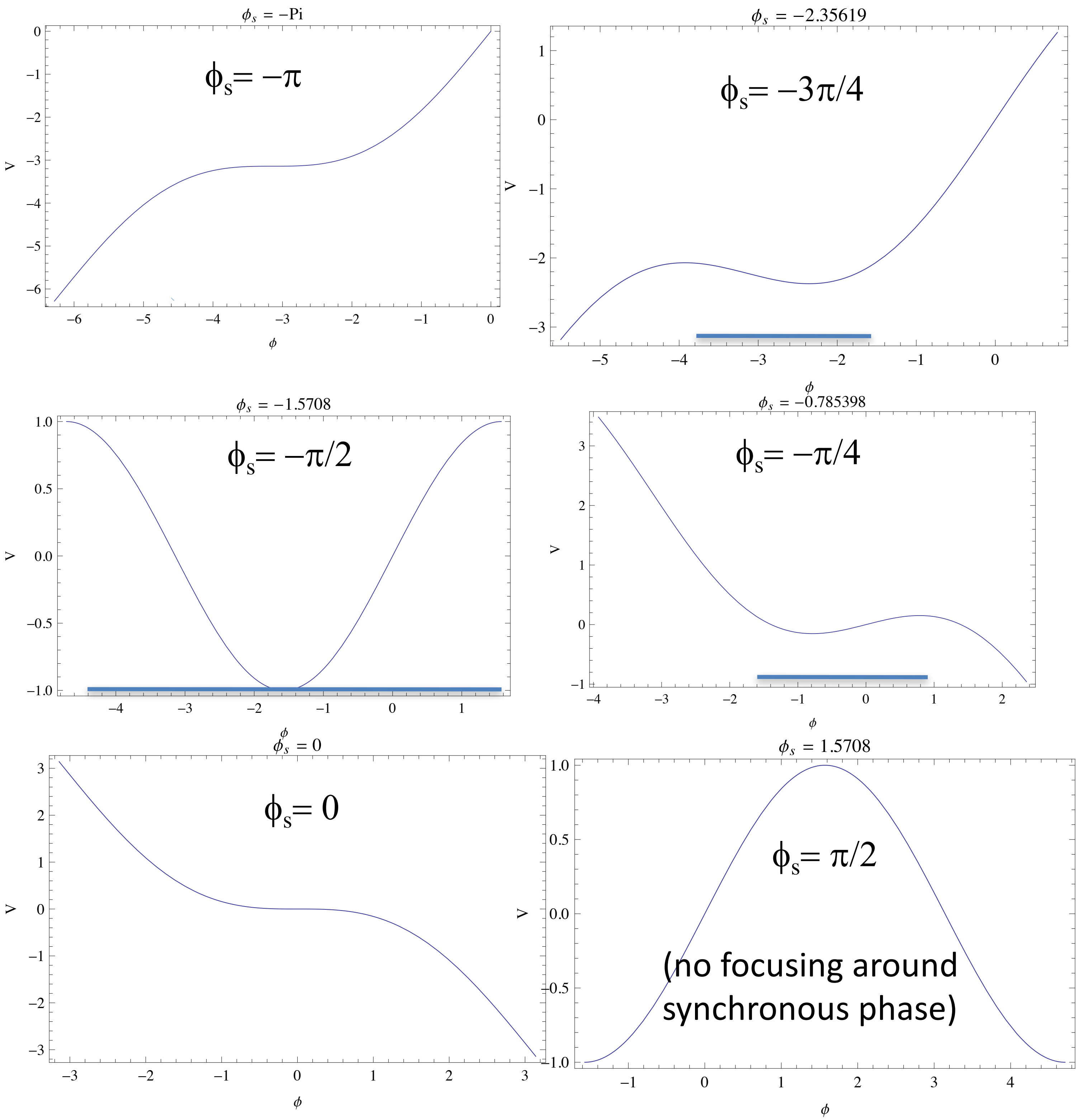
$$\frac{dV}{d\phi} = B(\cos\phi - \cos\phi_s)$$

$$\frac{d^2V}{d\phi^2} = -B\sin\phi$$

$> 0 \Rightarrow -\pi < \phi_s < 0$

FOR LONGITUDINAL FOCUSING





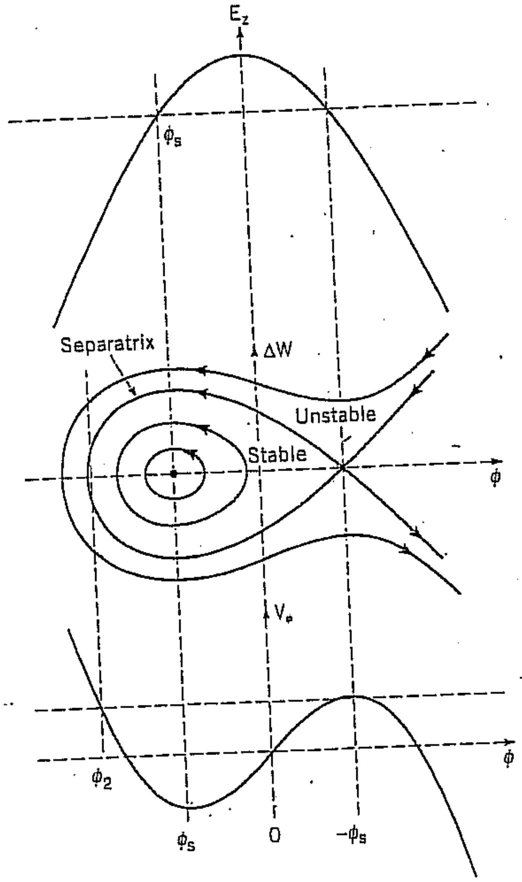
$$V(\phi) \sim B(\sin\phi - \phi\cos\phi_s)$$

For there to be a region of focusing around the synchronous phase, $-\pi < \phi_s < 0$.

7

simultaneous acceleration and a potential well when $-\pi/2 \leq \phi_s \leq 0$. The stable region for the phase motion extends from $\phi_2 < \phi < -\phi_s$, where the lower phase limit ϕ_2 can be obtained numerically by solving for ϕ_2 using $H_\phi(\phi_2) = H_\phi(-\phi_s)$. Figure 6.3 shows longitudinal phase space and the longitudinal potential well. At the potential maximum, where $\phi = -\phi_s$, we

from T. Wangler's "PRINCIPLES of RF LINEAR ACCELERATORS"



w-φ PHASE SPACE

Figure 6.3. At the top, the accelerating field is shown as a cosine function of the phase; the synchronous phase ϕ_s is shown as a negative number, which lies earlier than the crest where the field is rising in time. The middle plot shows some longitudinal phase-space trajectories, including the separatrix, the limiting stable trajectory, which passes through the unstable fixed point at $\Delta W = 0$, and $\phi = -\phi_s$. The stable fixed point lies at $\Delta W = 0$ and $\phi = \phi_s$, where the longitudinal potential well has its minimum, as shown in the bottom plot.

Electrostatic Potential of a uniform density ellipsoid in free space (cf Landau & Lifshitz, Classical Theory of Fields, p. 297)

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \frac{z^2}{r_z^2} = 1$$

It can be shown that:

$$\varphi = \frac{\rho}{4\varepsilon_0} r_x r_y r_z \int_{s_{min}}^{\infty} \left(1 - \frac{x^2}{r_x^2+s} - \frac{y^2}{r_y^2+s} - \frac{z^2}{r_z^2+s}\right) \frac{ds}{R_s}$$

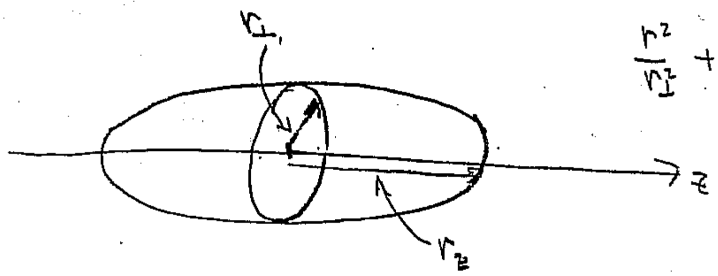
$$R_s = \sqrt{(r_x^2+s)(r_y^2+s)(r_z^2+s)}$$

$$s_{min} = \begin{cases} 0 & \text{if interior point } \left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \frac{z^2}{r_z^2} < 1\right) \\ \xi & \text{if exterior point } \left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \frac{z^2}{r_z^2} > 1\right) \end{cases}$$

Here ξ is the positive root of

$$\frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} + \frac{z^2}{r_z^2+\xi} = 1$$

SPACE-CHARGE FIELD OF AXISYMMETRIC BUNCHED BEAMS



$$\frac{r^2}{r_\perp^2} + \frac{z^2}{r_z^2} = 1$$

BOUNDARY OF ELLIPSOID

INTERIOR

THE POTENTIAL OF A UNIFORM DENSITY BUNCH IN FREE SPACE (A MACLAURIN SPHEROID) IS GIVEN BY:

(cf Landau & Lifshitz, Classical Theory of Fields, p 297)

$$\varphi = \frac{-\rho}{4\epsilon_0}\left(\alpha_\perp r^2 + \alpha_\parallel z^2 - \delta\right)$$

where $\alpha_\perp = r_\perp^2 r_z \int_0^\infty \frac{ds}{(r_\perp^2+s)\,\Delta}$

$$\alpha_\parallel = r_\perp^2 r_z \int_0^\infty \frac{ds}{(r_z^2+s)\,\Delta}$$

$$\delta = r_\perp^2 r_z \int_0^\infty \frac{ds}{\Delta}$$

where $\Delta^2 = (r_\perp^2+s)^2(r_z^2+s)$

FOR NON-RELATIVISTIC BEAM:

$$E_z = -\frac{\partial\phi}{\partial z} = f\,\frac{\rho}{\epsilon_0}\,z$$

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{(1-f)}{2}\,\frac{\rho}{\epsilon_0}\,r$$

$$f = f(\alpha) = \begin{cases} \frac{\alpha^2}{1-\alpha^2}\left[\frac{1}{\sqrt{1-\alpha^2}}\tanh^{-1}\sqrt{1-\alpha^2} - 1\right] & \alpha < 1 \\ \frac{1}{3} & \alpha = 1 \\ \frac{\alpha^2}{\alpha^2-1}\left[1 - \frac{1}{\sqrt{\alpha^2-1}}\tan^{-1}\sqrt{\alpha^2-1}\right] & \alpha > 1 \end{cases} \qquad \alpha \equiv \frac{r_\perp}{r_z}$$

Check: for a spherical bunch

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad 4\pi r^2 E_r = \frac{4\pi r^3}{3} \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow E_r = \frac{\rho}{3\varepsilon_0} r$$

THE FIELD FOR ALL RADII MAY BE WRITTEN:

$$E_r = \frac{\rho}{2\varepsilon_0}\left[\frac{\alpha^2}{(\alpha^2+\chi)(1+\chi)^{1/2}} - F(\chi,\alpha)\right] r$$

$$E_z = \frac{\rho}{\varepsilon_0}\left[\, F(\chi,\alpha)\right] z$$

$$F(\chi,\alpha) = \begin{cases} \dfrac{\alpha^2}{1-\alpha^2}\left[\dfrac{1}{\sqrt{1-\alpha^2}}\tanh^{-1}\dfrac{\sqrt{1-\alpha^2}}{\sqrt{1+\chi}} - \dfrac{1}{\sqrt{1+\chi}}\right] & \alpha < 1 \\ \dfrac{1}{3(1+\chi)^{3/2}} & \alpha = 1 \\ \dfrac{\alpha^2}{\alpha^2-1}\left[\dfrac{1}{\sqrt{1+\chi}} - \dfrac{1}{\sqrt{\alpha^2-1}}\tan^{-1}\left(\dfrac{\sqrt{\alpha^2-1}}{\sqrt{1+\chi}}\right)\right] & \alpha > 1 \end{cases}$$

χ satisfies:

$$\frac{\alpha^2 (r^2/r_\perp^2)}{\alpha^2+\chi} + \frac{(z^2/r_z^2)}{1+\chi} = 1 \qquad \text{for exterior particle}$$

$$\chi = 0 \qquad \text{for interior particle}$$

$$\alpha \equiv \frac{r_\perp}{r_z}$$

FOR AN EXTERIOR PARTICLE AT r & z $\Rightarrow$ χ CAN BE SOLVED FOR (QUADRATIC EQUATION FOR χ).

$\Rightarrow$ E_r & E_z ARE KNOWN ANALYTICALLY FOR ALL r, z.

EXAMPLE: $r = 0$ $\Rightarrow$ $\chi = \dfrac{z^2}{r_z^2} - 1$

(and $z > r_z$)

RELATIVISTIC TRANSFORMATION FROM BEAM FRAME TO LAB FRAME:
(see e.g., JACKSON, CLASSICAL ELECTRODYNAMICS)

$$\underline{E} = \gamma(\underline{E}' - c\underline{\beta} \times \underline{B}') - \frac{\gamma^2}{\gamma+1} \underline{\beta}(\underline{\beta} \cdot \underline{E}')$$

$$\underline{B} = \gamma(\underline{B}' + \frac{\underline{\beta}}{c} \times \underline{E}') - \frac{\gamma^2}{\gamma+1} \underline{\beta}(\underline{\beta} \cdot \underline{B}')$$

$'$ = BEAM FRAME

NO PRIME = LAB FRAME

$$\underline{B}' = 0$$

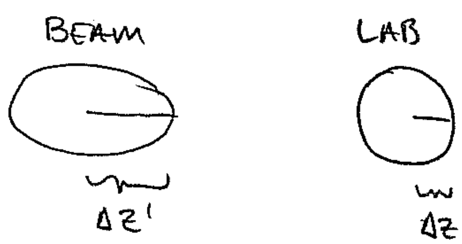
$$\Rightarrow \quad E_z = \gamma E_z' - \frac{\gamma^2}{\gamma+1}\beta^2 E_z' = E_z'$$

$$E_r = \gamma E_r'$$

$$B_\theta = \frac{\gamma\beta}{c} E_r'$$

$$F_z = q E_z = q E_z'$$

$$F_r = q(E_r - v_z B_\theta) = \frac{1}{\gamma} q E_r'$$



$$\Delta z = \frac{1}{\gamma} \Delta z'$$

$$\underline{x}_\perp = \underline{x}_\perp'$$

$$\rho = \gamma \rho'$$

$$\underline{F}_\perp = \frac{d\underline{p}_\perp}{dt} = \gamma_s m v_{zs}^2 \frac{d^2 \underline{x}_\perp}{ds^2} \quad \left(\text{NEGLECTING } \frac{d\gamma_s}{dt}, \frac{dv_{zs}}{dt}\right)$$

$$\Delta F_z = \frac{dp_{\parallel}}{dt} - \frac{dp_{\parallel s}}{dt} = \gamma_s^3 m v_{zs}^2 \frac{d^2(z - z_s)}{ds^2} \quad \left(\text{USING } \frac{d}{ds}\gamma\beta = \gamma^3 \frac{d\beta}{ds}\right)$$

FOR RELATIVISTIC BEAM

(cf. Barnard & Lund 1997 & Lund & Barnard 1997 PAC97 Conf Proceedings)

$$\frac{d^2 \underline{x}_\perp}{ds^2} = \frac{F_\perp}{\gamma_s \beta_s^2 m c^2} \qquad \underline{F}_{\perp s} = \frac{q\rho}{2\gamma_s^2 \epsilon_0}\left[1 - f(\alpha)\right] \underline{x}_\perp$$

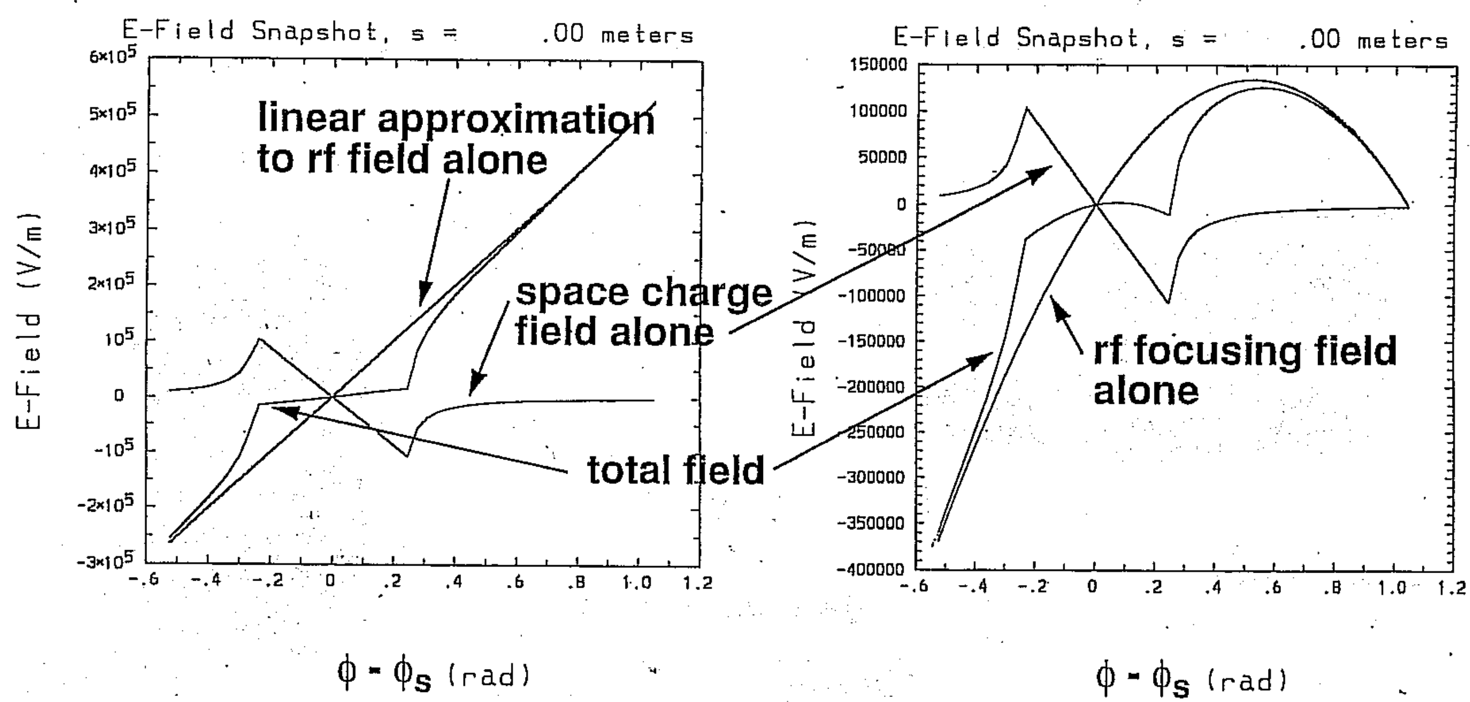
$$\frac{d^2 \Delta z}{ds^2} = \frac{F_z}{\gamma_s^3 \beta_s^2 m c^2} \qquad F_{zs} = \frac{q\rho}{\epsilon_0} f(\alpha)\, \Delta z$$

$$\alpha = \frac{r_\perp}{\gamma r_z} \qquad \left[\alpha = \frac{r_\perp}{(r_z \text{ in comoving frame})}\right]$$

COMBINING FOCUSING + SELF FIELDS

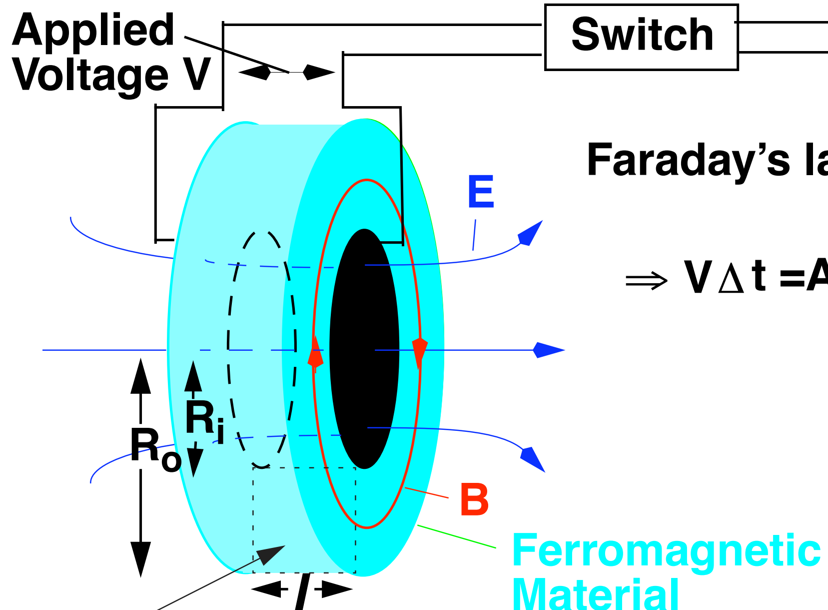
$$\frac{d^2}{ds^2}\Delta z = -k_{s0}^2 \Delta z + \frac{q\rho\, f(\alpha)}{\gamma_s^3 \beta_s^2 m c^2 \epsilon_0}\Delta z \qquad \text{(LINEAR rf)}$$

Total field seen by particle is sum of rf and spacecharge



here $\phi - \phi_s = -(2\pi/\beta_s\lambda)\,\Delta z$, where $\beta_s c$ is the longitudinal velocity of the synchronous particle and $\lambda = c/\nu$ is the rf vacuum wavelength

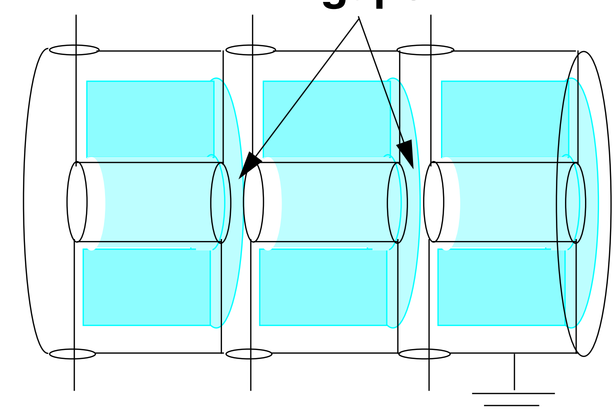
Induction acceleration: Volt-second limits



Cross-sectional area A
$A = (R_o - R_i)\ l$

Faraday's law, $\nabla \times E = -\frac{\partial B}{\partial t}$

$\Rightarrow V \Delta t = A \Delta B$

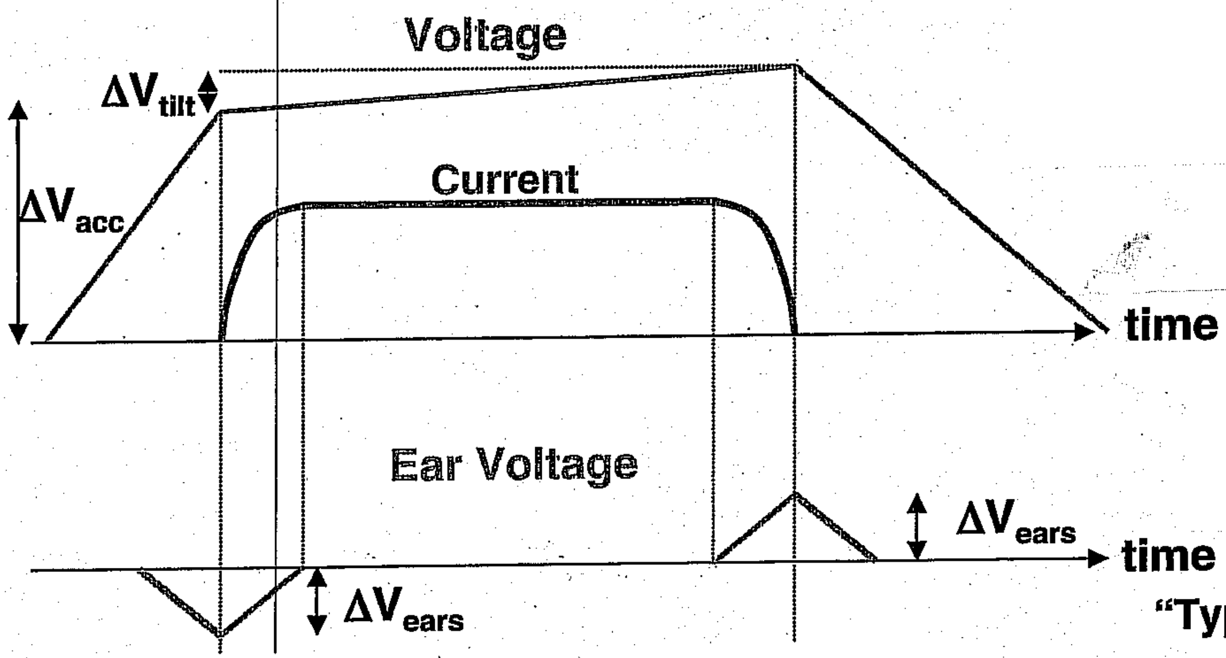


Volt-seconds per m: $(dV/dz)\ \Delta t = (R_o - R_i)\ \Delta B\ f_{radial}\ f_{longit.}$

~ 1 m $\quad \sim 2.5$T $\quad \sim 0.8 \quad \sim 0.8$

$\boxed{(dV/dz)\ \Delta t <\sim 1.6 \text{ V-s/m}}$

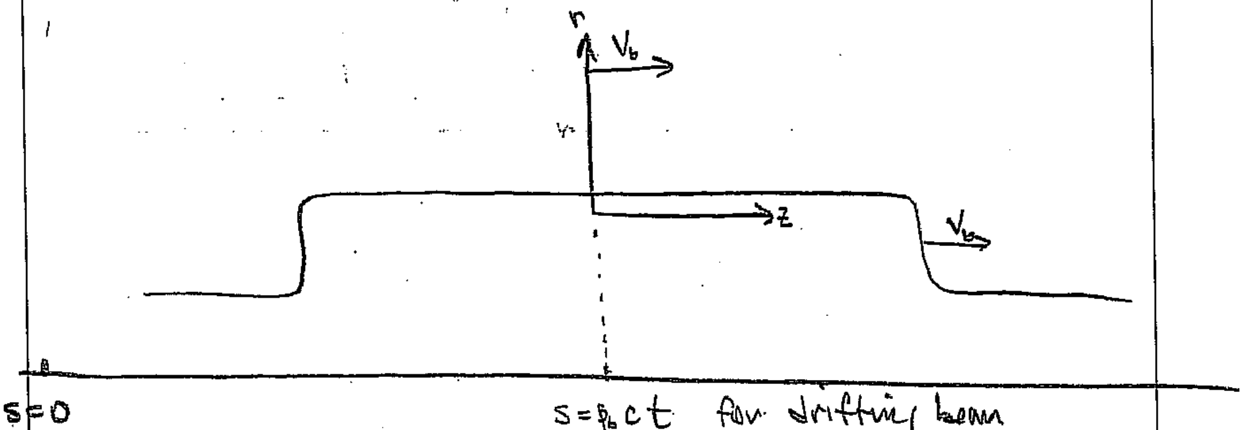
Several types of waveform are needed to accelerate, compress, and confine the beam



"Typical" numbers:

ΔV_{tilt} ~ 1 kV
ΔV_{ears} ~ 14 kV
ΔV_{acc} ~ 100 kV

COORDINATE SYSTEM



$s = \beta_b c t$ for drifting beam
= position of beam center in lab frame

$s \Longleftrightarrow t$ are related by $\beta_b c$ for drifting beam

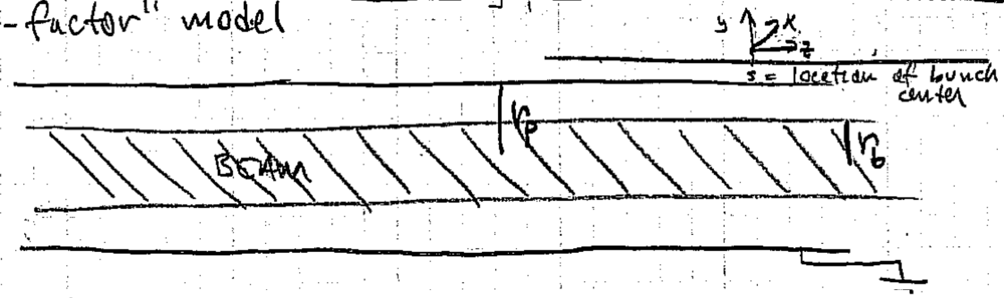
z = longitudinal coordinate in beam frame ($z=0$ = beam center)

r = radial coordinate in beam frame (or lab frame).

(This class will assume non-relativistic dynamics?
These are ions with $\beta < 0.2$).

LONGITUDINAL PHYSICS OF LONG PULSES (BUNCH LENGTH >> r_{pipe})

"g-factor" model



If $$\frac{\partial^2\phi}{\partial z^2} << \frac{1}{r}\left(\frac{\partial}{\partial r} r \frac{\partial\phi}{\partial r}\right) \quad \Rightarrow \quad \frac{\partial\phi}{\partial r} = -\frac{\lambda(r)}{2\pi\varepsilon_0 r}$$

Let $$\rho = \begin{cases} \rho_0 & 0 < r < r_b \\ 0 & r_b < r < r_p \end{cases} \quad \Rightarrow \quad \lambda = \lambda_0\left(\frac{r}{r_b}\right)^2$$

$$\phi = \int_r^{r_p} \frac{\partial\phi}{\partial r} dr = \begin{cases} \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{2}\left(1 - \frac{r^2}{r_b^2}\right) + \ln\frac{r_p}{r_b}\right] & 0 < r < r_b \\ \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_p}{r}\right) & r_b < r < r_p \end{cases}$$

$$\frac{\partial\phi}{\partial z} = \frac{1}{2\pi\varepsilon_0}\left[\frac{1}{2}\left(1-\frac{r^2}{r_b^2}\right) + \ln\frac{r_p}{r_b}\right]\frac{\partial\lambda}{\partial z} - \frac{1}{2\pi\varepsilon_0}\left[1 - \frac{r^2}{r_b^2}\right]\frac{\lambda}{r_b}\frac{\partial r_b}{\partial z}$$

__If__ $\rho = \text{const} \Rightarrow \frac{\lambda}{r_b^2} = \text{const} \qquad \frac{\partial\lambda}{\partial z} = \frac{-2\lambda}{r_b}\frac{\partial r_b}{\partial z}$

$$\Rightarrow \quad \frac{\partial\phi}{\partial z} = \frac{1}{2\pi\varepsilon_0}\ln\left(\frac{r_p}{r_b}\right)\frac{\partial\lambda}{\partial z}$$

$$E_z = \frac{-g}{4\pi\varepsilon_0}\frac{\partial\lambda}{\partial z}$$

where $g = 2\ln\left(\frac{r_p}{r_b}\right)$ [SPACE-CHARGE DOMINATED BEAM]

For space charge dom. beam:

$k_{\beta 0}^2\, r_b = Q/r_b$

$k_{\beta 0}^2 \sim \lambda/r_b^2 \sim \rho \sim \text{const}$

$$\frac{\partial \phi}{\partial z} = \frac{1}{2\pi\varepsilon_0}\left[\frac{1}{2}\left(1-\frac{r^2}{r_b^2}\right) + \ln\frac{r_p}{r_b}\right]\frac{\partial \lambda}{\partial z} - \frac{1}{2\pi\varepsilon_0}\cancel{\left[1-\frac{r^2}{r_b^2}\right]}\frac{\lambda}{r_b}\frac{\partial r_b}{\partial z} \qquad (1$$

FOR EMITTANCE DOMINATED BEAMS:

(with constant emittance and radius)

RADIUS NOT DETERMINED BY λ

SO $\frac{\partial r_b}{\partial z} \simeq 0$

$$\left\langle \frac{\partial \phi}{\partial z} \right\rangle = \frac{1}{2\pi\varepsilon_0}\left[\frac{1}{2}\left(1-\underbrace{\left\langle \frac{r^2}{r_b^2}\right\rangle}_{=1/2}\right) + \ln\frac{r_p}{r_b}\right]\frac{\partial \lambda}{\partial z}$$

$$\Rightarrow \quad g = 2\ln\left(\frac{r_p}{r_b}\right) + \frac{1}{2}$$ (EMITTANCE DOMINATED BEAMS)

(SEE REISER, SECTION 6.3 FOR DISCUSSION ON g-factor).

Vlasov-equation for a drifting beam:

$$\frac{\partial f}{\partial s} + x'\frac{\partial f}{\partial x} + x''\frac{\partial f}{\partial x'} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + z'\frac{\partial f}{\partial z} + z''\frac{\partial f}{\partial z'} = 0$$

Let $\tilde{f}(z,z',s) \equiv \iiiint f \, dx\, dx'\, dy\, dy'$

INTEGRATING VLASOV EQUATION: If $z'' \neq f(x,x',y,y')$:

$$\Rightarrow \frac{\partial \tilde{f}}{\partial s} + \iiint x' \underbrace{\int \frac{\partial f}{\partial x} dx}_{= f\big|_{-\infty}^{\infty} = 0} dx'\, dy\, dy' + \cdots \; z'\frac{\partial \tilde{f}}{\partial z} + z''\frac{\partial \tilde{f}}{\partial z'} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{f}}{\partial s} + z'\frac{\partial \tilde{f}}{\partial z} + z''\frac{\partial \tilde{f}}{\partial z'} = 0} \quad \text{1 D Vlasov}$$

Now let $\lambda \equiv q\int \tilde{f}\, dz'$; $\lambda \overline{z'} = \int \tilde{f} z'\, dz'$; $\lambda \overline{z'^2} \equiv \int \tilde{f} z'^2 dz'$

Also, let $\Delta z'^2 \equiv \overline{z'^2} - \left(\overline{z'}\right)^2$

FLUID EQUATIONS

INTEGRATING 1D VLASOV OVER z':

$$\boxed{\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \overline{z'}) = 0} \quad \text{(CONTINUITY EQUATION)}$$

MULTIPLYING BY z' & INTEGRATING VLASOV OR z':

$$\frac{\partial}{\partial s}\lambda \overline{z'} + \frac{\partial}{\partial z}\lambda \overline{z'^2} - \lambda \overline{z''} = 0$$

DIVIDING BY λ, USING CONTINUITY EQUATION & DEFINITION OF $\Delta z'^2$:

$$\boxed{\underbrace{\frac{\partial \overline{z'}}{\partial s} + \overline{z'}\frac{\partial}{\partial z}\overline{z'}}_{\text{INERTIAL}} + \underbrace{\frac{1}{\lambda}\frac{\partial}{\partial z}(\lambda \Delta z'^2)}_{\text{PRESSURE TERM}} = \underbrace{\overline{z''}}_{\text{FORCE}}} \quad \text{(MOMENTUM EQUATION)}$$

$$= \frac{1}{v_0^2}\frac{qE_z}{m}$$

Combining g-factor model with fluid equations

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \bar{z}') = 0$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \underbrace{\frac{1}{\lambda}\frac{\partial}{\partial z}(\lambda \Delta z'^2)}_{\text{PRESSURE TERM}} = \underbrace{\frac{-gq}{4\pi\epsilon_0 m V_0^2}\frac{\partial \lambda}{\partial z}}_{\text{SPACE CHARGE TERM}}$$

WHEN PRESSURE TERM << SPACE CHARGE TERM,

(& LET $C_s^2 \equiv \frac{qg\lambda_0}{4\pi\epsilon_0 m}$ = "(SPACE CHARGE WAVE SPEED)^2"

$$\Longrightarrow \boxed{\begin{aligned} &\frac{\partial \lambda}{\partial s} + \lambda \frac{\partial \bar{z}'}{\partial z} + \bar{z}' \frac{\partial \lambda}{\partial z} = 0 \\ \& \; &\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \frac{C_s^2}{\lambda_0 V_0^2}\frac{\partial \lambda}{\partial z} = 0 \end{aligned}} \qquad (g1)$$

LINEARIZING g1

LET $\lambda = \lambda_0 + \lambda_1 \qquad \bar{z}' = \bar{z}_0' + \bar{z}_1'$

EQUILIBRIUM λ_0 = CONSTANT

$\bar{z}_0' = 0$

LINEARIZING

$$\frac{\partial \lambda_1}{\partial s} + \lambda_0 \frac{\partial \bar{z}_1'}{\partial z} = 0 \qquad (g2a)$$

$$\frac{\partial \bar{z}_1'}{\partial s} + \frac{C_s^2}{\lambda_0 V_0^2}\frac{\partial \lambda_1}{\partial z} = 0 \qquad (g2b)$$

TAKING $\frac{\partial}{\partial s}$ of (g2a) & $\frac{\partial}{\partial z}$ of g2b and combining:

$$\Longrightarrow \boxed{\frac{\partial^2 \lambda_1}{\partial s^2} - \frac{C_s^2}{V_0^2}\frac{\partial^2 \lambda_1}{\partial z^2} = 0 \quad \Longrightarrow \text{WAVE EQUATION}}$$

SOLVING WAVE EQUATION

$$\frac{\partial^2 \lambda_1}{\partial s^2} - \frac{c_s^2}{V_0^2}\frac{\partial^2 \lambda_1}{\partial z^2} = 0$$

Let $\lambda_1 = \tilde{\lambda}_1 \exp\left[\frac{i\omega}{V_0}s \pm ikz\right]$

$$-\frac{\omega^2}{V_0^2} + \frac{k^2 c_s^2}{V_0^2} = 0 \quad \Rightarrow \quad \omega = c_s k$$

$\Rightarrow$ PHASE & GROUP VELOCITY OF WAVES $= C_s$
(in beam frame)

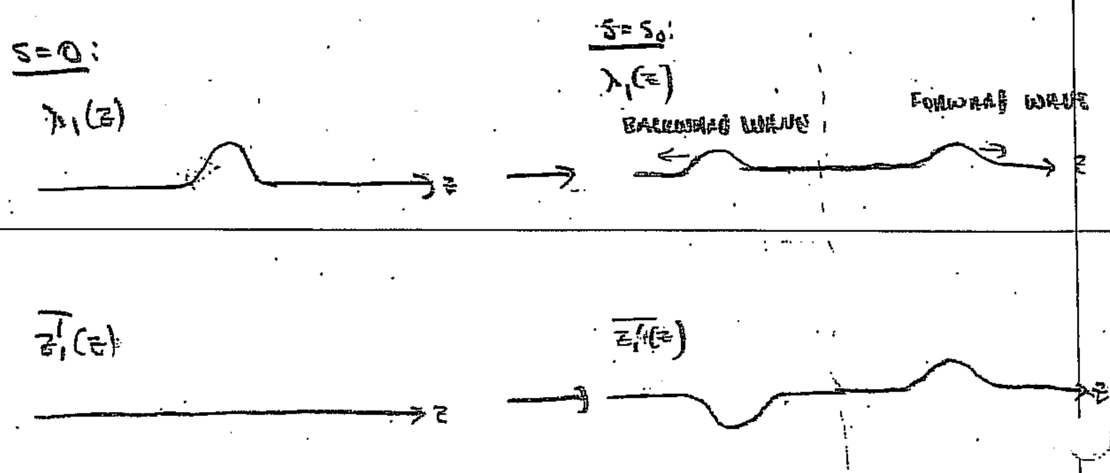
GENERAL SOLUTION

$$\lambda_1 = \lambda_0 f_+[u_+] + \lambda_0 f_-[u_-]$$

where $u_+ = z + \frac{c_s s}{V_0} + C_0$ & $u_- = z - \frac{c_s s}{V_0} + C_0$

& $f_+[u]$ & $f_-[u]$ are any functions of the argument.
& C_0 is an arbitrary constant

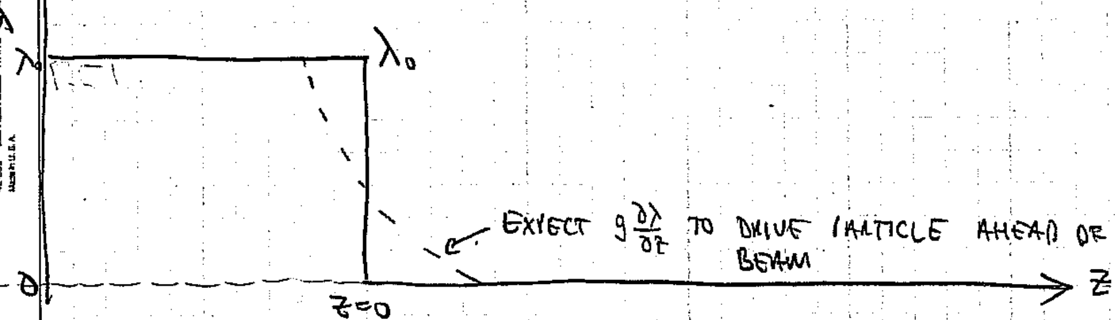
$$\overline{z_1'} = \frac{c_s}{V_0}\left[-f_+[u_+] + f_-[u_-]\right]$$



BEAM ENDS & RAREFACTION WAVES

(FALTENS & LEE, J. APP. PHYS. 61, 5219)
(ALSO LANDAU & LIFSHITZ, FLUID MECHANICS):

SUPPOSE YOU START WITH A PULSE THAT ENDS WITH A STEP FUNCTION IN λ. WHAT HAPPENS TO THE END?



TO ANALYZE: RETURN TO NON-LINEAR FLUID EQUATIONS (SINCE $\delta\lambda \sim \lambda_0$) (g1):

$$\frac{\partial\lambda}{\partial s} + \lambda\frac{\partial\overline{z'}}{\partial z} + \overline{z'}\frac{\partial\lambda}{\partial z} = 0 \qquad \text{(CONTINUITY)}$$

$$\frac{\partial\overline{z'}}{\partial s} + \overline{z'}\frac{\partial\overline{z'}}{\partial z} + \frac{C_s^2}{\lambda_0 v_0^2}\frac{\partial\lambda}{\partial z} = 0 \qquad \text{(momentum)}$$

1ST IT IS CONVENIENT TO DEFINE:

$$\Lambda \equiv \lambda/\lambda_0$$

$$V \equiv \frac{v_0}{C_s}\overline{z'}$$

$$\left(C_s^2 \equiv \frac{q}{m}\frac{g\lambda_0}{4\pi\epsilon_0}\right) \qquad \zeta \equiv \frac{v_0}{C_s} z$$

$$\Rightarrow \quad \frac{\partial\Lambda}{\partial s} + \Lambda\frac{\partial V}{\partial\zeta} + V\frac{\partial\Lambda}{\partial\zeta} = 0 \qquad \text{(CONTINUITY)}$$

$$\frac{\partial V}{\partial s} + V\frac{\partial V}{\partial\zeta} + \frac{\partial\Lambda}{\partial\zeta} = 0 \qquad \text{(momentum)} \qquad (r1)$$

TRY A SIMILARITY SOLUTION: $\Lambda = \Lambda(x)$ & $V = V(x)$

WHERE $x = \frac{z}{s} = \left(\frac{V_0 z}{C_s s}\right)$

$$\frac{\partial x}{\partial s} = -\frac{x}{s} \qquad \frac{\partial x}{\partial z} = \frac{x}{z}$$

$$\frac{\partial \Lambda}{\partial s} = -\frac{d\Lambda}{dx}\frac{x}{s} \qquad \frac{\partial \Lambda}{\partial z} = \frac{d\Lambda}{dx}\frac{x}{z} \qquad \frac{\partial V}{\partial s} = -\frac{dV}{dx}\frac{x}{s} \qquad \frac{\partial V}{\partial z} = \frac{dV}{dx}\frac{x}{z}$$

$$\left[-\frac{d\Lambda}{dx}\frac{x}{s} + \Lambda\frac{dV}{dx}\frac{x}{z} + V\frac{d\Lambda}{dx}\frac{x}{z}\right] = 0 \qquad \text{(CONTINUITY)}$$

$$\left[-\frac{dV}{dx}\frac{x}{s} + V\frac{dV}{dx}\frac{x}{z} + \frac{d\Lambda}{dx}\frac{x}{z}\right] = 0 \qquad \text{(momentum)}$$

MULTIPLY BY z/s & gather terms):

$$\Rightarrow \begin{bmatrix} V-x & \Lambda \\ 1 & V-x \end{bmatrix}\begin{bmatrix} d\Lambda/dx \\ dV/dx \end{bmatrix} = 0$$

FOR NON-TRIVIAL SOLUTION DETERMINANT MUST VANISH:

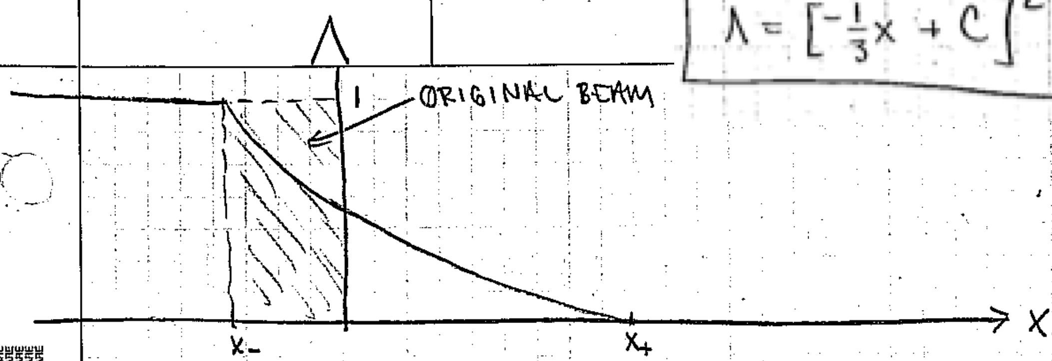
$$\boxed{\Lambda = [V-x]^2}$$

$$\Rightarrow \frac{d\Lambda}{dx} = 2[V-x]\left[\frac{dV}{dx} - 1\right] \quad \& \quad \frac{d\Lambda}{dx} = -[V-x]\frac{dV}{dx}$$

$$\Rightarrow -\frac{dV}{dx} = 2\frac{dV}{dx} - 2 \quad \Rightarrow \quad \frac{dV}{dx} = \frac{2}{3}$$

$$\Rightarrow \boxed{\begin{aligned} V &= \frac{2}{3}x + C \\ \Lambda &= \left[-\frac{1}{3}x + C\right]^2 \end{aligned}}$$

$$\boxed{\begin{aligned} V &= \tfrac{2}{3}x + C \\ \Lambda &= \left[-\tfrac{1}{3}x + C\right]^2 \end{aligned}}$$



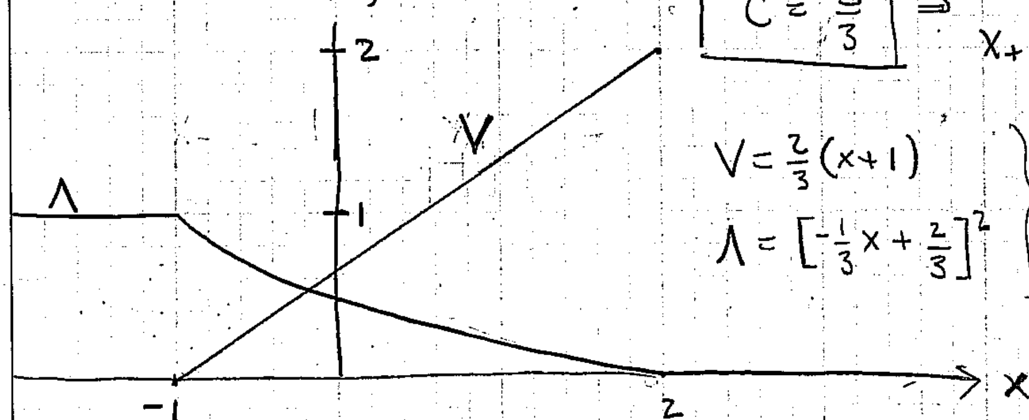
At x_+: $\Lambda = 0 \Rightarrow C = \frac{1}{3}x$ $\qquad x_+ = 3C$

At x_-: $\Lambda = 1 \Rightarrow C = \frac{1}{3}x + 1 \Rightarrow x_- = 3C - 3$

MASS CONSERVATION $\Rightarrow 1(-x_-) = -(3C-3) = \int_{3C-3}^{3C} \left[C - \frac{1}{3}x\right]^2 dx = -3\,\frac{\left[C - \frac{1}{3}x\right]^3}{3}\Bigg|_{3C-3}^{3C}$

$-(3C-3) = 1$

$\Rightarrow \boxed{C = \frac{2}{3}} \Rightarrow \begin{aligned} x_- &= -1 \\ x_+ &= 2 \end{aligned}$



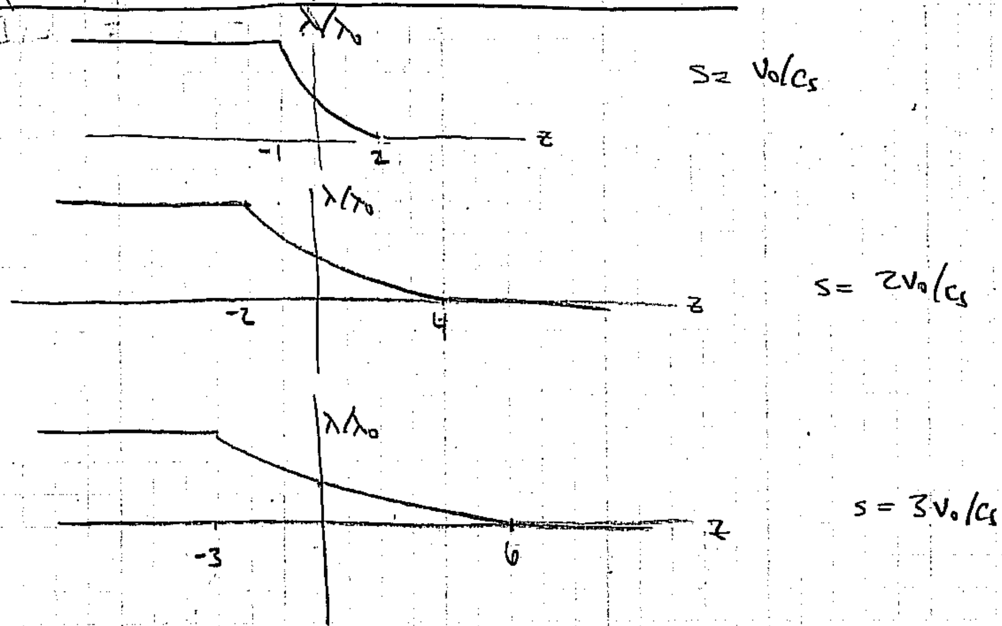
$$\left.\begin{aligned} V &= \tfrac{2}{3}(x+1) \\ \Lambda &= \left[-\tfrac{1}{3}x + \tfrac{2}{3}\right]^2 \end{aligned}\right\} \text{ for } -1 < x < 2$$

RECALL $x = \frac{V_0 z}{C_s s}$ $\qquad$ SO $x = 2 \Rightarrow z = 2C_s\left(\frac{s}{V_0}\right)$

$x = -1 \Rightarrow z = -C_s\left(\frac{s}{V_0}\right)$

SO BEAM END EXPANDS AT TWICE SPACE-CHARGE WAVE SPEED & RAREFACTION WAVE PROPAGATES INWARD AT THE SPACE-CHARGE WAVE SPEED.

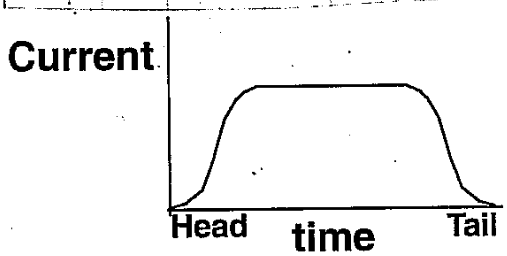
SNAPSHOTS OF λ/λ_0 VS z AT VARIOUS S

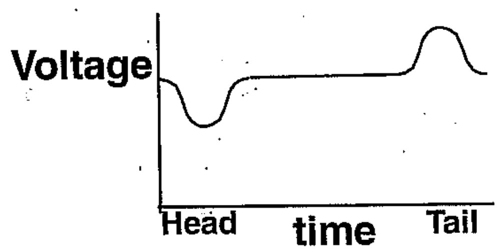


HOW DOES ONE PREVENT "END EROSION"?

APPLY EAR PULSES AT END OF BEAM:

$$V \sim E_z = \frac{-g}{4\pi\epsilon_0}\frac{\partial \lambda}{\partial z}$$





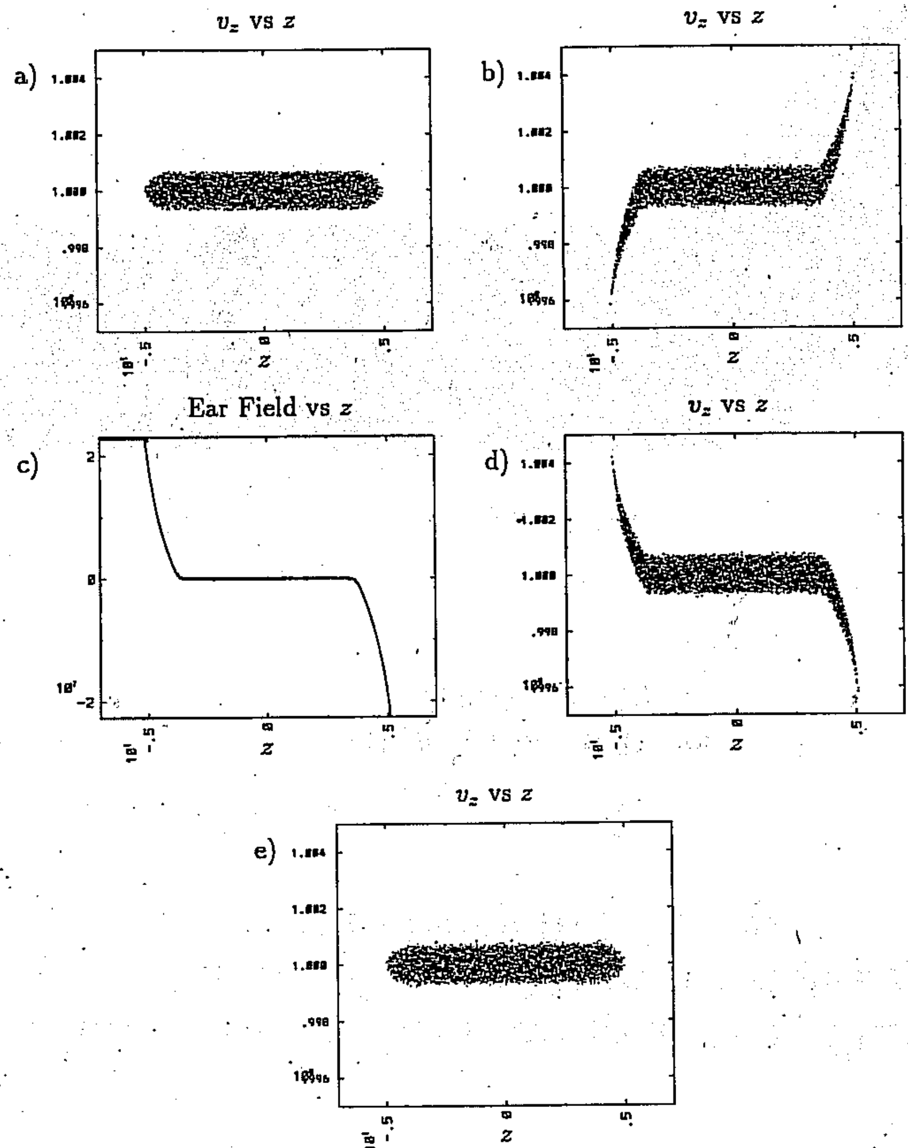


Figure 6.4: One cycle of intermittently-applied ears. (a) Initial phase space (b) Beam expands (c) Ear Field is applied (d) Beam is compressed (e) Beam expands back to its initial state

from D. Callahan Miller
PhD thesis, U.C. Davis, 1994.