Transverse Kinetic Stability*

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"Beam Physics with Intense Space-Charge"
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Transverse Kinetic Stability: Outline

Transverse Kinetic Stability: Detailed Outline

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9) Uniform Density Beams and Extreme Energy States
   - Variational Formulation
   - Self-Field Energy Minimization

10) Collective Relaxation of Space-Charge Nonuniformities and rms Emittance Growth
   - Conservation Constraints
   - Relaxation Processes
   - Emittance Growth Bounds from Space-Charge Nonuniformities

11) Emittance Growth from Envelope Mismatch Oscillations
   - To be added

12) Non-Tenuous Halo Induced Mechanism of Higher Order Instability in Quadrupole Focusing Channels
   - Halo Model for an Elliptical Beam
   - Pumping Mechanism
   - Stability Properties

Detailed Outline - 4

13) Non-Tenuous Halo Induced Instability in Solenoidal Focusing Systems
   - To be added

14) Phase Mixing and Landau Damping in Beams
   - (to be added, future editions)

Contact Information

References

Acknowledgments

S1: Overview: Machine Operating Points

Good transport of a single component beam with intense space-charge described by a Vlasov-Poisson type model requires:

1. Lowest Order:

   Stable single-particle centroid: $\sigma_0 < 180^\circ$ see: Transverse Particle Dynamics, Transverse Centroid and Env.

2. Next Order:

   Stable rms envelope: $\sigma_0,\ \sigma/\sigma_0$ both outside of envelope bands see: Transverse Centroid and Envelope Descriptions

3. Higher Order:

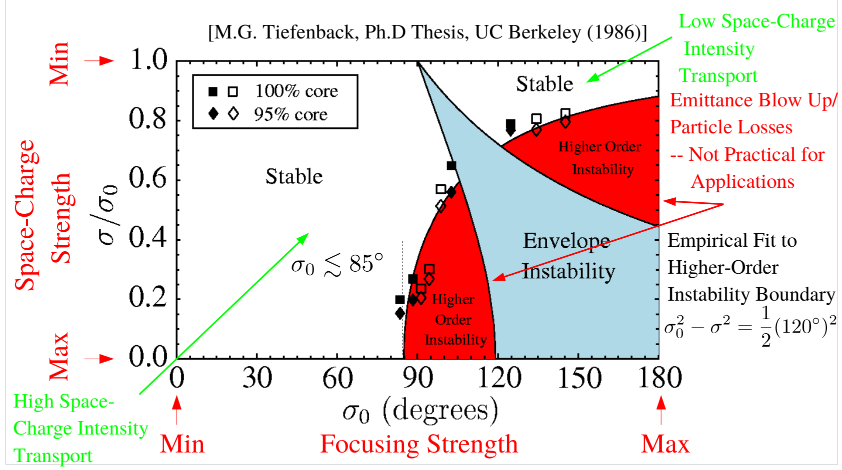
   "Stable" Vlasov description: To be covered these lectures

Transport of a relatively smooth initial beam distribution can fail or become "unstable" within the Vlasov model for several reasons:

- Collective modes internal to beam become unstable and grow
  - Large amplitudes can lead to statistical (rms) beam emittance growth
- Excessive halo generated
  - Increased statistical beam emittance and particle losses
- Combined processes above

Transport limits in periodic (FODO) quadrupole lattices that result from higher order processes have been measured in the SBTE experiment. These results had only limited theoretical understanding over 20+ years

Limits de ned with respect to reasonable (smooth) initial distributions



Comments:

In this schematic picture used only two parameters

$\sigma_0 \iff$ Provide a measure of focusing strength for xed form lattice functions $\kappa_x,\ \kappa_y$

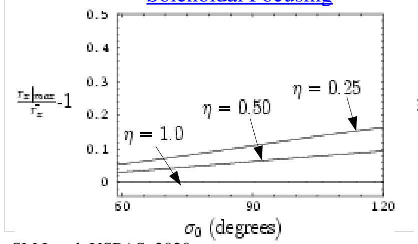
$\sigma/\sigma_0 \iff$ Normalized measure of space-charge intensity

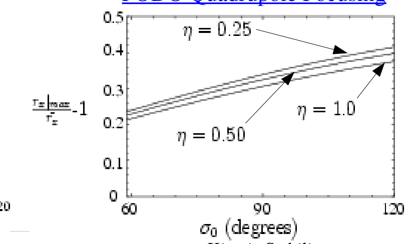
Depending on lattice and beam, these may not be the only relevant parameters

Example: Focusing strength measure from analysis of matched env equation

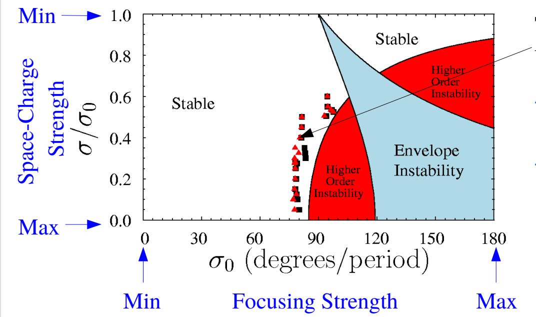
- Solenoid: Envelope utter relates to both σ_0 and occupancy η
- FODO Quadrupole: Envelope utter largely scales with σ_0, weak in η

$$\frac{r_x|_{\max}}{\bar{r_x}} - 1 \simeq \begin{cases} (1-\cos\sigma_0)\frac{(1-\eta)(1-\eta/2)}{6} & \text{Solenoidal Focusing} \\ (1-\cos\sigma_0)^{1/2}\frac{(1-\eta/2)}{2^{3/2}(1-2\eta/3)^{1/2}} & \text{Quadrupole Focusing} \end{cases}$$





Summary of beam stability with intense space-charge in a quadrupole transport lattice: centroid, envelope, and theory boundary based on higher order emittance growth/particle losses



Theory stability boundary points

- Lund and Chawla, NIMA **561** 203 (2006)
- Lund, Barnard, Bukh, Chawla, and Chilton, NIMA **277** 173 (2007)

Theory analyzes AG transport limits without equilibria

- Suggests near core, chaotic halo resonances driven by matched beam envelope utter can drive strong emittance growth and particle losses
- Results checked with fully self-consistent simulations

Analogous mechanisms (with much smaller region of parameters leading to "instability") exist for solenoidal transport

S2: Overview: Collective Modes and Transverse Kinetic Stability

In discussion of transverse beam physics we have covered to date:

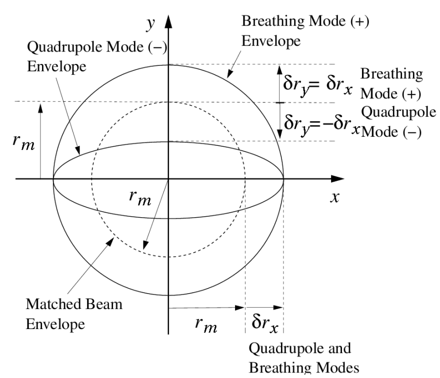
"Equilibrium" Matched Envelope

- Used to estimate balance of space-charge and focusing forces
  - KV model for periodic focusing
  - Continuous focusing equilibria for qualitative guide on space-charge e ects such as Debye screening and nonlinear equilibrium self- eld e ects

Centroid/Envelope Modes and Stability

- Lowest order collective oscillations of the beam
  - Analyzed assuming xed internal form of the distribution
- Model only exactly correct for KV equilibrium distribution
  - Should hold in a leading-order sense for a wide variety of real beams
- Predictions of instability regions are well veri ed by experiment
  - Signi cantly restricts allowed system parameters for periodic focusing lattices
- Envelope and Centroid instability can be avoided using focusing su ciently weak to avoid envelope instability by taking $\sigma_0 < 90^\circ$ for both solenoid and quadrupole focusing channels

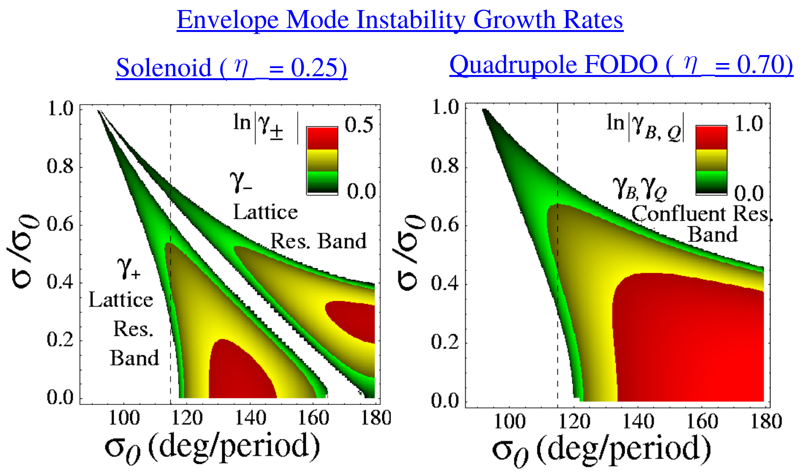
Example – Envelope Modes on a Round, Continuously Focused Beam



The rough analogs of these modes in a periodic focusing lattice can be destabilized

- Constrains system parameters to avoid band (parametric) regions of instability

Reminder (SM Lund lecture on Centroid and Envelope Descriptions of Beams): Instability bands of the KV envelope equation are well understood in periodic focusing channels



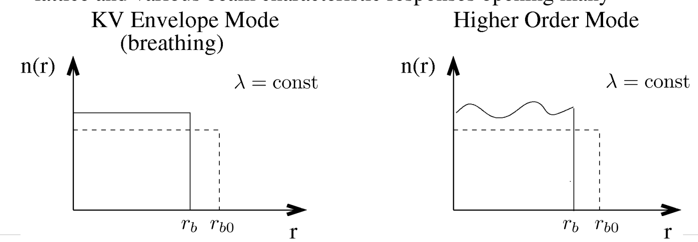
[S.M. Lund and B. Bukh, PRSTAB 024801 (2004)]

A lack of centroid and envelope instabilities is a *necessary* but *not su cient* condition for good transport.

Also need stability with respect to wave distortions in a more complete Vlasov model based kinetic theory including self-consistent space-charge

Higher-order Collective (internal) Mode Stability

- Perturbations will generally drive nonlinear space-charge forces
- Evolution of such perturbations can change the beam rms emittance
- Many possible internal modes of oscillation should be possible relative to moment (envelope) oscillations
  - Frequencies can di er signi cantly from envelope modes
  - Creates more possibilities for resonant exchanges with a periodic focusing lattice and various beam characteristic responses opening many



Plasma physics approach to beam physics:

Resolve:

$$f(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = f_\perp(\{C_i\}) + \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$$

equilibrium, perturbation, $f_\perp \gg |\delta f_\perp|$

and carry out equilibrium + stability analysis

Comments:

- Attraction is to parallel the impressive successes of plasma physics
  - Gain insight into preferred state of nature
- Beams are born o a source and may not be close to an equilibrium condition
  - Appropriate single particle constants of the motion unknown for periodic focusing lattices other than the KV distribution
  - Not clear if smooth equilibria exist for nite radius beams
- Intense beam self- elds and nite radial extent vastly complicate equilibrium description and analysis of perturbations relative to plasma physics
  - In uence of beam edge (nite plasma) and intense (generally nonlinear) self- elds complicate picture relative to neutral plasma physics which support (approximately) local force free thermal equilibrium.

Review: Transverse Vlasov-Poisson Model: for a coasting, single species beam with electrostatic self- elds propagating in a linear focusing lattice:

$\mathbf{x}_\perp, \mathbf{x}'_\perp$ transverse particle coordinate, angle

$q, \quad m$ charge, mass

$\gamma_b, \quad \beta_b$ axial relativistic factors

$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle distribution

$H_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle Hamiltonian

Vlasov Equation (see Barnard, Introductory Lectures; Lund, Transverse Eq. Dists.):

$$\frac{d}{ds} f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Particle Equations of Motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Hamiltonian (see: Lund, lectures on Transverse Equilibrium Distributions):

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}\kappa_x(s)x^2 + \frac{1}{2}\kappa_y(s)y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Poisson Equation:

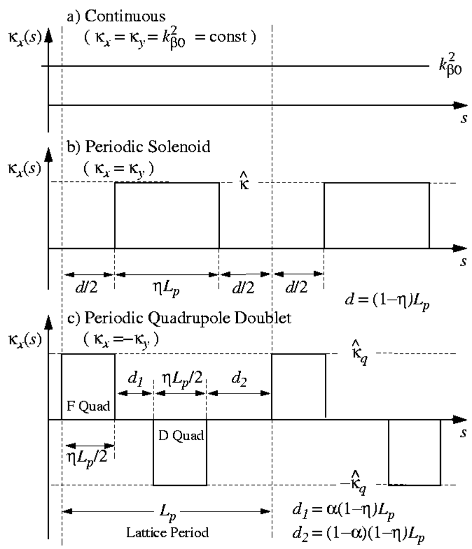
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int d^2\mathbf{x}'_\perp \; f_\perp$$

+ boundary conditions on ϕ

Beam charge density

$$\rho = qn = q\int d^2\mathbf{x}'_\perp \; f_\perp$$

Review: Focusing lattices, continuous and periodic (simple piecewise constant):



Lattice Period L_p

Occupancy η
$\eta \in [0, 1]$

Solenoid description carried out implicitly in Larmor frame
[see: S.M. Lund, lectures on Transverse Particle Dynamics]

Syncopation Factor α

$$\alpha \in [0, \frac{1}{2}]$$

$$\alpha = \frac{1}{2} \implies FODO$$

Continuous Focusing: $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Will be using primarily this form in these lectures

Solenoidal Focusing (in Larmor frame variables): $\kappa_x = \kappa_y = \kappa(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}\kappa\mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Quadrupole Focusing: $\kappa_x = -\kappa_y = \kappa_q(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}\kappa_q x^2 - \frac{1}{2}\kappa_q y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

We will concentrate (mostly) on the continuous focusing model in these lectures and will summarize some results on periodic focusing

- Kinetic theory is notoriously complicated even in this (simple) case
- By analogy with envelope mode results expect that kinetic theory of periodic focusing systems to have many more possible instabilities
- As in equilibrium analysis, the continuous model can give simpli ed insight on a range of relevant kinetic stability considerations

Expression of Vlasov Equation

See also lectures on Transverse Equilibrium Distributions

Hamiltonian expression of the Vlasov equation:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}_\perp'}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0$$

$$= \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0$$

Using the equations of motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'} = \mathbf{x}_\perp'$$

$$\frac{d}{ds}\mathbf{x}_\perp' = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} = -\left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right)$$

Gives the explicit form of the Vlasov equation:

- Use in these lectures with continuous focusing: $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$\frac{\partial f_\perp}{\partial s} + \mathbf{x}_\perp'\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \left(\hat{\mathbf{x}}\kappa_x x + \hat{\mathbf{y}}\kappa_y y + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right)\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0$$

Continuous Focusing $\implies$

$$\frac{\partial f_\perp}{\partial s} + \mathbf{x}_\perp'\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2\mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right)\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0$$

Comments on Vlasov-Poisson Model

- Collisionless Vlasov-Poisson model good for intense beams with many particles
  - Collisions negligible, see: J.J. Barnard, Introductory Lectures
- Vlasov-Poisson model is solved as an initial value problem
  1) $f_\perp(\mathbf{x}_\perp, \mathbf{x}_\perp', s = s_i) =$ Initial "condition" (function) specified
  2) Vlasov-Poisson model solved for subsequent evolution in s for $f_\perp(\mathbf{x}_\perp, \mathbf{x}_\perp', s)$ for $s \geq s_i$
- The Vlasov distribution function $f_\perp \geq 0$ can be thought of as a probability distribution evolving in $\mathbf{x}_\perp - \mathbf{x}_\perp'$ phase-space.
  - Particles/probability neither created nor destroyed
  - Flows along characteristic particle trajectories in $\mathbf{x}_\perp - \mathbf{x}_\perp'$ phase-space given by the particle equations of motion
  - Vlasov equation can be thought of as a higher-dimensional continuity equation describing incompressible ow in $\mathbf{x}_\perp - \mathbf{x}_\perp'$ phase-space
- Normalization of the 4D (transverse) distribution $f_\perp$ is chosen such that:
  - See also discussion in Transverse Equilibrium Distributions

$$\rho = q\int d^2x_\perp' \; f_\perp \qquad \lambda = q\int d^2x_\perp \int d^2x_\perp' \; f_\perp = \text{const}$$

- The coupling to the self- eld via the Poisson equation makes the Vlasov-Poisson model *highly* nonlinear

$$\rho = q \int d^2x'_\perp \; f_\perp \qquad\qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0}$$

+ aperture boundary condition on ϕ

- Vlasov-Poisson system is written without acceleration, but the transforms developed to identify the normalized emittance in the lectures on Transverse Particle Dynamics can be exploited to generalize all result presented to (weakly) accelerating beams (interpret in tilde variables)

- For solenoidal focusing the system can be interpreted in the rotating Larmor frame, see: lectures on Transverse Particle Dynamics

- System as expressed applies to 2D (unbunched) beam as expressed
  - Considerable di culty in analysis for 3D version for transverse/longitudinal physics

S3: Linearized Vlasov Equation

Because of the complexity of kinetic theory, we will limit discussion to a simple continuous focusing model Vlasov-Poisson system for a coasting beam within a round pipe $\qquad \kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$\frac{df_\perp}{ds} = \left\{\frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right)\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\right\} f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = 0$$

$$\nabla_\perp^2 \phi(\mathbf{x}_\perp, s) = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$$

$$\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const}$$

Then expand the distribution and eld as:

$$\begin{aligned} f_\perp &= f_0(H_0) + \delta f_\perp \\ \phi &= \phi_0 + \delta\phi \end{aligned}$$

equilibrium perturbation

Comment:
The Poisson equation connects $f_\perp$ and ϕ so, $\delta f_\perp$ and $\delta\phi$ *cannot be independently speci ed.* We quantify the connection shortly.

At present, there is *no assumption* that the perturbations are small

- Use subscript zeros to distinguish equilibrium quantities in the absence of perturbations to set up perturbation analysis

The equilibrium satis es:
(see: S.M. Lund, lectures on Transverse Equilibrium Distributions)

$$H_0 = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi_0$$

$f_0(H_0) \geq 0$ (any non-negative function)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_0}{\partial r}\right) = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; f_0(H_0)$$

The unperturbed distribution must then satisfy the equilibrium Vlasov equation:

$$\left\{\frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp}\right)\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\right\} f_0(H_0) = 0$$

(with $\frac{\partial}{\partial s} \to 0$)

$$\left\{\mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp}\right)\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\right\} f_0(H_0) = 0$$

Because the Poisson equation is linear, and ϕ_0 satis es the equilibrium Poisson equation, the Perturbed Poisson Equation for $\delta\phi$ is:

$$\nabla_\perp^2 \delta\phi(\mathbf{x}_\perp, s) = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$$

$$\delta\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const}$$

Insert the perturbations in Vlasov's equation and expand terms:

$$\left\{\frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp}\right)\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\right\} f_0(H_0)$$

equilibrium term (= 0)

$$+\left\{\frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp}\right)\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\right\} \delta f_\perp$$

equilibrium characteristics of perturbed distribution

$$= \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \delta\phi}{\partial \mathbf{x}_\perp}\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}f_0(H_0) + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \delta\phi}{\partial \mathbf{x}_\perp}\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\delta f_\perp$$

perturbed eld linear correction term; nonlinear term $\sim \delta^2$

Take the perturbations to be small-amplitude:

$$f_0(H_0) \gg |\delta f_\perp| \qquad \phi_0 \gg \delta\phi$$

<--- follows automatically from distribution/Poisson Eqn

and neglect the nonlinear terms to obtain the linearized Vlasov-Poisson system:

$$\left\{\frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp}\right)\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\right\} \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \delta\phi(\mathbf{x}_\perp, s)}{\partial \mathbf{x}_\perp}\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}f_0(H_0)$$

$$\nabla_\perp^2 \delta\phi(\mathbf{x}_\perp, s) = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) \qquad \delta\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const}$$

Solution of the Linearized Vlasov Equation: Use the method of characteristics to recast in a more manageable form for beam applications

The linearized Vlasov equation is an integral-partial differential equation system

- Highly nontrivial to solve!
- The structure of the equations suggests that the Method of Characteristics can be employed to simplify analysis

Note that the equilibrium Vlasov equation is:

$$\left\{\frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp}\right)\cdot \frac{\partial}{\partial \mathbf{x}'_\perp}\right\} f_0 = 0$$

$$\left.\frac{d}{ds}\right|_{\text{eq. orbit}} f_0 = 0$$

Interpret:

$$\left\{\frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp}\right)\cdot \frac{\partial}{\partial \mathbf{x}'_\perp}\right\} = \left.\frac{d}{ds}\right|_{\text{eq. orbit}}$$

as a total derivative evaluated along an equilibrium particle orbit in the continuum approximation beam equilibrium. This suggests employing the *method of characteristics.*

Method of Characteristics:

Orbit equations of motion of a "characteristic particle" in equilibrium:

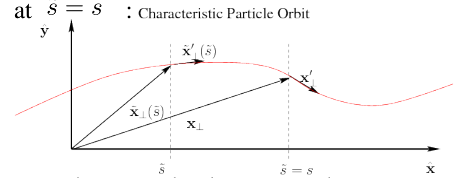
$$\frac{d}{d\tilde{s}}\tilde{\mathbf{x}}_\perp(\tilde{s}) = \tilde{\mathbf{x}}'_\perp(\tilde{s})$$

$$\frac{d}{d\tilde{s}}\tilde{\mathbf{x}}'_\perp(\tilde{s}) = -k_{\beta 0}^2\tilde{\mathbf{x}}_\perp(\tilde{s}) - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi_0(\tilde{\mathbf{x}}_\perp(\tilde{s}))}{\partial\tilde{\mathbf{x}}_\perp(\tilde{s})}$$

"Initial" conditions of characteristic orbit chosen such that particle passes through phase-space coordinates $\mathbf{x}_\perp, \ \mathbf{x}'_\perp$ at $\tilde{s} = s$: Characteristic Particle Orbit

$$\tilde{\mathbf{x}}_\perp(\tilde{s} = s) = \mathbf{x}_\perp$$

$$\tilde{\mathbf{x}}'_\perp(\tilde{s} = s) = \mathbf{x}'_\perp$$



Then the linearized Vlasov equation can be equivalently expressed as:

$$\frac{d}{d\tilde{s}}\delta f_\perp(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s}), \tilde{s}) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\delta\phi(\tilde{\mathbf{x}}_\perp(\tilde{s}))}{\partial\tilde{\mathbf{x}}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\tilde{\mathbf{x}}'_\perp}f_0(H_0(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s})))$$

Integrate:

$$\int_{-\infty}^{s} d\tilde{s}\,\frac{d}{d\tilde{s}}\delta f_\perp(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s}), \tilde{s}) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\int_{-\infty}^{s} d\tilde{s}\,\frac{\partial\delta\phi(\tilde{\mathbf{x}}_\perp(\tilde{s}))}{\partial\tilde{\mathbf{x}}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\tilde{\mathbf{x}}'_\perp}f_0(H_0(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s})))$$

Neglect initial conditions at $\tilde{s} \to -\infty$ to analyze perturbations that grow in s:

$$\int_{-\infty}^{s} d\tilde{s}\,\frac{d}{d\tilde{s}}\delta f_\perp(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s}), \tilde{s}) = \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) - \lim_{\tilde{s}\to-\infty}\delta f_\perp(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s}), \tilde{s})$$

$$\simeq \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$$

Giving:

$$\delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\int_{-\infty}^{s} d\tilde{s}\,\frac{\partial\delta\phi(\tilde{\mathbf{x}}_\perp(\tilde{s}))}{\partial\tilde{\mathbf{x}}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\tilde{\mathbf{x}}'_\perp}f_0(H_0(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s})))$$

Insert this expression in the perturbed Poisson equation:

$$\nabla_\perp^2\delta\phi(\mathbf{x}_\perp, s) = -\frac{q}{\epsilon_0}\int d^2x'_\perp\ \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$$

$$\delta\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const}$$

To obtain the characteristic form of the perturbed Vlasov equation:

$$\nabla_\perp^2\delta\phi(\mathbf{x}_\perp, s) = \frac{-q^2}{m\epsilon_0\gamma_b^3\beta_b^2c^2}\int d^2x'_\perp\int_{-\infty}^{s} d\tilde{s}\,\frac{\partial\delta\phi(\tilde{\mathbf{x}}_\perp(\tilde{s}))}{\partial\tilde{\mathbf{x}}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\tilde{\mathbf{x}}'_\perp}f_0(H_0(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s})))$$

$$\delta\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const}$$

Summary:

Linearized Vlasov-Poisson system expressed in the method of characteristics

$$\nabla_\perp^2\delta\phi(\mathbf{x}_\perp, s) = \frac{-q^2}{m\epsilon_0\gamma_b^3\beta_b^2c^2}\int d^2x'_\perp\int_{-\infty}^{s} d\tilde{s}\,\frac{\partial\delta\phi(\tilde{\mathbf{x}}_\perp(\tilde{s}))}{\partial\tilde{\mathbf{x}}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\tilde{\mathbf{x}}'_\perp}f_0(H_0(\tilde{\mathbf{x}}_\perp(\tilde{s}), \tilde{\mathbf{x}}'_\perp(\tilde{s})))$$

$$\delta\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const}$$

With characteristic orbits in the equilibrium beam satisfying:

Eqns of Motion:

$$\frac{d}{d\tilde{s}}\tilde{\mathbf{x}}_\perp(\tilde{s}) = \tilde{\mathbf{x}}'_\perp(\tilde{s})$$

$$\frac{d}{d\tilde{s}}\tilde{\mathbf{x}}'_\perp(\tilde{s}) = -k_{\beta 0}^2\tilde{\mathbf{x}}_\perp(\tilde{s}) - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi_0(\tilde{\mathbf{x}}_\perp(\tilde{s}))}{\partial\tilde{\mathbf{x}}_\perp(\tilde{s})}$$

Initial Conditions:

$$\tilde{\mathbf{x}}_\perp(\tilde{s} = s) = \mathbf{x}_\perp$$

$$\tilde{\mathbf{x}}'_\perp(\tilde{s} = s) = \mathbf{x}'_\perp$$

Gives the self-consistent evolution of the perturbations

- Similar statement for nonlinear perturbations (Homework problem)

Effectively restates the Poisson equation as a differential-integral equation that is solved to understand the evolution of perturbations

- Simpler to work with .... but still *very* complicated to solve in general cases due to nonlinear equilibrium characteristics which, other than special (KV) cases, are difficult to solve analytically

Formulation can be applied with no modification to any equilibrium distribution

- Need not be continuous focusing
- Method is used with a periodic focused KV equilibrium distribution to analyze the stability of normal mode perturbations about a KV equilibrium
  - Equilibrium function of linear field Courant-Snyder invariants
  - Formulation very difficult to solve

To apply method of characteristics to construct linear normal mode perturbations:

1) Take harmonic variation with s dependence

$$\delta\phi(\mathbf{x}_\perp, s) = \delta\phi_n(\mathbf{x}_\perp)e^{-iks} \qquad i = \sqrt{-1}$$

$$\delta\phi_n(\mathbf{x}_\perp) = \text{eigenfunction}$$

2) Find (via expansion) form of $\delta\phi_n(\mathbf{x}_\perp)$ that satisfies the integral-differential equation and boundary conditions

- Expect solutions to exist only for certain values of k (dispersion relation) linked to specific symmetry eigenfunctions $\delta\phi_n(\mathbf{x}_\perp)$
- Corresponding solutions will be "normal modes" that describe the transverse collective oscillations of the beam

S4: Collective Modes on a KV Equilibrium Beam

Unfortunately, calculation of normal modes is complicated even in continuous focusing. Nevertheless, the normal modes of the KV distribution can be analytically calculated and give insight on the expected collective response of a beam with intense space-charge.

Review: Continuous Focusing KV Equilibrium

- see: SM Lund, lectures on Transverse Equilibrium Distributions

$$f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta\left(H_\perp - \frac{\varepsilon^2}{2r_b^2}\right)$$

$k_{\beta 0} =$ Undepressed betatron wavenumber
$r_b =$ Beam edge radius
$\hat{n} =$ Beam number density
$Q =$ Dimensionless perveance
$\varepsilon =$ rms edge emittance

Express equilibrium parameters in normalized forms as before to provide a "guide" to other systems:

Applied Focusing:

$$k_{\beta 0}^2 = \left(\frac{\sigma_0}{L_p}\right)^2 = \text{const}$$

Matched Envelope:

$$r_b = \left(\frac{Q + \sqrt{4k_{\beta 0}^2\varepsilon^2 + Q^2}}{2k_{\beta 0}^2}\right)^{1/2} = \text{const}$$

$$\Longrightarrow \quad \sigma = \sqrt{\sigma_0^2 - \frac{Q}{(r_b/L_p)^2}} = \frac{\varepsilon L_p}{r_b^2}$$

$$\frac{k_{\beta 0}^2\varepsilon^2}{Q^2} = \frac{\sigma_0^2\varepsilon^2}{Q^2L_p^2} = \frac{(\sigma/\sigma_0)^2}{[1-(\sigma/\sigma_0)^2]^2}$$

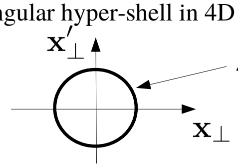
Further comments on the KV equilibrium: Distribution Structure

Equilibrium distribution for non-continuous focusing channels:

$$\text{f}_\perp \sim \delta[\text{Courant-Snyder invariants}]$$

Forms a highly singular hyper-shell in 4D phase-space

Schematic:



4D singular hyper-shell surface

- Singular distribution has large "Free-Energy" to drive many instabilities
  - Low order envelope modes are physical and highly important (see: S.M. Lund, lectures on Centroid and Envelope Descriptions of Beams)
- Perturbative analysis shows strong collective instabilities
  - Hofmann, Laslett, Smith, and Haber, Part. Accel. **13**, 145 (1983)
  - Higher order instabilities (collective modes) have unphysical aspects due to (delta-function) structure of distribution and must be applied with care (see following lecture material)
  - Instabilities can cause problems if the KV distribution is employed as an initial beam state in self-consistent simulations

A full kinetic stability analysis of the elliptical beam KV equilibrium distribution is complicated and uncovers many strong instabilities

[I. Hofmann, J.L. Laslett, L. Smith, and I. Haber, Particle Accel. 13, 145 (1983); R. Gluckstern, Proc. 1970 Proton Linac Conf., Batavia 811 (1971)]

Expand Vlasov's equation to linear order with:

$$f_\perp \rightarrow f_\perp(\text{ C.S. Invariant}) \;+\; \delta f_\perp$$

$f_\perp(\text{C.S. Invariant})$ = equilibrium
$\delta f_\perp$ = perturbation

Solve the Poisson equation:

$$\nabla_\perp^2\delta\phi = -\frac{q}{\epsilon_0}\int d^2x'\;\delta f_\perp$$

using truncated polynomials for $\delta\phi$ internal to the beam to represent a "normal mode" with pure harmonic variation, i.e., $\delta\phi \sim \text{func}(x,y)e^{-iks}$

$$\delta\phi = e^{-iks}\left\{\sum_{m=0}^{n}A_m^{(0)}(s)x^{n-m}y^m + \sum_{m=0}^{n-2}A_m^{(1)}(s)x^{n-m-2}y^m + \cdots\right\}$$

$k = \text{const} = \text{Mode Wavenumber}$ $\quad n = 2, 3, 4, \cdots$ "order" of mode
$i = \sqrt{-1}$ $\quad m$ can be restricted to even or odd terms

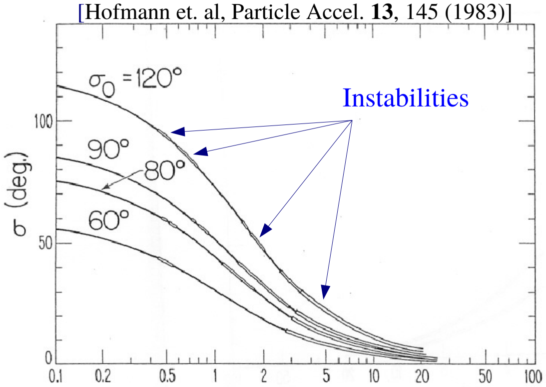
- Truncated polynomials can meet all boundary conditions (Glukstern, Hofmann)
- Eigenvalues of a Floquet form transfer matrix analyzed for stability properties
  - Lowest order results reproduce KV envelope instabilities
  - Higher order results manifest many strong instabilities

Higher order kinetic instabilities of the KV equilibrium are strong and cover a wide parameter range for periodic focusing lattices

Example: FODO Quadrupole Stability

4^{th} order ($n = 4$) even mode

[Hofmann et. al, Particle Accel. **13**, 145 (1983)]



$\sigma/\sigma_0 \to 1$ Q' $\sigma/\sigma_0 \to 0$

(space-charge parameter)

(undepressed) --> increasing space-charge --> (fully depressed)

Comment:

Hofmann et al notation on space-charge parameter:

$$Q' \neq \frac{dQ}{ds}$$

$$\neq \text{ Our } Q$$

Q' scale not defined in paper

The continuous focusing limit can be analyzed to better understand properties of internal modes on a KV beam

[S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998): see Appendix B, C]

Continuous focusing, KV equilibrium beam: $\varepsilon_x = \varepsilon_y \equiv \varepsilon$

$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$ $r_x = r_y \equiv r_b$

Search for *axisymmetric* $(\partial/\partial\theta = 0)$ normal mode solutions with $\sim e^{-iks}$ variations with:

$k = \text{ const } = \text{ Mode Wavenumber}$ (generally complex)

$\delta\phi = \delta\phi_n(r)e^{-iks}$ $\delta\phi_n(r) = \text{Truncated Polynomial in } r$

Find after some analysis:

- See Appendix A, derived using method of characteristics and solving a radial eigenvalue equation

Mode Eigenfunction (2n "order" in the sense of Hoffman et. al.):

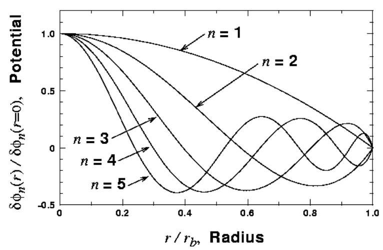
$$\delta\phi_n = \begin{cases} \frac{A_n}{2}\left[P_{n-1}\left(1-2\frac{r^2}{r_b^2}\right) + P_n\left(1-2\frac{r^2}{r_b^2}\right)\right], & 0 \le r \le r_b \\ 0, & r_b < r \end{cases}$$

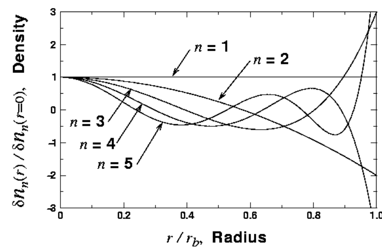
$n = 1,\ 2,\ 3,\ \cdots$ $A_n = \text{const}$

$P_n(x) =$ n^{th} order Legendre polynomial

Plots of radial eigenfunction help illustrate normal mode structure:

Potential Density ($\delta n_n = -\epsilon_0 \nabla_\perp^2 \delta\phi_n/q$)





- Polynomial eigenfunction has n-1 density profile "wiggles" and tends to vary more rapidly near beam edge for higher n values
- Eigenfunction structure suggestive of wave perturbations often observed internal to the beam in simulations for a variety of beam distributions

Corresponding dispersion relation has degenerate branches for each eigenfunction some of which go strongly unstable for $n \geq 2$

Dispersion Relation

$$2n + \frac{1-(\sigma/\sigma_0)^2}{(\sigma/\sigma_0)^2}\left[B_{n-1}\left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0}\right) - B_n\left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0}\right)\right] = 0$$

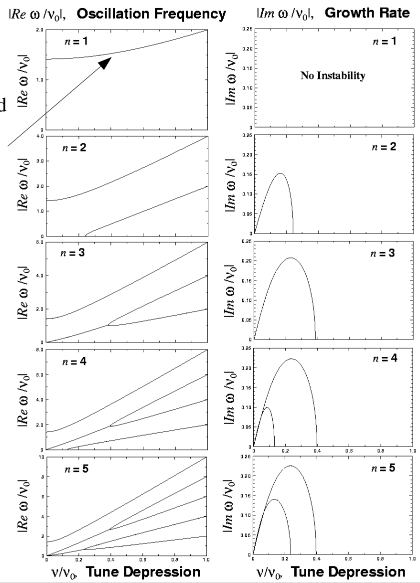
where:

$$B_j(\alpha) \equiv \begin{cases} 1, & j = 0 \\ \frac{(\alpha/2)^2-0^2}{(\alpha/2)^2-1^2}\frac{(\alpha/2)^2-2^2}{(\alpha/2)^2-3^2}\cdots\frac{(\alpha/2)^2-(j-1)^2}{(\alpha/2)^2-j^2} & j = 1,3,5,\cdots \\ \frac{(\alpha/2)^2-1^2}{(\alpha/2)^2-2^2}\frac{(\alpha/2)^2-3^3}{(\alpha/2)^2-4^2}\cdots\frac{(\alpha/2)^2-(j-1)^2}{(\alpha/2)^2-j^2} & j = 2,4,6,\cdots \end{cases}$$

- n distinct branches for nth order (real coefficient) polynomial dispersion relation in $(k/k_{\beta 0})^2$
- Some range of σ/σ_0 unstable for all $n > 1$
  - Instability exists for some n for $\sigma/\sigma_0 < 0.3985$
  - Growth rates are strong

Plot dispersion relation roots in real and imaginary parts to analyze stability properties of each eigenmode

Continuous focusing limit dispersion relation results for KV beam stability



n = 1, and envelope mode curves overlap

Notation Change:

$$k/k_{\beta 0} \equiv \omega/\nu_0$$

$$\sigma/\sigma_0 \equiv \nu/\nu_0$$

Envelope Mode:
(breathing mode)

- See lectures on: Transverse Centroid and Envelope Models

$$\sigma_+ = \sqrt{2\sigma_0^2 + 2\sigma^2}$$

[S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998): see Appendix B, C]

Summary stability results for a continuously focused KV beam with axisymmetric perturbations

Stability results are highly pessimistic and inconsistent with simulation and experiment which show:

- Internal collective waves with at times strong similarity to stable branches of the KV distribution but without the strong instabilities predicted
- Smooth initial distributions likely to be present in the lab transport well with no instability or pronounced growth of phase-space area
  - Particularly true in ideal continuous focusing systems
  - Lesser degree of stability found for periodic focusing systems (see S12).

If we take the KV results literally transport would be precluded by one or more collective mode being unstable when $\sigma/\sigma_0 < 0.3985$

For continuous focusing,   uid theory shows that some branches and features of the KV kinetic dispersion relation *are* physical

[S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998)]

KV model kinetic instabilities are a paradox:

Low-order features physical:

- Envelope equations well veri ed and assoc instabilities must be avoided in design

Higher-order collective modes:

- Perturbations seen in simulations/lab similar in form to the normal mode radial eigenfunctions
- BUT perturbations on real, smooth beam core not typically unstable where the KV model predicts strong bands of parametric instability

How is this situation resolved? Partial answer suggested by a   uid theory model of the KV equilibrium that eliminates unphysical aspects of the singular KV equilibrium core

Fluid theory:

- KV equilibrium distribution is reasonable in   uid theory
  - No singularities
  - Flat density and parabolic radial temperature pro les
- Theory truncated by assuming zero heat   ow

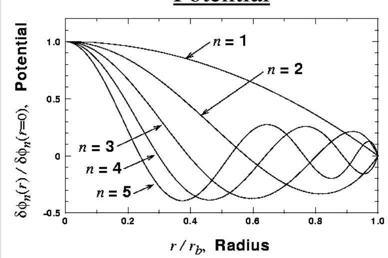
Results of normal mode analysis based on a   uid theory:

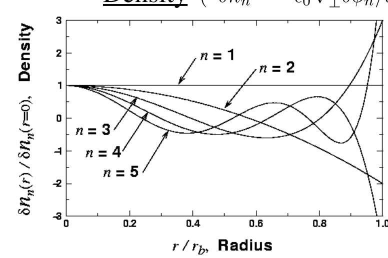
Mode eigenfunctions:

Exactly the same as derived under kinetic theory!

Potential

Density ($\delta n_n = -\epsilon_0 \nabla_\perp^2 \delta\phi_n / q$)





Mode dispersion relation:

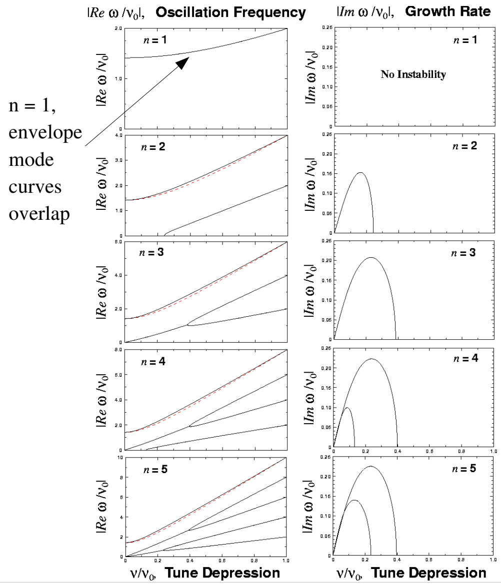
$$\frac{k}{k_{\beta 0}} = \sqrt{2 + 2\left(\frac{\sigma}{\sigma_0}\right)^2 (2n^2 - 1)}$$

$$n = 1,\ 2,\ 3,\ \cdots$$

- Agrees well with the stable high frequency branch in kinetic theory

Results show that aspects of higher-order KV internal modes are physical!

Continuous focusing limit dispersion relation results for KV beam stability



Notation Change:

$k/k_{\beta 0} \equiv \omega/\nu_0$

$\sigma/\sigma_0 \equiv \nu/\nu_0$

Red: Fluid Theory

(no instability)

Black: Kinetic Theory

(unstable branches)

[S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998)]

Appendix A: Solution of the Small Amplitude Perturbed Vlasov Equation for a Continuously Focused KV Beam

Not yet typeset. See handwritten note supplement:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2020/lec_set_08/tks_sup.pdf

S5: Global Conservation Constraints

Apply for any initial distribution, equilibrium or not.

- Strongly constrain nonlinear evolution of the system.
- Valid even with a beam pipe provided that particles are not lost from the system and that symmetries are respected.
- Useful to bound perturbations, but yields no information on evolution timescales.

1) Generalized Entropy

$$U_G = \int d^2x_\perp \int d^2x'_\perp \; G(f_\perp) \;= \text{const}$$

$G(f_\perp)$ = Any diffrentiable functions satisfying $G(f_\perp \to 0) = 0$

- Applies to *all* Vlasov evolutions
  - Need not be continuous focusing here!

// Examples

Line-charge: $G(f_\perp) = q f_\perp$ → $\lambda = q \int d^2x_\perp \int d^2x'_\perp \; f_\perp \;= \text{const}$

Entropy: $G(f_\perp) = -\frac{f_\perp}{f_0}\ln\left(\frac{f_\perp}{f_0}\right)$ f_0 positive constant

→ $\mathcal{S} = -\int d^2x_\perp \int d^2x'_\perp \; \frac{f_\perp}{f_0}\ln\left(\frac{f_\perp}{f_0}\right) \;= \text{const}$ //

2) Transverse Energy in continuous focusing systems

$$U_{\mathcal{E}} = \int d^2x_\perp \int d^2x'_\perp \left\{\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp\right\} f_\perp \; + \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2} \;= \text{const}$$

Here,

$\int d^2x'_\perp \int d^2x_\perp \; \frac{1}{2}\mathbf{x}'^2_\perp f_\perp$ ~ Kinetic Energy

$\int d^2x'_\perp \int d^2x_\perp \; \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp f_\perp$ ~ Potential Energy of applied focusing forces

$\int d^2x_\perp \; \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2}$ ~ Self-Field Energy (Electrostatic)

- Does not hold when focusing forces vary in *s*
  - Can still be *approximately* valid for rms matched beams where energy will regularly pump into and out of the beam
- Self eld energy term diverges in radially unbounded 2D systems (no aperture)
  - Still useful if an appropriate in nite constant is subtracted (to regularize)
  - Expression adequate as expressed for system with a round conducting, perfectly conducting aperture

Comments on system energy form:

$$U_{\mathcal{E}} = \int d^2x'_\perp \int d^2x_\perp \left\{ \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp \right\} f_\perp + \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp \phi|^2}{2m\gamma_b^3\beta_b^2c^2} = \text{const}$$

Analyze the energy term:

zero for grounded aperture
in nite system

$$\int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int d^2x_\perp \frac{1}{2}\nabla_\perp \cdot (\epsilon_0\phi\nabla_\perp\phi) - \int d^2x_\perp \frac{1}{2}\phi\epsilon_0\nabla^2_\perp\phi$$

or in nite constant
in free space

Employ the Poisson equation:

$$\nabla^2_\perp \phi = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; f_\perp$$

Giving: $\Longrightarrow \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int d^2x_\perp \int d^2x'_\perp \; \frac{1}{2}q\phi f_\perp$

$$U_{\mathcal{E}} = \int d^2x'_\perp \int d^2x_\perp \left\{ \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{1}{2}\frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \right\} f_\perp = \text{const}$$

symmetry factor

- Note the relation to the system Hamiltonian with a symmetry factor to not double count particle contributions $H_\perp = \frac{1}{2}\mathbf{x}'^{\,2}_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$

Comments on self- eld energy divergences:

In unbounded (free space) systems, far from the beam the eld must look like a line charge:

$$-\frac{\partial\phi}{\partial r} \sim \frac{\lambda}{2\pi\epsilon_0 r} \qquad r > r_{\text{large}}$$

Resolve the total eld energy into a nite (near) term and a divergent term:

$$\int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int_{r\le r_{\text{large}}} d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} + \frac{\lambda^2}{4\pi\epsilon_0}\int_{r_{\text{large}}}^{\infty} dr\frac{1}{r}$$

total — nite term — logarithmically divergent term

- This divergence can be subtracted out to thereby regularized the system energy
  - Renders energy constraint useful for application to equilibria in radially unbounded systems such as thermal equilibrium
  - Details on regulating self- eld divergences can be found in: Lund, Barnard, and Miller, PAC 1995, p. 3278

3) Angular Momentum

$$U_\theta = \int d^2x_\perp \int d^2x'_\perp \; (y'x - x'y)f_\perp = \text{const}$$

- Can apply to periodic (solenoidal and Einzel lens focusing) systems
- Focusing and beam pipe (if present) must be axisymmetric
  - Useful for typical solenoidal magnetic focusing with a round beam pipe
- Does not apply to alternating gradient quadrupole focusing since such systems do not have the required axisymmetry
- Subtle point: This form is really a Canonical Angular Momentum and applies to solenoidal magnetic focusing when the variables are expressed in the rotating Larmor frame (i.e., in the "tilde" variables)
  - see: S.M. Lund, lectures on Transverse Particle Dynamics

4) Axial Momentum

$$U_z = \int d^2x_\perp \int d^2x'_\perp \; m\gamma_b\beta_b c \; f_\perp = \text{const}$$

- Trivial in present model, but useful when equations of motion are generalized to allow for a spread in axial momentum

Comments on applications of the global conservation constraints:

- Global invariants strongly constrain the nonlinear evolution of the system
  - Only evolutions consistent with Vlasov's equation are physical
  - Constraints consistent with the model can bound kinematically accessible evolutions
- Application of the invariants does not require (di cult to derive) normal mode descriptions
  - But cannot, by itself, provide information on evolution timescales
- Use of global constraints to bound perturbations has appeal since distributions in real machines may be far from an equilibrium. Used to:
  - Derive su cient conditions for stability
  - Bound particle losses [O'Neil, Phys. Fluids **23**, 2216 (1980)] in nonneutral single-species, plasma columns (important for antimatter storage).
  - Bound changes of system moments (for example the rms emittance) under assumed relaxation processes; see S10

S6: Kinetic Stability Theorem for continuous focusing equilibria

[Fowler, J. Math Phys. **4,** 559 (1963); Gardner, Phys. Fluids **6**, 839 (1963);
R. Davidson, Physics of Nonneutral Plasmas, Addison-Wesley (1990)]

Resolve:

$$f_\perp = f_0(H_0) + \delta f_\perp$$

$f_0(H_0) =$ Equilibrium (subscript 0) distribution

$\delta f_\perp =$ Perturbation about equilibrium

Denote the equilibrium potential as $\phi = \phi_0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_0}{\partial r}\right) = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; f_0(H_0) \qquad \rho = q\int d^2x'_\perp \; f_0(H_0)$$

$$\phi_0(r = r_p) = \text{const}$$

Then by the linearity of Poisson's equation,

$$\nabla_\perp^2 \phi = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; f_\perp$$

$$\phi(r = r_p) = \text{const}$$

the perturbed potential $\delta\phi \equiv \phi - \phi_0$ must satisfy,

$$\nabla_\perp^2 \delta\phi = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; \delta f_\perp$$

$$\delta\phi(r = r_p) = \text{const}$$

Employ generalized entropy and transverse energy global constraints (S5):

$$U_G = \int d^2x_\perp \int d^2x'_\perp \; G(f_\perp) \;= \text{const}$$

$$U_\mathcal{E} = \int d^2x'_\perp \int d^2x_\perp \; \left\{\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp\right\} f_\perp \; + \; \int d^2x_\perp \; \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2} \;= \text{const}$$

Apply to equilibrium and full distribution to form an e ective "free-energy" F:

$$\Delta U_G = U_G - U_{G0} = \text{const}$$

$$\Delta U_\mathcal{E} = U_\mathcal{E} - U_{\mathcal{E}0} = \text{const}$$

Both total and equilibrium hold individually, so can subtract

$$F \equiv \Delta U_\mathcal{E} + \Delta U_G = \text{const}$$

$$= \int d^2x'_\perp \int d^2x_\perp \; \left\{\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp\right\}[f_\perp - f_0(H_0)]$$

$$+ \; \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2}\int d^2x_\perp \; \left\{\frac{|\nabla_\perp\phi|^2}{2} - \frac{|\nabla_\perp\phi_0|^2}{2}\right\} \; + \; \int d^2x_\perp \int d^2x'_\perp \; [G(f_\perp) - G(f_0)]$$

Conservation of free energy applies to any initial distribution for any smooth, di erentiable function G

- Use freedom in choice of G and constant value of F to make choices to allow us to bound perturbations

First manipulate self- eld energy term in F:

substitute $\phi = \phi_0 + \delta\phi$ in

$$\frac{1}{2}\int d^2x_\perp \; \{|\nabla_\perp\phi|^2 - |\nabla_\perp\phi_0|^2\} = \frac{1}{2}\int d^2x_\perp \; \{|\nabla_\perp\delta\phi|^2 + 2\nabla_\perp\phi_0\cdot\nabla_\perp\delta\phi\}$$

$$= \frac{1}{2}\int d^2x_\perp \; |\nabla_\perp\delta\phi|^2 \; + \; \int d^2x_\perp \; \{\nabla_\perp\cdot(\phi_0\nabla_\perp\delta\phi) - \phi_0\nabla_\perp^2\delta\phi\}$$

0 — Div Theorem, and free* to take $\phi_0(r = r_p) = 0$

using the Poisson equation: $\nabla_\perp^2\delta\phi = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; \delta f_\perp$

$$= \frac{1}{2}\int d^2x_\perp \; |\nabla_\perp\delta\phi|^2 \; + \; \frac{q}{\epsilon_0}\int d^2x \int d^2x'_\perp \; \phi_0\delta f_\perp$$

*Note: Can take other ref on pipe and works but then need to use more care. Also works in free-space.

Using these results, the free energy expansion is then equivalently expressed as:

$$F = \int d^2x'_\perp \int d^2x_\perp \; \left\{\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q\phi_0}{m\gamma_b^3\beta_b^2c^2}\right\}\delta f_\perp$$

$$+ \; \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2}\int d^2x_\perp \; \frac{|\nabla_\perp\delta\phi|^2}{2} \; + \; \int d^2x_\perp \int d^2x'_\perp \; [G(f_\perp) - G(f_0)]$$

$$= \text{const}$$

Up to this point, no assumptions whatsoever have been made on the magnitude of the perturbations:

Take $|\delta f_\perp| \ll f_0$ and Taylor expand G to 2[nd] order:

$$G(f_\perp) = G(f_0 + \delta f_\perp) = G(f_0) + \frac{dG(f_0)}{df_0}\delta f_\perp + \frac{d^2G(f_0)}{df_0^2}\frac{(\delta f_\perp)^2}{2} + \Theta(\delta f_\perp^3)$$

Without loss of generality,we *choose* to eliminate the term $\propto \int d^2_\perp{}' \cdots \delta f$

$$\frac{dG(f_0)}{df_0} = -H_0 = -\left(\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q\phi}{m\gamma_b^2\beta_b^2c^2}\right)$$

- This choice can *always* be realized

Then

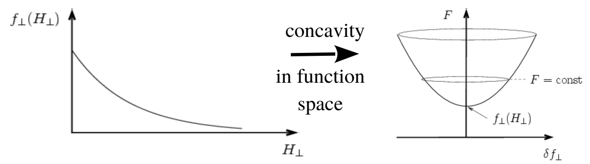
$$\frac{d^2G(f_0)}{df_0^2} = -\frac{\partial H_0}{\partial f_0} = \frac{-1}{\partial f_0(H_0)/\partial H_0}$$

and the expression for the free energy reduces to:

$$F = \int d^2x_\perp \; \left\{\frac{\epsilon_0|\nabla_\perp\delta\phi|^2}{2m\gamma_b^3\beta_b^2c^2} - \int d^2x'_\perp \; \frac{(\delta f_\perp)^2}{\partial f_0(H_0)/\partial H_0}\right\} \; + \Theta(\delta f_\perp^3) \;= \text{const}$$

- If $\partial f_0(H_0)/\partial H_0 < 0$ then F is a sum of two positive de nite terms and perturbations are bounded by F = const

$$F = \int d^2x_\perp \left\{ \frac{\epsilon_0 |\nabla_\perp \delta\phi|^2}{2m\gamma_b^3\beta_b^2 c^2} - \int d^2x'_\perp \frac{(\delta f_\perp)^2}{\partial f_0(H_0)/\partial H_0} \right\} = \text{const}$$



Value of F set by initial perturbations and concavity bounds excursions

Drop zero subscripts in statement of stability bound result:

Kinetic Stability Theorem

If $f_\perp(H_\perp)$ is a monotonic decreasing function of $H_\perp$ with $\partial f_\perp(H_\perp)/\partial H_\perp < 0$ then the equilibrium de ned by $f_\perp(H_\perp)$ is stable to arbitrary small-amplitude perturbations.

- Kinetic stability theorem is a su cient condition for stability
  - Equilibria that violate the theorem satisfy a *necessary condition for instability* but *may or may not* be stable
  - But intuitively expect energy transfer to drive instability in such cases
- Mean value theorem can be used to generalize conclusions for arbitrary amplitude
  - see R. Davidson proof

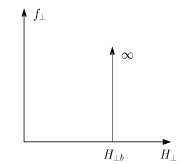
// Example Applications of Kinetic Stability Theorem

KV Equilibrium: $H_{\perp b} = \text{const}$

$$f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})$$

$\partial f_\perp/\partial H_\perp$ changes sign

inconclusive stability by theorem

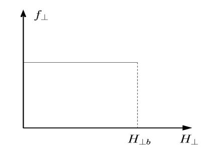


- Full normal mode analysis in Kinetic theory shows many strong instabilities when space-charge becomes strong
- Instabilities *not* surprising: delta function represents a highly inverted population in phase-space with “free-energy” to drive instabilities

Waterbag Equilibrium: $f_0 = \text{const} > 0$

$$f_\perp(H_\perp) = f_0\Theta(H_{\perp b} - H_\perp)$$

$$\partial f_\perp/\partial H_\perp = -f_0\delta(H_\perp - H_{\perp b}) \leq 0$$

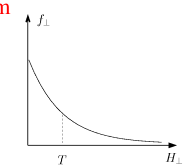


monotonic decreasing (marginal satis ed), stable by theorem

Thermal Equilibrium: $\beta = \text{const} > 0$

$$f_\perp(H_\perp) = f_0 \exp(-\beta H_\perp),$$

$$\partial f_\perp/\partial H_\perp = -f_0\beta \exp(-\beta H_\perp) \leq 0$$



monotonic decreasing (strongly satis ed), stable by theorem //

Implications of density inversion theorem and the kinetic stability theorem

In the SM Lund lectures on Transverse Equilibrium Distributions, we showed in a continuous focusing channel that knowledge of the beam density pro le $n(r)$ is equivalent to knowledge of the equilibrium distribution function $f_\perp(H_\perp)$ which generates the density pro le if the density pro le is a monotonic decreasing function of r

- Consequence of Poisson's equation for the equilibrium and the connection between $f_\perp(H_\perp)$ and the density $n(r)$

Density Inversion Theorem

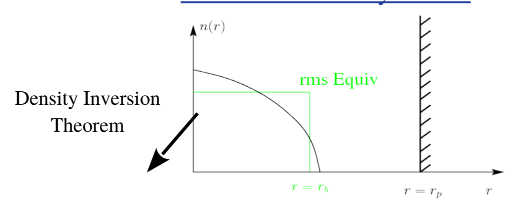
$$f_\perp(H_\perp) = -\frac{1}{2\pi}\frac{\partial n}{\partial \psi}\bigg|_{\psi=H_\perp} = -\frac{1}{2\pi}\frac{\partial n(r)/\partial r}{\partial \psi(r)/\partial r}\bigg|_{\psi=H_\perp}$$

$$\psi(r) = \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2}$$

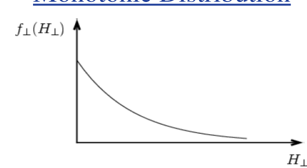
Expect for a distribution with su ciently rapid fall-o   in the radial density pro le from concavity and this result that

$$\frac{df_\perp(H_\perp)}{dH_\perp} < 0 \quad \Longrightarrow \quad \text{Stability (Kinetic Stability Theorem)}$$

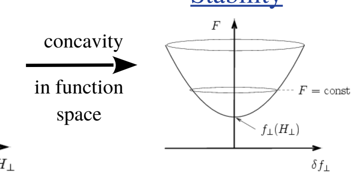
Monotonic Density Pro le



Monotonic Distribution



Stability



Comment:

- Result does not apply to periodic focusing systems
  - Still expect more benign stability if beam density projection fall o   monotonically in the radial coordinate
  - Density fall-o   can be abrupt consistent with Debye screening for a cold beam core
- Stability does not follow for radially hollowed beam density pro les
  - However, does not prove instability

S7: rms Emittance Growth and Nonlinear Forces

Fundamental theme of beam physics is to minimize statistical beam emittance growth in transport to preserve focusability on target

Return to the full transverse beam model with:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial x} \quad + \quad \text{Applied Nonlinear Field Terms}$$

and express as:

$$x''(s) + \kappa_x(s)x(s) = f_x^L(s)x(s) + F_x^{NL}(x,y,s)$$

$f_x^L(s) =$ Linear Space-Charge Coe cient

$F_x^{NL}(x,y,s) =$ Nonlinear Forces or Linear Skew Coupled Forces (Applied and Space-Charge)

// Examples:

$$f_x^L(s) = \frac{Q}{r_b^2(s)}$$

Self- eld forces within an axisymmetric (mismatched) KV beam core in a continuous focusing model

$$F_x^{NL}(x,y,s) \propto \text{Re}\left[b_3\left(\frac{x+iy}{r_p}\right)^2\right]$$

Electric (with normal and skew components) sextupole optic based on multipole expansions (see: lectures on Particle Equations of Motion) //

From the de nition of the statistical (rms) emittance:

$$\varepsilon_x \equiv 4[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2}$$

it is clear that it will be easier to derive an evolution equation for the square of the emittance and that will give us the evolution equation for the emittance since

$$\frac{d}{ds}\varepsilon_x^2 = 2\varepsilon_x \frac{d}{ds}\varepsilon_x$$

Di erentiate the squared emittance ε_x^2 moments and apply the chain rule:

cancel

$$\frac{d}{ds}\varepsilon_x^2 \equiv 32[\langle xx'\rangle_\perp \langle x'^2\rangle_\perp + \langle x^2\rangle_\perp \langle x'x''\rangle_\perp - \langle xx'\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp \langle xx''\rangle_\perp]$$
$$= 32[\langle x^2\rangle_\perp \langle x'x''\rangle_\perp - \langle xx'\rangle_\perp \langle xx''\rangle_\perp]$$

Apply the equation of motion:

$$x'' + \kappa_x x = f_x^L x + F_x^{NL}$$

To eliminate x'' in the moments and simplify. The linear terms $\propto x$ cancel to show *for any beam distribution* that:

$$\frac{d}{ds}\varepsilon_x^2 = 32\left[\langle x^2\rangle_\perp \langle x' F_x^{NL}\rangle_\perp - \langle xx'\rangle_\perp \langle x F_x^{NL}\rangle_\perp\right]$$

Implications of:

$$\frac{d}{ds}\varepsilon_x^2 = 32\left[\langle x^2\rangle_\perp \langle x' F_x^{NL}\rangle_\perp - \langle xx'\rangle_\perp \langle x F_x^{NL}\rangle_\perp\right]$$

- Emittance evolution/growth driven by nonlinear or linear skew coupling forces
  - Nonlinear terms can result from applied or space-charge elds
  - More detailed analysis shows that skew coupled forces cause *x-y* plane transfer oscillations but there is still a 4D quadratic invariant
- Minimize nonlin/skew forces to preserve emittance and maintain focusability
- This result (essentially) has already been demonstrated in the problem sets for JJ Barnard's Introductory Lectures and SM Lund lectures on Centroid and Envelope Descriptions

If the beam is accelerating, the equations of motion become:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = f_x^L x + F_x^{NL}$$

and the result above can be generalized (see homework problems) in terms of the normalized emittance to account for *x-x*' phase space area damping with accel.

- No need to use normalized coordinates: straightforward direct proof

$$\varepsilon_{nx} \equiv \gamma_b\beta_b\varepsilon_x$$
$$\frac{d}{ds}\varepsilon_{nx}^2 = 32(\gamma_b\beta_b)^2\left[\langle x^2\rangle_\perp \langle x' F_x^{NL}\rangle_\perp - \langle xx'\rangle_\perp \langle x F_x^{NL}\rangle_\perp\right]$$

These results motivate that if nonlinear and skew focusing terms are minimized that the envelope equations can be integrated with:

No Accel: Constant rms edge emittance

$$\varepsilon_x \equiv 4[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2} = \text{const}$$

Accel: Constant normalized rms edge emittance

$$\varepsilon_{nx} \equiv \gamma_b\beta_b\varepsilon_x = \gamma_b\beta_b 4[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2} = \text{const}$$

- Special case of solenoid focusing symmetry skew coupling is removable by using Larmor frame variables
  - If Larmor frame variables are not used regular emittances are expected to strongly evolve when the beam enters and exits a solenoid

S8: rms Emittance Growth and Nonlinear Space-Charge Forces

[Wangler et. al, IEEE Trans. Nuc. Sci. 32, 2196 (1985), Reiser, *Charged Particle Beams*, (1994)]

In the continuous focusing model , all nonlinear forces are from space-charge:

$$x'' + \kappa_{\beta 0}^2 x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \qquad \Longrightarrow \qquad F_x^{NL} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

Insert this F_x^{NL} in the emittance evolution formula of S7 to obtain:

- Any *linear* self-field component in F_x^{NL} will subtract out (see steps in S7)

$$\frac{d}{ds}\varepsilon_x^2 = -\frac{32q}{m\gamma_b^3\beta_b^2c^2}\left[\langle x^2\rangle_\perp\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp - \langle xx'\rangle_\perp\langle x\frac{\partial\phi}{\partial x}\rangle_\perp\right]$$

For any *axisymmetric* $(\partial/\partial\theta = 0)$ beam it can be shown (see following slides) that:

1) $\langle x\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2}\langle r\frac{\partial\phi}{\partial r}\rangle_\perp = -\frac{\lambda}{8\pi\epsilon_0}$

2) $\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2}\langle r'\frac{\partial\phi}{\partial r}\rangle_\perp = \frac{1}{2\lambda}\frac{dW}{ds}$

3) $\langle xx'\rangle_\perp = \frac{1}{2}\langle rr'\rangle_\perp = -\frac{4\pi\epsilon_0}{\lambda^2}\langle x^2\rangle_\perp\frac{dW_u}{ds}$

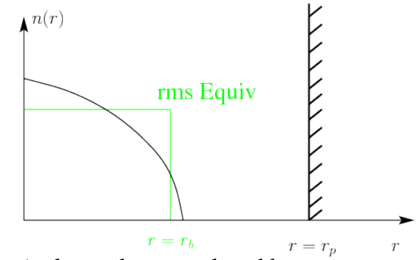
$$W = \frac{\epsilon_0}{2}\int d^2x\,|\nabla_\perp\phi|^2$$

$=$ self-field energy (per unit axial length)

$W_u = W$ for an *rms equivalent uniform density beam*

Reminder: rms Equivalent Beam Definition

An rms equivalent beam is a uniform density (KV) beam with the same 2nd order moments as the physical beam. For an rms equivalent axisymmetric $(\partial/\partial\theta = 0)$ beam:



Rms equivalance: beam replaced by a uniform density KV beam with same (at location measured) 2nd order moments as physical beam

$$\langle x^2\rangle_\perp = \langle x^2\rangle_\perp|_{\rm KV}$$

$$\langle x'^2\rangle_\perp = \langle x'^2\rangle_\perp|_{\rm KV}$$

$$\Longrightarrow \quad r_b = 2\langle x^2\rangle_\perp^{1/2} = r_b|_{\rm KV}$$

$$\varepsilon = 4\sqrt{\langle x^2\rangle_\perp\langle x'^2\rangle_\perp} = \varepsilon|_{\rm KV}$$

- The KV replacement for rms equivalance will generally evolve in s

Energy of rms Equivalent Beam:

For a uniform density beam the Poisson equation can be directly solved, or more simply, apply Gauss' Law in 2D to obtain the radial electric field as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{\lambda}{\pi r_b^2\epsilon_0}\begin{cases}1, & r < r_b\\ 0, & r > r_b\end{cases} \qquad \phi(r = r_p) = 0$$

$$E_r = -\frac{\partial\phi}{\partial r} = \begin{cases}\frac{\lambda r}{2\pi\epsilon_0 r_b^2}, & r < r_b\\ \frac{\lambda}{2\pi\epsilon_0 r}, & r > r_b\end{cases}$$

Using this result the energy of the uniform density rms equivalent beam can be calculated as:

$$W = \frac{\epsilon_0}{2}\int d^2x\,|\nabla_\perp\phi|^2 = \pi\epsilon_0\int_0^{r_b} dr\,r\left(\frac{\partial\phi}{\partial r}\right)^2$$

$$= \frac{\lambda^2}{4\pi\epsilon_0}\left[\int_0^{r_b} dr\,r\frac{r^2}{r_b^4} + \int_{r_b}^{r_p} dr\,r\frac{1}{r^2}\right]$$

$$= \frac{\lambda^2}{4\pi\epsilon_0}\left[\frac{1}{4} + \ln\left(\frac{r_p}{r_b}\right)\right]$$

Giving the energy of the uniform density rms equivalent beam as:

$$W|_{\rm rms\ eq} \equiv W_u = \frac{\lambda^2}{4\pi\epsilon_0}\left[\frac{1}{4} + \ln\left(\frac{r_p}{r_b}\right)\right]$$

- This expression can also be applied for a beam in free space by appropriately interpreting with $r_p \to \infty$ (giving infinite constant term)

// Aside: Derivation of moment relations 1), 2), 3) for Wangler's Theorem

1) Proof: $\langle x\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2}\langle r\frac{\partial\phi}{\partial r}\rangle_\perp = -\frac{\lambda}{8\pi\epsilon_0}$

- Has been derived in homework problems: will review here

From axisymmetry:

$$\langle x\frac{\partial\phi}{\partial x}\rangle_\perp = \langle y\frac{\partial\phi}{\partial y}\rangle_\perp = \frac{1}{2}\langle r\frac{\partial\phi}{\partial r}\rangle_\perp$$

From Poisson's equation:

Integrate:

Line charge within radius $r = r_b$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{qn}{\epsilon_0} \quad \Longrightarrow \quad -r\frac{\partial\phi}{\partial r} = \frac{\lambda_r(r)}{2\pi\epsilon_0} \qquad \lambda_r(r) \equiv 2\pi q\int_0^r d\tilde{r}\,\tilde{r}n(\tilde{r})$$

Using this expression for $\frac{\partial\phi}{\partial r}$ in the moment along with $n(r) = \int d^2x'\,f_\perp$ and $\frac{d\lambda_r(r)}{dr} = 2\pi rn(r)$

$$\left\langle r\frac{\partial\phi}{\partial r}\right\rangle_\perp = \frac{\int d^2x_\perp\int d^2x'_\perp\,r\frac{\partial\phi}{\partial r}\,f_\perp}{\int d^2x_\perp\int d^2x'_\perp\,f_\perp} = \frac{\int d^2x_\perp r\frac{\partial\phi}{\partial r}n}{\int d^2x_\perp\,n} = \frac{2\pi\int_0^{r_p}dr\,r\left(r\frac{\partial\phi}{\partial r}\right)n(r)}{\lambda/q}$$

$$= -\frac{1}{2\pi\epsilon_0\lambda}\int_0^{r_p}dr\lambda_r(r)\frac{d\lambda_r(r)}{dr} = -\frac{1}{4\pi\epsilon_0}\lambda_r(r)|_{r=0}^{r=r_p}$$

But:

λ 0

$$\left\langle r\frac{\partial\phi}{\partial r}\right\rangle_\perp = -\frac{1}{4\pi\epsilon_0}\lambda_r(r)|_{r=0}^{r=r_p} = -\frac{1}{4\pi\epsilon_0}[\lambda_r(r_p) - \lambda_r(0)] = -\frac{\lambda}{4\pi\epsilon_0}$$

This proves the quoted result for an arbitrary axisymmetric beam:

$$\langle x\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2}\langle r\frac{\partial\phi}{\partial r}\rangle_\perp = -\frac{\lambda}{8\pi\epsilon_0}$$

Proving 1)

2) Proof: $\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2}\langle r'\frac{\partial\phi}{\partial r}\rangle_\perp = \frac{1}{2\lambda}\frac{dW}{ds}$

From axisymmetry:

$$\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp = \langle x'\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x}\rangle_\perp = \langle\frac{xx'}{r}\frac{\partial\phi}{\partial r}\rangle_\perp$$

$$\langle y'\frac{\partial\phi}{\partial y}\rangle_\perp = \langle y'\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y}\rangle_\perp = \langle\frac{yy'}{r}\frac{\partial\phi}{\partial r}\rangle_\perp$$

and:

$$\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp = \langle y'\frac{\partial\phi}{\partial y}\rangle_\perp = \frac{1}{2}\langle\frac{xx'+yy'}{r}\frac{\partial\phi}{\partial r}\rangle_\perp = \frac{1}{2}\langle r'\frac{\partial\phi}{\partial r}\rangle_\perp$$

since:

$$r = \sqrt{x^2+y^2} \qquad \Longrightarrow \qquad r' = \frac{dr}{ds} = \frac{xx'+yy'}{\sqrt{x^2+y^2}} = \frac{xx'+yy'}{r}$$

We can apply Poisson's equation and integration by parts to recast W as

- Take reference $\phi = 0$ on pipe at $r = r_p$ without loss of generality

0 $\qquad \nabla^2\phi = -\frac{qn}{\epsilon_0}$

$$W = \frac{\epsilon_0}{2}\int d^2x\,|\nabla_\perp\phi|^2 = \frac{\epsilon_0}{2}\int d^2x\,\{\nabla_\perp\cdot[\phi\nabla_\perp\phi] - \phi\nabla_\perp^2\phi\}$$

$$= \frac{q}{2}\int d^2x\,\phi n$$

In an axisymmetric system we can regard the unbunched beam as a collection of charged charged cylindrical shells with density

$\mathcal{N} =$ Number Cylindrical shells $\qquad \lambda_q\mathcal{N} = \lambda$

$\lambda_q =$ Charge per unit axial length of shell

$r_i =$ radius ith shell

$$qn(r) = \lambda_q\sum_{i=1}^{N}\frac{\delta(r-r_i)}{2\pi r}$$

$$\Longrightarrow \qquad W = \frac{q}{2}\int d^2x\,\phi n = \frac{\lambda_q}{2}\sum_{i=1}^{\mathcal{N}}\phi(r=r_i)$$

Di erentiate this expression for W with respect to s:

$$\frac{dW}{ds} = \frac{\lambda_q}{2}\sum_{i=1}^{\mathcal{N}}r_i'\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i} = \frac{\lambda}{2\mathcal{N}}\sum_{i=1}^{\mathcal{N}}r_i'\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i}$$

But within this charged shell picture we also can express the moment directly as:

$$\langle r'\frac{\partial\phi}{\partial r}\rangle_\perp = \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}r_i'\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i}$$

Which proves the result quoted:

$$\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2}\langle r'\frac{\partial\phi}{\partial r}\rangle_\perp = \frac{1}{2\lambda}\frac{dW}{ds}$$

Proving 2)

3) Proof: $\langle xx'\rangle_\perp = -\frac{4\pi\epsilon_0}{\lambda^2}\langle x^2\rangle_\perp\frac{dW_u}{ds}$

For a uniform density beam, we explicitly calculated the eld energy:

$$W_u = \frac{\lambda^2}{4\pi\epsilon_0}\left[\frac{1}{4} + \ln\left(\frac{r_p}{r_b}\right)\right]$$

Di erentiate: Result A)

$$\frac{dW_u}{ds} = -\frac{\lambda^2}{4\pi\epsilon_0}\frac{d}{ds}\ln r_b = -\frac{\lambda^2}{4\pi\epsilon_0}\frac{1}{r_b}\frac{dr_b}{ds} \qquad \Longrightarrow \qquad \frac{dr_b}{ds} = -\frac{4\pi\epsilon_0 r_b}{\lambda^2}\frac{dW_u}{ds}$$

Use: Result B)

$$r_b \equiv 2\langle x^2\rangle_\perp^{1/2} \quad \Longrightarrow \quad \frac{dr_b}{ds} = \frac{2\langle xx'\rangle_\perp}{\langle x^2\rangle_\perp^{1/2}} \quad \Longrightarrow \quad \langle xx'\rangle_\perp = \frac{1}{2}\langle x^2\rangle_\perp^{1/2}\frac{dr_b}{ds}$$

Eliminating $\frac{dr_b}{ds}$ in result B) with result A) then gives the result quoted:

$$\langle xx'\rangle_\perp = -\frac{4\pi\epsilon_0}{\lambda^2}\langle x^2\rangle_\perp\frac{dW_u}{ds}$$

Proving 3)

//

Using moment expressions 1), 2), and 3) in the emittance evolution equation :

$$\frac{d}{ds}\varepsilon_x^2 = -\frac{32q}{m\gamma_b^3\beta_b^2c^2}\left[\langle x^2\rangle_\perp \overset{2)}{\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp} - \overset{3)}{\langle xx'\rangle_\perp}\overset{1)}{\langle x\frac{\partial\phi}{\partial x}\rangle_\perp}\right]$$

1) $\langle x\frac{\partial\phi}{\partial x}\rangle_\perp = -\frac{\lambda}{8\pi\epsilon_0}$ 2) $\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2\lambda}\frac{dW}{ds}$

3) $\langle xx'\rangle_\perp = -\frac{4\pi\epsilon_0}{\lambda^2}\langle x^2\rangle_\perp\frac{dW_u}{ds}$

$$\Longrightarrow \quad \frac{d}{ds}\varepsilon_x^2 = -\frac{32q}{m\gamma_b^3\beta_b^2c^2}\left[\langle x^2\rangle_\perp\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp - \langle xx'\rangle_\perp\langle x\frac{\partial\phi}{\partial x}\rangle_\perp\right]$$

$$= -\frac{32q}{m\gamma_b^3\beta_b^2c^2}\left[\langle x^2\rangle_\perp\frac{1}{2\lambda}\frac{dW}{ds} - \frac{4\pi\epsilon_0}{\lambda^2}\langle x^2\rangle_\perp\frac{dW_u}{ds}\frac{\lambda}{8\pi\epsilon_0}\right]$$

$$= -32\pi\epsilon_0\left[\frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2}\right]\langle x^2\rangle_\perp\frac{d}{ds}\left[\frac{W-W_u}{\lambda^2}\right]$$

$$= -32\pi\epsilon_0 Q\langle x^2\rangle_\perp\frac{d}{ds}\left[\frac{W-W_u}{\lambda^2}\right]$$

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const} \qquad \lambda = \text{const}$$

Could also pull λ through *s* derivative but group with *W* to set scale since $W \sim \lambda^2$

This derives Wangler's Theorem describing the emittance evolution of a nonuniform density beam due to nonlinear space-charge forces:

$$\frac{d}{ds}\varepsilon_x^2 = -32\pi\epsilon_0 Q\langle x^2\rangle_\perp\frac{d}{ds}\left(\frac{W-W_u}{\lambda^2}\right)$$

$W =$ Field energy (nonuniform) beam

$W_u =$ Field energy of rms equivalent uniform density beam

Alternatively, without the scale grouping, this can be expressed as:

$$\frac{d}{ds}\varepsilon_x^2 = -\frac{16q/\lambda}{m\gamma_b^3\beta_b^2c^2}\langle x^2\rangle_\perp\frac{d}{ds}(W-W_u)$$

- Result sometimes called "Wangler's Theorem" in honor of extensive work by Wangler on the topic
  - Also derived by Laposolle earlier but less was done with the result
- Applies to both radially bounded and radially infinite systems
- Result does *not* require an equilibrium for validity – only axisymmetry
- Result can be partially generalizable [J. Struckmeier and I. Hofmann, Part. Accel. **39**, 219 (1992)] to an unbunched elliptical beam
  - Result may have implications to the structure of nonuniform density Vlasov equilibria (if they exist) in periodic focusing channels: $W \neq W_u$ implies that equilibrium emittance must vary periodically in *s* when unless the density profile evolves self-similarly (see later analysis)

Application: Using Wangler's theorem to estimate emittance changes from the relaxation of space-charge nonuniformities

Wangler's theorem:

$$\frac{d}{ds}\varepsilon_x^2 = -32\pi\epsilon_0 Q\langle x^2\rangle_\perp\frac{d}{ds}\left(\frac{W-W_u}{\lambda^2}\right)$$

$W =$ Field energy (nonuniform) beam

$W_u =$ Field energy of rms equivalent uniform density beam

If the rms radius does not change much in the beam evolution:

$$r_b \equiv 2\langle x^2\rangle^{1/2} \simeq \text{const} \qquad \Longrightarrow \qquad \langle x^2\rangle_\perp = \frac{r_b^2}{4} \simeq \text{const}$$

Then the equation can be trivially integrated, showing that:

$$\Delta_{fi}(\varepsilon_x^2) \simeq -32\pi\epsilon_0 Q r_b^2\Delta_{fi}\left(\frac{W-W_u}{\lambda^2}\right)$$

$$\Delta_{fi}(\cdots) \equiv \text{Final State Value} - \text{Initial State Value}$$

So if the initial and final density profiles are known, the change in beam emittance can be simply estimated by calculating associated field energies for the initial and final nonuniform and rms equivalent uniform beams

- Change in space-charge energy is converted to thermal energy (emittance)
- Will find in most reasonable cases this effect should be small (see S10)

Is it reasonable to assume that the beam radius may not change much?

Consider the rms envelope equation for a continuous focusing system to better understand what is required for $r_b = 2\langle x^2\rangle_\perp^{1/2} \simeq \text{const}$

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

- Valid in an rms equivalent sense with $\varepsilon_x \neq \text{const}$ for a non-KV beam

If the emittance term is small relative to the perveance term

$$\frac{Q}{r_b} \gg \frac{\varepsilon_x^2}{r_b^3} = 0$$

and the initial beam starts out as matched we can approximate the equation as

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} \simeq 0 \qquad \Longrightarrow \qquad r_b \simeq \sqrt{\frac{Q}{k_{\beta 0}^2}}$$

then it is reasonable to expect the beam radius to remain nearly constant with modest emittance growth factors for a space-charge dominated beam. This ordering must be checked after estimating the emittance change based the final to initial state energy differences. See S9 and S10 analysis for a better understanding on the range of validity of this ordering.

Results to help better understand signi cance of Wangler's Theorem

Wangler's theorem:

$$\frac{d}{ds}\varepsilon_x^2 = -32\pi\epsilon_0 Q\langle x^2\rangle_\perp \frac{d}{ds}\left(\frac{W-W_u}{\lambda^2}\right)$$

$W =$ Field energy (nonuniform) beam

$W_u =$ Field energy of rms equivalent uniform density beam

KV Beam:

(axisymmetric focus/beam, matched or mismatched, cont or *s*-varying focusing)

Then

$$W = W_u \iff \text{KV beam}$$

$$\frac{d}{ds}\varepsilon_x^2 = 0 \iff \varepsilon_x = \text{const}$$

This shows that Wangler's theorem is consistent with the known result that a KV distribution evolves with rms edge emittance $\varepsilon_x = \text{const}$

- Result holds whether or not the (axisymmetric) KV beam is matched to the applied focusing lattice or whether the focusing is constant or not

Self-Similarly Evolving Beam:

It can be shown that $W = W_u$ for a beam with a self similarly evolving density pro le and this holds regardless of the form of evolution!

$$\frac{d}{ds}\varepsilon_x^2 = 0 \iff \varepsilon_x = \text{const}$$

- See derivation next pages
- Generalizes KV result

Proof: $\frac{d}{ds}(W - W_u) = 0$ for a self-similarly evolving beam

Consider a beam evolving with a self-similarly evolving density pro le:

$$qn(r) = \frac{\lambda}{\pi r_b^2} g\left(\frac{r^2}{r_b^2}\right) \qquad r_b = r_b(s) \quad \text{with arb evolution}$$

with

$$\lambda = q\int d^2x_\perp\, n \qquad r_b = 2\langle x^2\rangle_\perp^{1/2} = \sqrt{2\langle r^2\rangle_\perp}$$

and $g(x)$ is *any* shape function satisfying the two constraints

$$1) \quad \int_0^{r_p^2/r_b^2} dx\, g(x) \;= 1$$

For consistency with speci ed charge $\lambda = \text{const}$ satis ed

$$\Longrightarrow \lambda = \int d^2x_\perp\, qn \;= \frac{2\lambda}{r_b^2}\int_0^{r_p} dr\, rg\left(\frac{r^2}{r_b^2}\right) \;= \lambda\int_0^{r_p^2/r_b^2} dx\, g(x)$$

$$2) \quad \int_0^{r_p^2/r_b^2} dx\, xg(x) \;= \frac{1}{2}$$

For consistency with speci ed (evolving) rms edge radius $r_b(s)$ satis ed

$$\Longrightarrow \frac{r_b^2}{4} = \langle x^2\rangle_\perp = \frac{\frac{1}{2}\int_0^{r_p} dr\, r^3 g\left(\frac{r^2}{r_b^2}\right)}{\int_0^{r_p} dr\, rg\left(\frac{r^2}{r_b^2}\right)} = \frac{r_b^2}{2}\frac{\int_0^{r_p^2/r_b^2} dx\, xg(x)}{\int_0^{r_p^2/r_b^2} dx\, g(x)}$$

Calculate the electrostatic energy for the self-similar pro le:

$$W = \frac{\epsilon_0}{2}\int d^2x_\perp\, |\nabla_\perp\phi|^2 \;= \pi\epsilon_0\int_0^{r_p} dr\, r\left(\frac{\partial\phi}{\partial r}\right)^2$$

Using the axisymmetric solution to Poisson's equation [see steps in moment 1)]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{qn}{\epsilon_0} \quad \Longrightarrow \quad -\frac{\partial\phi}{\partial r} = \frac{\lambda_r(r)}{2\pi\epsilon_0 r} \qquad \lambda_r(r) \equiv 2\pi q\int_0^r d\tilde{r}\,\tilde{r}n(\tilde{r})$$

Gives:

$$\text{W} = \pi\epsilon_0\int_0^{r_p} dr\, r\left(\frac{\partial\phi}{\partial r}\right)^2 \;= \frac{1}{4\pi\epsilon_0}\int_0^{r_p}\frac{dr}{r}\,\lambda_r^2(r)$$

But:

$$\lambda_r(r) = 2\pi q\int_0^r d\tilde{r}\,\tilde{r}n(\tilde{r}) \;= 2\pi\int_0^r d\tilde{r}\,\tilde{r}\frac{\lambda}{\pi r_b^2}g\left(\frac{\tilde{r}^2}{r_b^2}\right) = \lambda\int_0^{r^2/r_b^2} dx\, g(x)$$

$$= \lambda G\left(\frac{r^2}{r_b^2}\right)$$

Giving:

$$W = \frac{1}{8\pi\epsilon_0}\lambda^2\int_0^{r_p^2/r_b^2}\frac{dx}{x}G^2(x) \qquad G(x) \equiv \int_0^x d\tilde{x}\, g(\tilde{x})$$

Di erentiating the eld expression *W* with respect to s gives

$$W = \frac{1}{8\pi\epsilon_0}\lambda^2\int_0^{r_p^2/r_b^2}\frac{dx}{x}G^2(x) \quad \Longrightarrow \quad \frac{dW}{ds} = \frac{\lambda^2}{8\pi\epsilon_0}\left.\frac{G^2(x)}{x}\right|_{x=r_p^2/r_b^2}\frac{d}{ds}r_p^2/r_b^2$$

and employing the normalization condition

$$\int_0^{r_p^2/r_b^2} dx\, g(x) \;= G(r_p^2/r_b^2) = 1$$

obtains

$$\frac{dW}{ds} = -\frac{1}{4\pi\epsilon_0}\frac{\lambda^2}{r_b}\frac{dr_b}{ds}$$

independent of the speci c form of the charge distribution $qn(r) = \frac{\lambda}{\pi r_b^2}g\left(\frac{r^2}{r_b^2}\right)$

This result also applies to a uniform density beam with $W = W_u$

$$\frac{dW_u}{ds} = -\frac{1}{4\pi\epsilon_0}\frac{\lambda^2}{r_b}\frac{dr_b}{ds}$$

- Can also verify this directly by di erentiating the expression for the energy of a uniform density beam [see steps in moment derivation 3)]

Inserting these results in Wangler's theorem shows there is no emittance change for a self-similarly evolving beam

$$\frac{d}{ds}\varepsilon_x^2 = -32\pi\epsilon_0 Q\langle x^2\rangle_\perp \frac{d}{ds}\left(\frac{W - W_u}{\lambda^2}\right) \qquad \frac{dW}{ds} = -4\pi\epsilon_0 \frac{\lambda^2}{r_b}\frac{dr_b}{ds}$$

$$\frac{dW_u}{ds} = -4\pi\epsilon_0 \frac{\lambda^2}{r_b}\frac{dr_b}{ds}$$

$$\Longrightarrow \quad \frac{d}{ds}\varepsilon_x^2 = 0$$

Showing that $\varepsilon_x = \text{const}$ for an arbitrary self-similar evolution in the density pro le of the beam core

Comments:

- Shows it is not only a uniform density KV beam that can have constant emittance but self-similar density evolutions also: regardless of amplitude
  - Adds to cases supporting evolution of emittance can be small !
- Implies that if density evolution is nearly self-similar like might be the case with a low over collective mode distortion in the core that there would be little emittance evolution
  - However, collective modes do not evolve self-similarly

S9: Uniform Density Beams and Extreme Energy States

Variationally construct minima of the self- eld energy per unit axial length (W):

$$W = \frac{\epsilon_0}{2}\int d^2x_\perp\, |\nabla_\perp\phi|^2$$

for an *axisymmetric* beam ($\partial/\partial\theta = 0$) which need not be continuously focused:

subject to:

$\lambda = \text{const}$ ... xed line-charge

$r_b = \sqrt{2\langle r^2\rangle_\perp} = \text{const}$ ... xed rms equivalent beam radius

Use the method of Lagrange multipliers to incorporate the xed rms-radius constraint, by varying (Helmholtz free energy):

$$F = W - \mu(\lambda/q)\langle r^2\rangle_\perp \propto \int d^2x_\perp \left\{\epsilon_0\frac{|\nabla_\perp\phi|^2}{2} - \mu r^2 n\right\} \qquad \mu = \text{const}$$

and require that variations satisfy the Poisson equation and conserve charge to satisfy the xed line-charge constraint.

$\phi \Longrightarrow$ Variation $\delta\phi$

$n \Longrightarrow$ Variation δn

Poisson equation relates $\delta\phi, \; \delta n$

Poisson equation:

$$\nabla_\perp^2\delta\phi = -\frac{q}{\epsilon_0}\delta n$$

$$\delta\phi|_{\text{boundary}} = 0$$

Charge conservation:

$$\int d^2x_\perp\, \delta n = 0 \qquad \Longrightarrow \qquad \text{Variations satisfy } \lambda = \text{const}$$

Take variations of F (*terminate* at 2nd order) giving:

- In nite order result: No approximation!

$$\delta F \propto -\int d^2x_\perp\, \{\mu r^2 + \text{const}\}\,\delta n \;+\; \epsilon_0\int d^2x_\perp \nabla_\perp\phi\cdot\nabla_\perp\delta\phi \;+\; \frac{\epsilon_0}{2}\int d^2x_\perp|\nabla_\perp\delta\phi|^2$$

Here, we added zero to the equation:

$$\text{const}\int d^2x_\perp\, \delta n = 0$$

to help clarify a reference choice in ϕ in steps that follow

Integrating the 2nd term by parts and employing the Poisson equation then gives:

$$\delta F \propto \int d^2x_\perp\, \{q\phi - \mu r^2 - \text{const}\}\,\delta n \;+\; \frac{\epsilon_0}{2}\int d^2x_\perp|\nabla_\perp\delta\phi|^2$$

For an extremum, the rst order variation term must vanish, giving *within the beam*:

$$q\phi = \mu r^2 + \text{const}$$

From Poisson's equation within the beam, this constraint on ϕ gives:

$$\nabla_\perp^2\phi = -\frac{q}{\epsilon_0}n \;\Longrightarrow\; \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{q}{\epsilon_0}n \;\Longrightarrow\; n = -\frac{4\epsilon_0\mu}{q^2} = \text{const}$$

This is the density of a uniform, axisymmetric beam, which implies that a uniform density axisymmetric beam is the extreme value state of W

This extremum is a global minimum since all variations about the extremum (2nd term of boxed equation above) are positive de nite:

$$\delta F|_{\text{uniform beam}} \propto \int d^2x_\perp|\nabla_\perp\delta\phi|^2 \quad \geq 0$$

Result:

At xed line charge and rms (envelope) radius, a uniform density beam minimizes the electrostatic self- eld energy

The result:

At fixed line charge and rms radius, a uniform density beam minimizes the electrostatic self-field energy

combined with Wangler's Theorem (see S8):

$$\frac{d}{ds}\varepsilon_x^2 = -32\pi\epsilon_0 Q\langle x^2\rangle_\perp \frac{d}{ds}\left(\frac{W-W_u}{\lambda^2}\right)$$

$W =$ Field energy (nonuniform) beam

$W_u =$ Field energy of rms equivalent uniform density beam

with $\langle x^2\rangle_\perp = r_b^2/4 \simeq \text{const}$ shows that:

- Self-field energy changes from beam nonuniformity drives emittance evolution
- Expect the following local in s trends in an evolving beam density profile
  - *Nonuniform* density => *more uniform* density <=> local emittance *growth*
    $W - W_u$ decreasing in s, $-d(W-W_u)/ds > 0$
  - *Uniform* density => *more nonuniform* density <=> local emittance *reduction*
    $W - W_u$ increasing in s, $-d(W-W_u)/ds < 0$
- Should attempt to:
  maintain beam density uniformity to preserve beam emittance and focusability
- Results can be partially generalized to 2D elliptical beams
  - See: J. Struckmeier and I. Hofmann, Part Accel. **39**, 219 (1992)

S10: Collective Relaxation of Space-Charge Nonuniformities and rms Emittance Growth

The space-charge profile of intense beams can be born highly nonuniform out of nonideal (real) injectors or become nonuniform due to a variety of (error) processes. Also, low-order envelope matching of the beam may be incorrect due to focusing and/or distribution errors.

How much emittance growth and changes in other characteristic parameters may be induced by relaxation of characteristic perturbations?

- Employ Global Conservation Constraints of system to bound possible changes
- Assume full relaxation to a final, uniform density state for simplicity

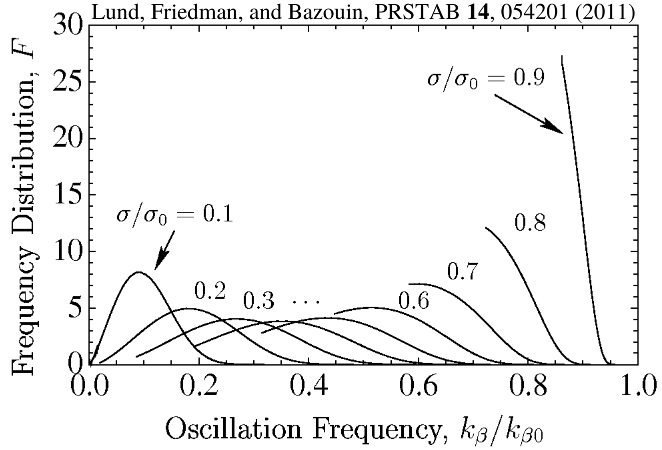
What is the mechanism for the assumed relaxation?

- Collective modes launched by errors will have a broad spectrum
  - Phase mixing can smooth nonuniformities – mode frequencies incommensurate
- Nonlinear interactions, Landau damping, interaction with external errors, ...
- Certain errors more/less likely to relax:
  - Internal wave perturbations expected to relax due to many interactions
  - Envelope mismatch will not (coherent mode) unless amplitudes are very large producing copious halo and nonlinear interactions

Motivation for rapid phase-mixing mechanism for beams with intense space-charge: strong spread in distribution of particle oscillation frequencies in the core of the beam

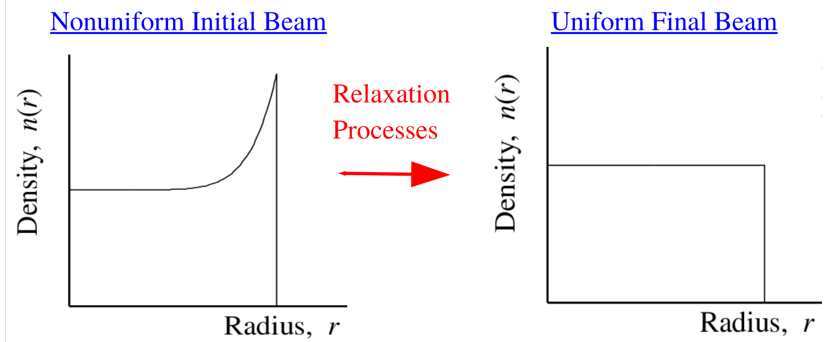
Thermal equilibrium beam core results, see S.M. Lund lectures on Transverse Equilibrium Distributions, S7

Lund, Friedman, and Bazouin, PRSTAB **14**, 054201 (2011)



Estimate emittance increases from relaxation of nonlinear space-charge waves if an initial nonuniform beam to a uniform density beam

- Should result in max estimate since uniform density beam has lowest energy as shown in S9



Reference: High resolution self-consistent PIC simulations shown in class

- Continuous focusing and a more realistic FODO transport lattice
  - Relaxation more complete in FODO lattice due to a richer frequency spectrum
- Relaxations surprisingly rapid: few undepressed betatron wavelengths observed in simulations

Initial Nonuniform Beam Parameterization

$$n(r) = \begin{cases} \hat{n}\left[1 + \frac{1-h}{h}\left(\frac{r}{r_e}\right)^p\right], & 0 \leq r \leq r_e \\ 0, & r_e < r \leq r_p \end{cases}$$

$h =$ hollowing parameter $= n(r=0)/n(r=r_e)$

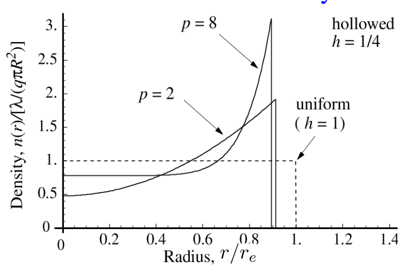
$p =$ radial index

$r_e =$ edge radius

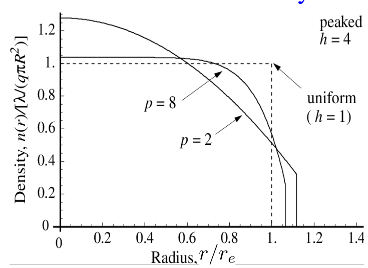
$$\lambda = \int d^2x_\perp \, n = \pi q \hat{n} r_e^2 \left[\frac{(ph+2)}{(p+2)h}\right] \qquad r_b = 2\langle x^2\rangle_\perp^{1/2} = \sqrt{\frac{(p+2)(ph+4)}{(p+4)(ph+2)}}\, r_e$$

Normalize pro les to compare common rms radius (r_b) and total charge (λ)

Hollowed Initial Density



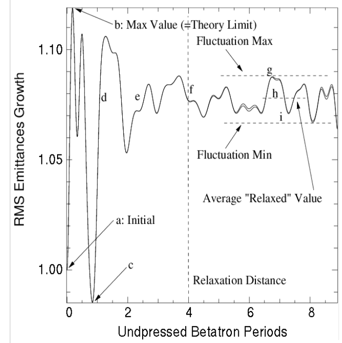
Peaked Initial Density

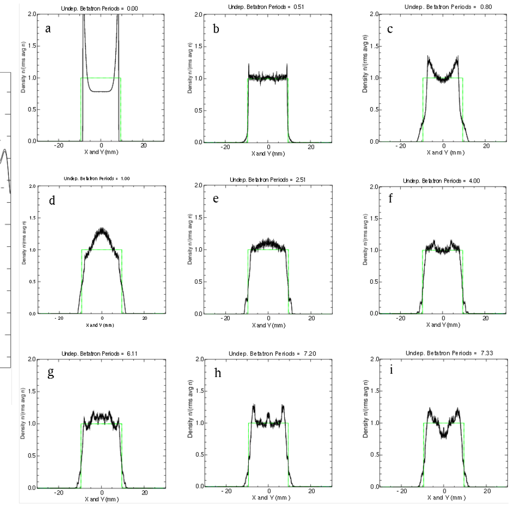


- Analogous de nitions are made for the radial temperature pro le of the beam

Example Simulation, Initial Nonuniform Beam

$\sigma/\sigma_0 = 0.2$ Initial density: h=1/4, p=8 Initial Temp: h = in nity, p=2





Show movie of evolution

[Lund, Grote, and Davidson, Nuc. Instr. Meth. A 544, 472 (2005)]

Hollowed beam simulation/theory results for strong space-charge

- Peaked beam shows very small emittance growth

Initial beam					Relaxed and transient beam		
σ_i/σ_0	Density		Temperature		Emittance growth		Undep. betatron periods to relax
	h	p	h	p	Theory	Simulation	
0.1	0.25	4	1	arb.	1.57	1.42 (1.57, 1.31–1.52)	3.5
			∞	2		1.45 (1.57, 1.38–1.52)	3.0
			0.5			1.41 (1.57, 1.30–1.52)	3.0
	0.25	8	1	arb.	1.43	1.33 (1.43, 1.28–1.38)	3.5
			∞	2		1.35 (1.43, 1.30–1.40)	4.5
			0.5			1.32 (1.43, 1.26–1.38)	4.0
0.20	0.25	4	1	arb.	1.17	1.11 (1.16, 1.09–1.13)	4.5
			∞	2		1.12 (1.16, 1.10–1.13)	3.0
			0.5			1.11 (1.16, 1.09–1.13)	4.0
	0.25	8	1	arb.	1.12	1.08 (1.12, 1.06–1.09)	5.5
			∞	2		1.08 (1.12, 1.07–1.09)	4.0
			0.5			1.08 (1.12, 1.06–1.09)	4.5

Theory results based on conservation of system charge and energy used to calculate the change in rms edge radius between initial (i) and nal (f) matched beam states

$$\frac{(r_{bf}/r_{bi})^2 - 1}{1-(\sigma_i/\sigma_0)^2} + \frac{p(1-h)[4+p+(3+p)h]}{(p+2)(p+4)(2+ph)^2} - \ln\left[\sqrt{\frac{(p+2)(ph+4)}{(p+4)(ph+2)}}\frac{r_{bf}}{r_{bi}}\right] = 0$$

Ratios of nal to initial emittance are then obtainable from the matched envelope eqns:

$$\frac{\varepsilon_{xf}}{\varepsilon_{xi}} = \frac{r_{bf}}{r_{bi}}\sqrt{\frac{(r_{bf}/r_{bi})^2 - [1-(\sigma_i/\sigma_0)^2]}{(\sigma_i/\sigma_0)^2}}$$

Movies (mpg format) shown in class are on the course web site:

Continuous Focusing:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2020/lec_set_08/tks_relax_cf.mpg

- Evolution case similar to one detailed on previous slides

Periodic Quadrupole Focusing:

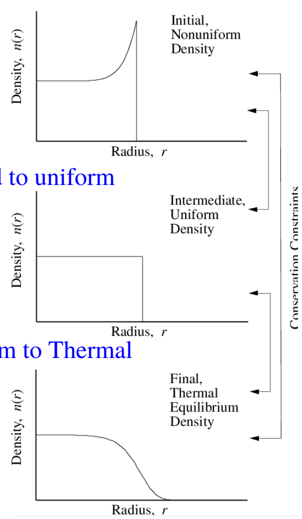
https://people.nscl.msu.edu/~lund/uspas/bpisc_2020/lec_set_08/tks_relax_ag.mpg

- Via D.P. Grote, LLNL: Evolution case for FODO quadrupole case with strong space-charge, and an extremely hollowed density initial beam that is rms envelope matched to the focusing lattice. The initial temperature spread is uniform. Speci c parameters unknown.
- Note that relaxation may be more complete than for the continuous focusing case
  - Likely a much broader spectrum of modes launched in periodic focus case

Theory estimates from global conservation constraints work well. What changes if the beam relaxes to a smooth thermal equilibrium instead? -- Very little change

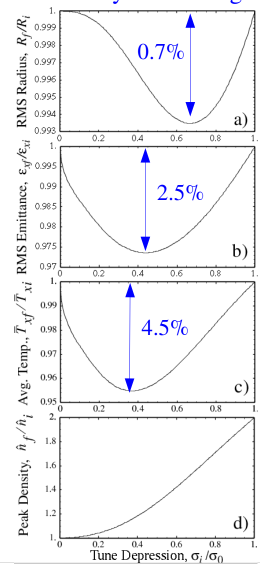
Step 1:
Peaked to uniform

Step 2:
Uniform to Thermal



Regard as a two-step relaxation and apply conservation steps between each step

Essentially no rms changes in 2nd step



Lund, Barnard, and Miller, PAC 1995, p. 3278

Comments:

- Due to such small changes in rms radius and emittance undergoing relaxation from a uniform density beam to a smooth equilibrium profile (thermal equilibrium case shown) we can neglect the small changes induced by the 2nd step when estimating emittance growth
- Note that changes are *maximum* at intermediate values of $\sigma_0 \sim 0.5$ rather than for small with $\sigma_0 \ll 0.5$ where space-charge is strongest
  - Space charge stronger but there is less change in profile under relaxation when $\sigma_0 \ll 0.5$
- Not surprisingly, changes are also small for weak space charge with $\sigma_0 \simeq 1$ since the strength of the space -charge field is weak
  - Result in spite of the density profile being far from uniform since space charge too weak for significant Debye screening of the applied foucus force
- Emittance *decreasing* on relaxation from uniform density distribution to a nonuniform density distribution is consistent with the expected trends predicted by Wanglers' Theorem discussed in S10

S11: Emittance Growth from Envelope Mismatch Oscillations

Emittance growth from envelope mismatch oscillations

Similar energy conservation methods can be applied to estimate the effect on emittance growth if the initial beam is envelope mismatched and the energy of the mismatch oscillation is converted into emittance if the beam relaxes

- See Reiser, *Theory and Design of Charged Particle Beams*, 1994, 2008

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

$$r_b'' \sim \text{Max}[(r_b - r_{b0})] k_B^2$$ Term can be large

$$r_{b0} = \text{Matched Radius}$$
$$k_B = \text{Breathing Mode Wave Number} \quad \left(k_B^2 = 4k_{\beta 0}^2 - 2\frac{Q}{r_{b0}^2} \right)$$

Large emittance increases can result from the relaxation of mismatch oscillations, but simulations of beams with high space-charge intensity suggest there is no mechanism to rapidly induce this relaxation

- Envelope oscillations are low-order collective modes of the beam and are thereby more likely to be difficult to damp.
- Possible exception: Lattice with large nonlinear applied focusing forces

S12: Non-Tenuous Halo Induced Mechanism of Higher Order Instability in Quadrupole Focusing Channels

In periodic focusing with alternating gradient quadrupole focusing (most common case), it has been observed in simulations and the laboratory that good transport in terms of little lost particles or emittance growth is obtained when the applied focusing strength satisfies:

$$\sigma_0 \lesssim 85^\circ$$ little dependence on σ/σ_0

For many years it was unclear what primary mechanism(s) cause this transport limit in spite of the effect being strongly expressed in simulations and laboratory experiments. It was long thought that collective modes coupled to the lattice were responsible. However:

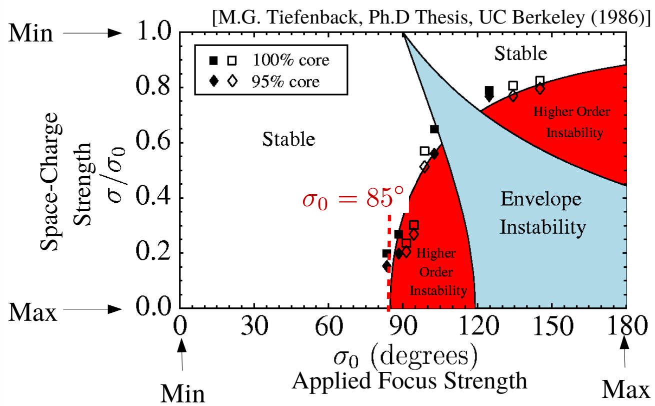
- Modes carry little free energy (see S10) to drive strong emittance growth
- Particle losses and strong halo observed when stability criterion is violated
- Collective internal modes likely also pumped but hard to explain on the basis of KV mode instabilities

The theory outlined here clarifies how this limit comes about via a strong halo-like resonance mechanism affecting near edge particles

- Does *not* require an equilibrium core beam

Review: In the SBTE experiment at LBNL:

Higher order Vlasov instability with strong emittance growth/particle losses observed in broad parametric region below envelope band



Results summarized by $\sigma_0 \lesssim 85°$ for strong space-charge

- Reliably applied design criterion in the lab
- Limited theory understanding for 20+ years; Haber, Laslett simulations supported

Self consistent Vlasov stability simulations were carried out with a wide range of parameters/distributions to quantify characteristics of instability

- Carried out using the WARP PIC code from LLNL/LBNL
- High resolution/stat 2D *x-y* slice simulations time-advanced to *s*-plane
- Non-singular, rms matched distributions loaded:
  - semi-Gaussian
  - Continuous focusing equilibrium $f_\perp(H_\perp)$ with self-consistent space-charge canonically transformed to alternating-gradient symmetry: (see Transverse Equilibrium Distributions, S10B)
    - waterbag
    - parabolic
    - Gaussian/Thermal
- Singular KV also loaded - only to check instability resolutions

More Details:

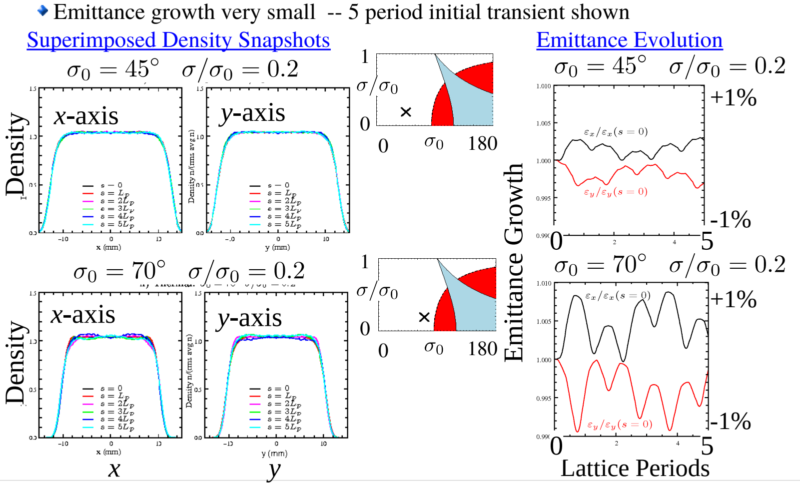
Stability simulations:
Lund and Chawla, "Space-charge transport limits of ion beams in periodic quadrupole focusing channels," *Nuc. Instr. Meth. A* **561**, 203 (2006)

Initial Loads applied:
Lund, Kikuchi, Davidson, "Generation of initial distributions for simulations with high space-charge intensity," PRSTAB **14**, 054201 (2011)

Parametric simulations of non-singular, initially rms matched distributions have little emittance evolution outside of instability regions experimentally observed

Example: initial thermal equilibrium distribution

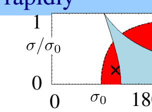
- Density along *x*- and *y*-axes for 5 periods
- Emittance growth very small -- 5 period initial transient shown

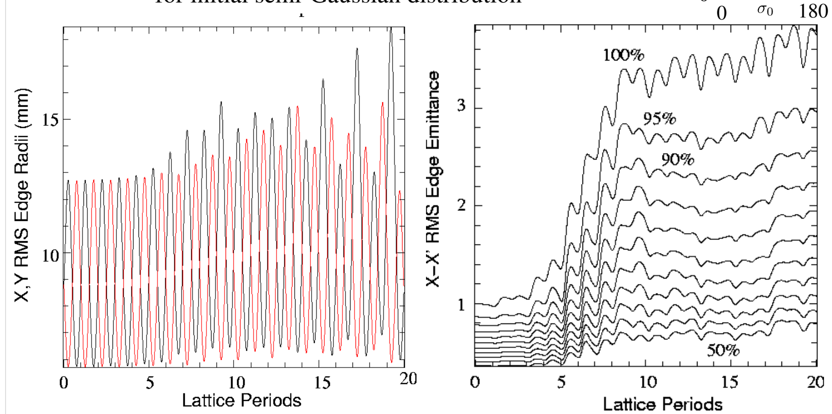


Parametric PIC simulations of quadrupole transport agree with experimental observations and show that large rms emittance growth can occur rapidly

Parameters: $\sigma_0 = 110°$, $\sigma/\sigma_0 = 0.2$ ($L_p = 0.5$ m, $\eta = 0.5$)

for initial semi-Gaussian distribution



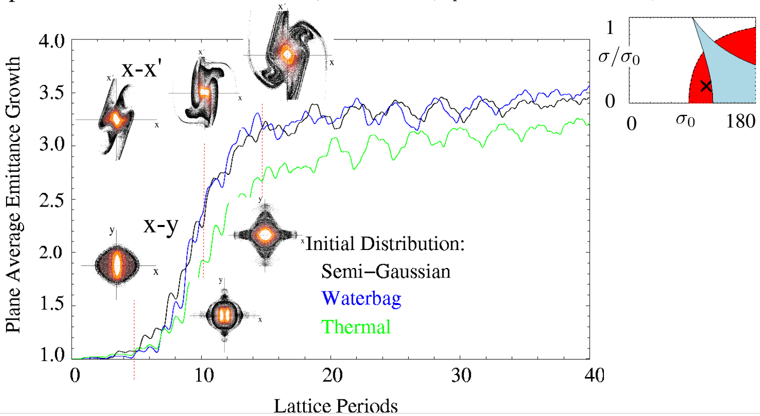


Higher $\sigma_0 \lesssim 85°$ makes the onset of emittance growth larger and more rapid

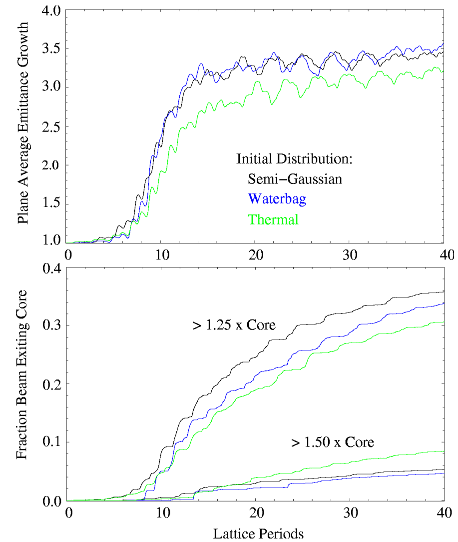
Parametric simulations nd broad instability region to the left of the envelope band -- features relatively insensitive to the form of the (non-singular) matched initial distribution

- Where unstable, growth becomes larger and faster with increasing σ_0

Example Parameters: $\sigma_0 = 110°, \ \sigma/\sigma_0 = 0.2 \ (L_p = 0.5 \text{ m}, \ \eta = 0.5)$



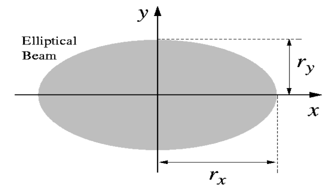
Essential instability feature -- particles evolve outside core of the beam precludes pure "internal mode" description of instability



Instantaneous, rms equivalent measure of beam core:

$$r_x = 2\langle x^2 \rangle_\perp^{1/2}$$

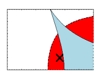
$$r_y = 2\langle y^2 \rangle_\perp^{1/2}$$



"tag" particles that evolve outside core at any s in simulation

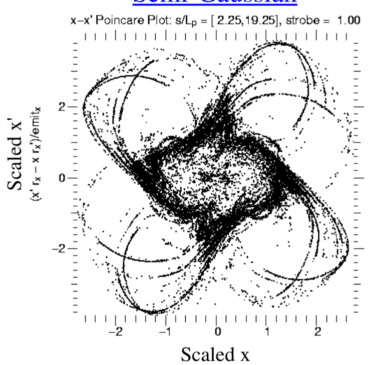
Self-consistent Poincare plots generated for the case of instability show large oscillation amplitude particles have halo-like resonant structure -- qualitative features relatively insensitive to the initial distribution

Lattice period Poincare strobe

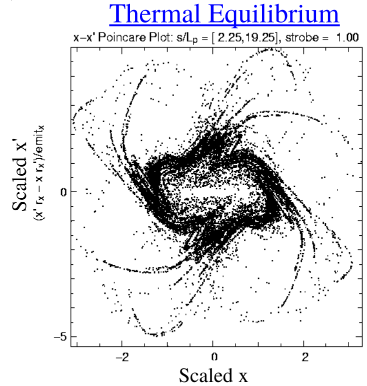


$\sigma_0 = 110° \quad \sigma/\sigma_0 = 0.2$

Semi-Gaussian



Thermal Equilibrium



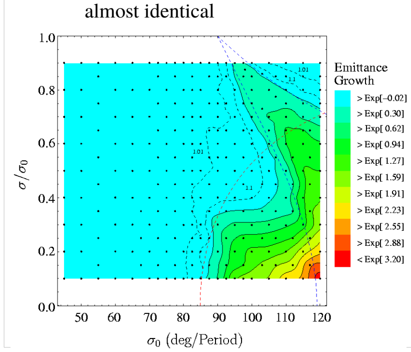
- Only particles evolving nearly along x-axis accumulated to generate clearer picture
  - Including o   axis particles does *not* change basic conclusions

Extensive simulations carried out to better understand the parametric region of strong emittance growth

- All simulations advanced 6 undepressed betatron periods
  - Enough to resolve transition boundary: transition growth can be larger if run longer
- Strong growth regions of initial distributions all similar (threshold can vary)
  - Irregular grid contouring with ~200 simulations (dots) thoroughly probe instabilities

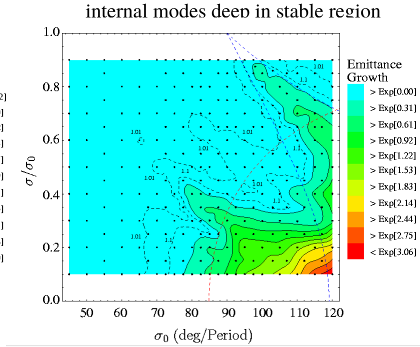
initial semi-Gaussian

- Initial thermal/Gaussian almost identical



initial Waterbag

- Initial KV similar with extra unstable internal modes deep in stable region



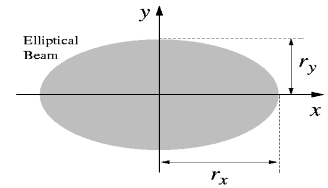
Motivated by simulation results -- explore "halo"-like mechanisms to explain observed space-charge induced limits to quadrupole transport

- Resonances can be *strong*: driven by matched envelope utter and strong space-charge
- *Not* tenuous halo:
  Near edge particles can easily evolve outside core due to:
  - Lack of equilibrium in core
  - Collective waves
  - Focusing errors, ...

  Most particles in beam core oscillate near edge
- Langiel rst attempted to apply halo mechanism to space-charge limits
  Langiel, *Nuc. Instr. Meth. A* **345**, 405 (1994) $\sigma_0 < 60^\circ$
  Appears to concluded overly restrictive stability criterion:
- Re ne analysis: examine halo properties of particles launched just outside the rms equivalent beam core and analyze in variables to reduce " utter" associated with the matched core oscillations in periodic focusing
  Lund and Chawla, *Nuc. Instr. Meth. A* **561**, 203 (2006)
  Lund, Barnard, Bukh, Chawla, and Chilton, *Nuc. Instr. Meth. A* **577**, 173 (2007)

Core-Particle Model --- Transverse particle equations of motion for a test particle moving inside and outside a uniform density elliptical beam envelope

$$x'' + \kappa_x x = \frac{2QF_x}{(r_x + r_y)r_x} x$$
$$y'' + \kappa_y y = \frac{2QF_y}{(r_x + r_y)r_y} y$$



$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^2 \beta_b^2 c^2}$$ ..... dimensionless perveance

Where:

Inside the beam

$F_x = 1$

$F_y = 1$

Outside the beam:

$$F_x = (r_x + r_y)\frac{r_x}{x} Re[\tilde{S}]$$
$$F_y = -(r_x + r_y)\frac{r_y}{y} Im[\tilde{S}]$$

with

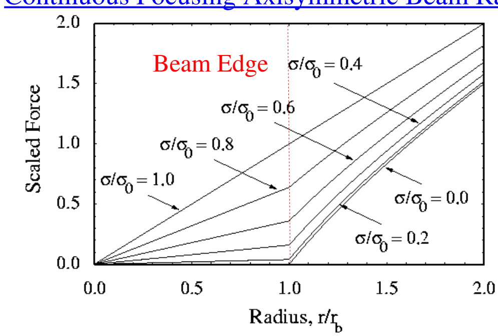
$$\tilde{S} \equiv \frac{\tilde{z}}{r_x^2 - r_y^2}\left[1 - \sqrt{1 - \frac{(r_x^2 - r_y^2)}{\tilde{z}^2}}\right]$$
$$= \frac{1}{2\tilde{z}}\left[1 + \frac{1}{2}\frac{r_x^2 - r_y^2}{\tilde{z}^2} + \frac{1}{8}\frac{(r_x^2 - r_y^2)^2}{\tilde{z}^4} + \cdots\right]$$

$\tilde{z} = x + iy$

$i = \sqrt{-1}$

Particles oscillating radially outside the beam envelope will experience oscillating nonlinear forces that vary with space-charge intensity and can drive resonances

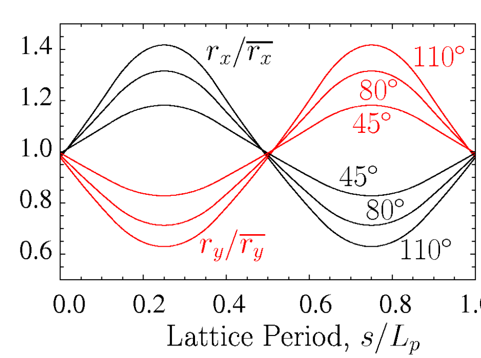
Continuous Focusing Axisymmetric Beam Radial Force



- Nonlinear force transition at beam edge larger for strong space-charge
- Edge oscillations of matched beam enhance nonlinear e ects acting on particles moving outside the envelope
- In AG focusing envelope oscillation amplitude scales strongly with σ_0

For quadrupole transport, relative matched beam envelope excursions increase with applied focusing strength

- Larger edge utter increases nonlinearity acting on particles evolving outside the core



$$\overline{r_x} = \int_0^{L_p} \frac{ds}{L_p} r_x(s)$$

$\eta = 0.5 \quad L_p = 0.5$ m

$Q = 5 \times 10^{-4}$

$\varepsilon_x = \varepsilon_y = 50$ mm-mrad

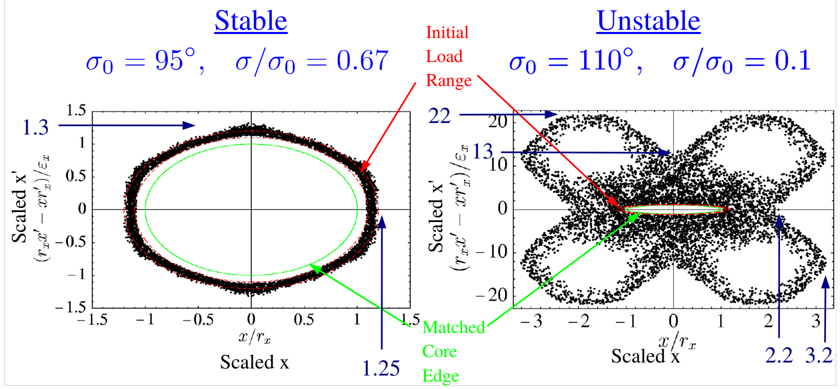
σ_0	σ/σ_0
45°	0.20
80°	0.26
110°	0.32

Space-charge nonlinear forces and *matched* envelope utter strongly drive resonances for particles evolving outside of beam edge

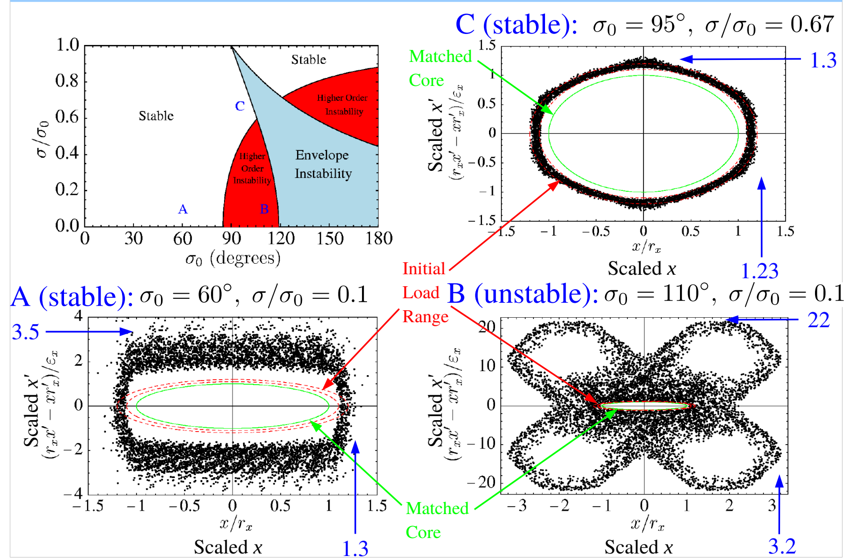
Core-particle simulations: Poincare plots illustrate resonances associated with higher-order halo production near the beam edge for FODO quadrupole transport

- High order resonances near the core are strongly expressed
- Resonances stronger for higher σ_0 and stronger space-charge
- Can overlap and break-up (strong chaotic transition) allowing particles launched *near the core* to rapidly increase in oscillation amplitude

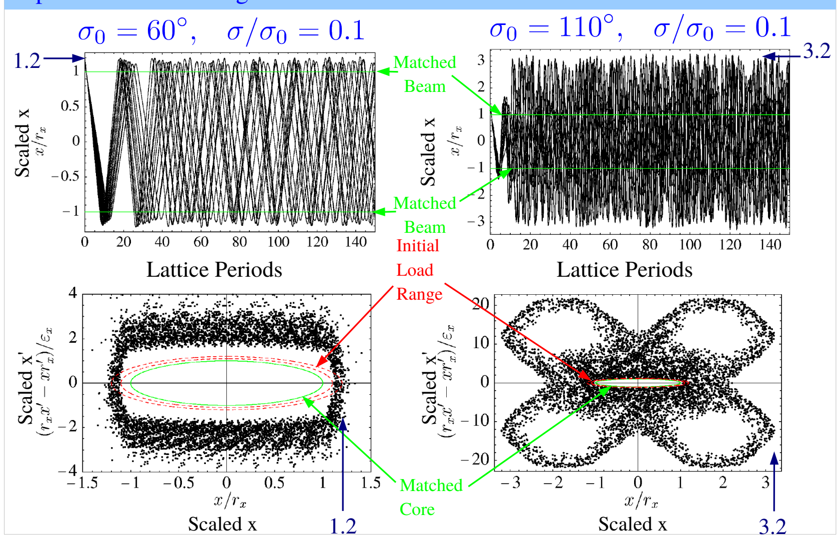
Lattice Period Poincare Strobe, particles launched [1.1,1.2] times core radius



Core-particle simulations: Poincare phase-space plots illustrate stability regions where near edge particles grow in oscillation amplitude: launch [1.1,1.2]x core

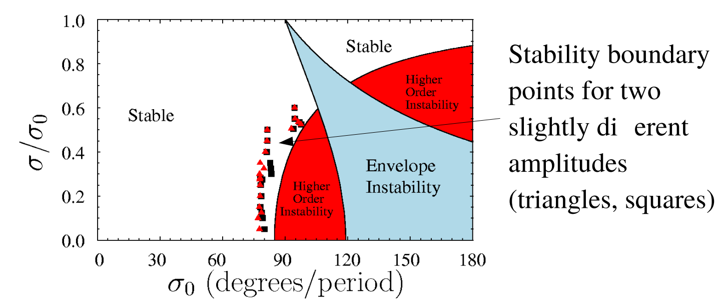


Core-particle simulations: Amplitude pumping of characteristic "unstable" phase-space structures is typically rapid and saturates whereas stable cases experience little or no growth



Core particle simulations: Stability boundary data from a "halo" stability criterion agree with experimental data for quadrupole transport limits

- Start at a point (σ_0, σ) deep within the stable region
- While increasing σ_0 vary σ to nd a point (if it exists) where initial launch groups [1.05, 1.10] outside the matched beam envelope are pumped to max amplitudes of 1.5 times the matched envelope
  - Boundary position relatively insensitive to speci c group and amplitude growth choices



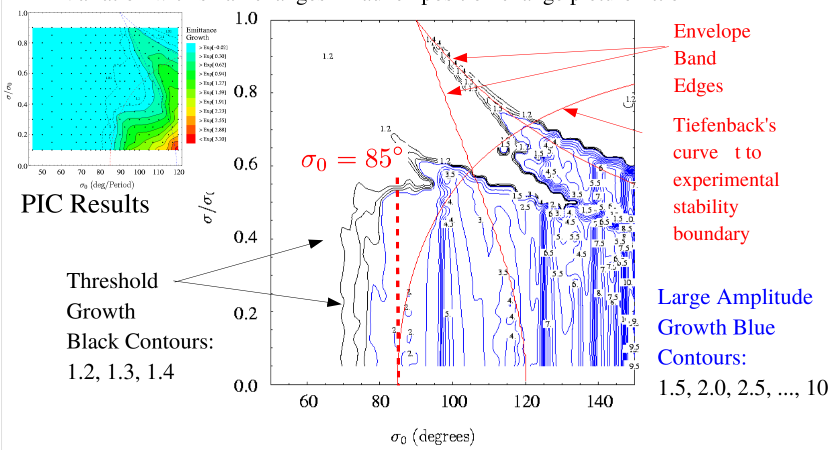
Other halo analyses of transport limits conclude overly restrictive limits:

[Lagniel, Nuc. Instr. Meth. A **345**, 405 (1994)]

Contours of max particle amplitudes in core particle model suggest stability regions consistent with self-consistent simulations and experiment

Max amplitudes achieved for particles launched [1.05,1.1] times the core radius:

- Variation with small changes in launch position change picture little



Note: consistent with PIC results, instability well above envelope band not found

Discussion: Higher order space-charge stability limits in periodic quadrupole transport

High-order space-charge related emittance growth observed in intense beam transport in quadrupole focusing channels with $\sigma_0 \gtrsim 85°$:

- SBTE Experiment at LBNL [M.G. Tiefenback, Ph.D Thesis, UC Berkeley (1986)]
- Simulations by Haber, Laslett, and others

A core-particle model suggests these space-charge transport limits result from a strong halo-like mechanism:

- Space-Charge and Envelope Flutter driven
- Results in large oscillation amplitude growth -- strongly chaotic resonance chain which limits at large amplitude rapidly increases oscillations of particles just outside of the beam edge
- Not weak: many particles participate -- Lack of core equilibrium provides pump of significant numbers of particles evolving sufficiently outside the beam edge
- Strong statistical emittance growth and lost particles (with aperture)

Mechanism consistent with other features observed:

- Stronger with envelope mismatch: consistent with mismatched beams more unstable
- Weak for high occupancy solenoid transport: less envelope flutter suppresses

More Details:

Lund and Chawla, *Space-charge transport limits of ion beams in periodic quadrupole focusing channels*, *Nuc. Instr. Meth. A* **561**, 203 (2006)

Lund, Barnard, Bukh, Chawla, and Chilton, *A core-particle model for periodically focused ion beams with intense space-charge*, *Nuc. Instr. Meth. A* **577**, 173 (2007)

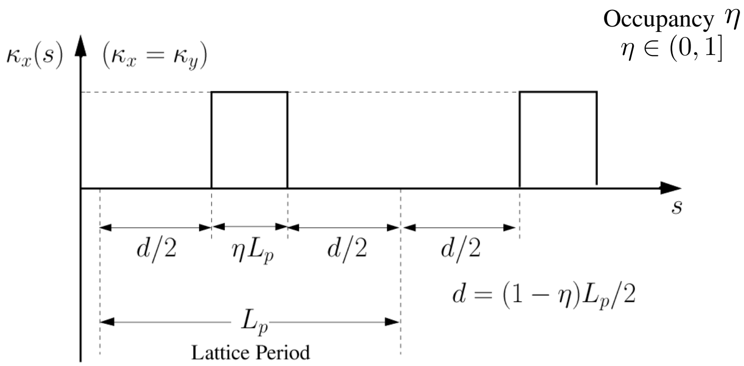
Lund, Kikuchi, and Davidson, *Generation of intial kinetic distributions for simulation of long-pulse charged particle beams with high space-charge intensity*, PRSTAB **12**, 114801 (2009)

S13: Non-Tenuous Halo Induced Instability in Solenoidal Focusing

Here we will briefly outline application of the core particle procedure applied in S12 for quadrupole focusing to analyze whether analogous transport limits appear in solenoidal focusing

- Will find limits occur but are much more benign than for quadrupole focusing and do not appear to introduce significant additional parameter restrictions beyond those occurring for envelope modes
  - Reason: Solenoids have lesser degree of envelope flutter to drive

Analogous core-particle stability studies have been carried out for periodic solenoidal transport channels



Solenoidal focusing weaker than quadrupole focusing:

- Less focusing strength than AG quadrupole for similar total eld energies as beam Kinetic energy increases
- Matched envelope utter less, and scales strongly with η
- Limit $\eta = 1$ stable (continuous focusing) with no envelope utter

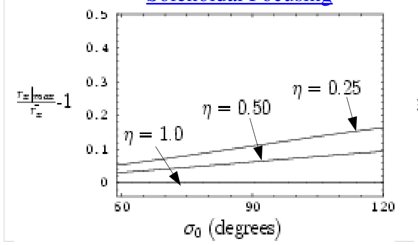
Flutter scaling of the matched beam envelope varies for quadrupole and solenoidal focusing

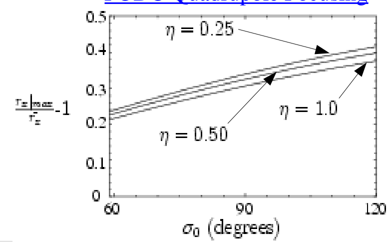
$$\frac{r_x|_{max}}{\bar{r}_x} - 1 \simeq \begin{cases} (1-\cos\sigma_0)\frac{(1-\eta)(1-\eta/2)}{6} & \text{Solenoidal Focusing} \\ (1-\cos\sigma_0)^{1/2}\frac{(1-\eta/2)}{2^{3/2}(1-2\eta/3)^{1/2}} & \text{Quadrupole Focusing} \end{cases}$$

Based on: E.P. Lee, Phys. Plasmas, **9** 4301 (2002) for limit $\sigma/\sigma_0 \to 0$

- Solenoids:
  - Varies signi cant with both σ_0 and η
- Quadrupoles:
  - Phase advance σ_0 variation signi cant
  - Occupancy η variation weak

Solenoidal Focusing | FODO Quadrupole Focusing

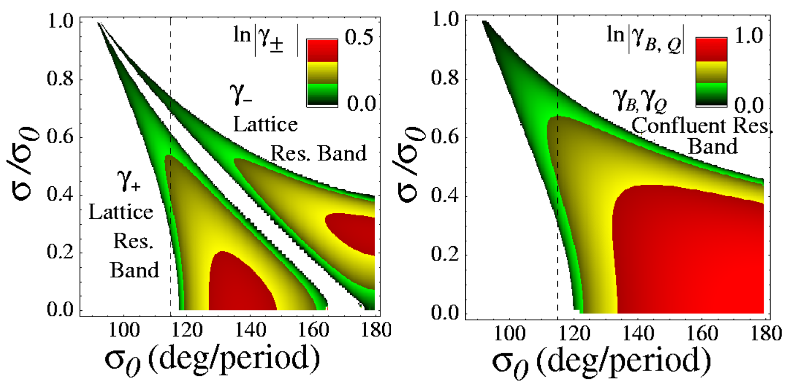




Envelope band instabilities and growth rates for periodic solenoidal and quadrupole doublet focusing lattices

Envelope Mode Instability Growth Rates

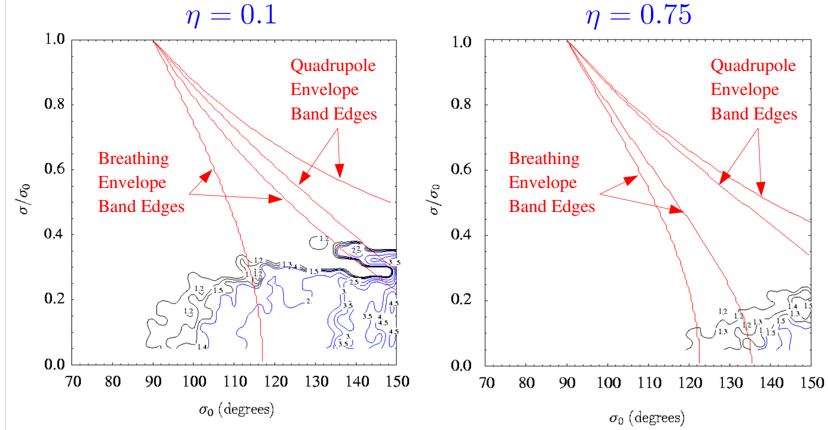
Solenoid (η = 0.25) | Quadrupole FODO (η = 0.70)



[S.M. Lund and B. Bukh, PRSTAB 024801 (2004)]

Similar space-charge dependent amplitude growth is observed as in quadrupole focusing, *but the e ect is weaker and occupancy dependent* due to di erent matched envelope utter scaling in solenoidal focusing

Carry out core particle study analogous to FODO quadrupole focus case launching test particles [1.05,1.1]x outside the matched core and calculating max amplitudes



S14: Phase Mixing and Landau Damping in Beams

May cover in future editions of class notes

- Likely inadequate time in lectures
- Simulation illustration?

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2020

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References: For more information see:

These course notes are posted with updates, corrections, and supplemental material at:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2020

Materials associated with previous and related versions of this course are archived at:

JJ Barnard and SM Lund, *Beam Physics with Intense Space-Charge,* USPAS:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2017 2017 Version
https://people.nscl.msu.edu/~lund/uspas/bpisc_2015 2015 Version
http://hifweb.lbl.gov/USPAS_2011 2011 Lecture Notes + Info
http://uspas.fnal.gov/programs/past-programs.shtml (2008, 2006, 2004)

JJ Barnard and SM Lund, *Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields*, UC Berkeley, Nuclear Engineering NE290H

http://hifweb.lbl.gov/NE290H 2009 Lecture Notes + Info

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M. Reiser, *Theory and Design of Charged Particle Beams*, Wiley (1994)

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F. Sacherer, *Transverse Space-Charge E ects in Circular Accelerators*, Univ. of California Berkeley, Ph.D Thesis (1968)

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S. Lund, A. Friedman, and G. Bazouin, "Sheet beam model for intense space charge: Application to Debye screening and the distribution of particle oscillation frequencies in a thermal equilibrium beam," PRSTAB **14**, 054201 (2011)