Electromagnetic wave propagation in Particle-In-Cell codes

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Self-Consistent Simulations of Beam and Plasma Systems

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Electromagnetic Waves: Outline

1. Numerical dispersion and Courant limit
    - Dispersion and Courant limit in 1D
    - Dispersion and Courant limit in 3D
    - Spectral solvers and numerical dispersion
2. Open boundaries conditions
    - Silver-Müller boundary conditions
    - Perfectly Matched Layers

1D discrete propagation equation in vacuum

Reminder: 1D discrete Maxwell equations in vacuum

$$\frac{{B_y}_{\ell+1/2}^{n+1/2} - {B_y}_{\ell+1/2}^{n-1/2}}{\Delta t} = -\frac{{E_x}_{\ell+1}^{n} - {E_x}_{\ell}^{n}}{\Delta z} \quad (\text{from } \partial_t \boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{E})$$

$$\frac{1}{c^2}\frac{{E_x}_{\ell}^{n+1} - {E_x}_{\ell}^{n}}{\Delta t} = -\frac{{B_y}_{\ell+1/2}^{n+1/2} - {B_y}_{\ell-1/2}^{n+1/2}}{\Delta z} \quad (\text{from } \frac{1}{c^2}\partial_t \boldsymbol{E} = \boldsymbol{\nabla} \times \boldsymbol{B})$$

These equations can be combined into a **propagation equation** for E_x:

$$\begin{aligned}
\frac{1}{c^2}\frac{{E_x}_{\ell}^{n+1} - {E_x}_{\ell}^{n}}{\Delta t^2} - \frac{1}{c^2}\frac{{E_x}_{\ell}^{n} - {E_x}_{\ell}^{n-1}}{\Delta t^2} &= -\frac{{B_y}_{\ell+1/2}^{n+1/2} - {B_y}_{\ell-1/2}^{n+1/2}}{\Delta z \Delta t} + \frac{{B_y}_{\ell+1/2}^{n-1/2} - {B_y}_{\ell-1/2}^{n-1/2}}{\Delta z \Delta t} \\
&= -\frac{{B_y}_{\ell+1/2}^{n+1/2} - {B_y}_{\ell+1/2}^{n-1/2}}{\Delta z \Delta t} + \frac{{B_y}_{\ell-1/2}^{n+1/2} - {B_y}_{\ell-1/2}^{n-1/2}}{\Delta z \Delta t} \\
&= \frac{{E_x}_{\ell+1}^{n} - {E_x}_{\ell}^{n}}{\Delta z^2} - \frac{{E_x}_{\ell}^{n} - {E_x}_{\ell-1}^{n}}{\Delta z^2}
\end{aligned}$$

1D discrete propagation equation in vacuum

$$\frac{1}{c^2}\frac{{E_x}_{\ell}^{n+1} - 2{E_x}_{\ell}^{n} + {E_x}_{\ell}^{n-1}}{\Delta t^2} = \frac{{E_x}_{\ell+1}^{n} - 2{E_x}_{\ell}^{n} + {E_x}_{\ell-1}^{n}}{\Delta z^2} \quad i.e. \frac{1}{c^2}\partial_t^2 E_x|_{\ell}^{n} = \partial_z^2 E_x|_{\ell}^{n}$$

1D dispersion relation

1D discrete propagation equation in vacuum

$$\frac{1}{c^2}\,\frac{E_{x\,\ell}^{\,n+1}-2E_{x\,\ell}^{\,n}+E_{x\,\ell}^{\,n-1}}{\Delta t^2}=\frac{E_{x\,\ell+1}^{\,n}-2E_{x\,\ell}^{\,n}+E_{x\,\ell-1}^{\,n}}{\Delta z^2}$$

→ **Von Neumann analysis:** assume the solutions of this equation are of the form $E_0 e^{ikz-i\omega t}$ (propagating wave), i.e.

$$E_{x\,\ell}^{\,n}=E_0\,e^{ik\,\ell\Delta z-i\omega n\Delta t}$$

Replacing this ansatz into the discrete progagation equation yields

$$\frac{e^{ik\ell\Delta z}}{c^2}\,\frac{e^{-i\omega(n+1)\Delta t}-2e^{-i\omega n\Delta t}+e^{-i\omega(n-1)\Delta t}}{\Delta t^2}=e^{-i\omega n\Delta t}\,\frac{e^{ik(\ell+1)\Delta z}-2e^{ik\ell\Delta z}+e^{ik(\ell-1)\Delta z}}{\Delta z^2}$$

$$\frac{e^{ik\ell\Delta z-i\omega n\Delta t}}{c^2}\,\frac{e^{-i\omega\Delta t}-2+e^{i\omega\Delta t}}{\Delta t^2}=e^{ik\ell\Delta z-i\omega n\Delta t}\,\frac{e^{ik\Delta z}-2+e^{-ik\Delta z}}{\Delta z^2}$$

$$\frac{1}{c^2}\,\frac{(e^{-i\omega\Delta t/2}-e^{i\omega\Delta t/2})^2}{\Delta t^2}=\frac{(e^{ik\Delta z/2}-e^{-ik\Delta z/2})^2}{\Delta z^2}$$

1D dispersion relation

$$\frac{1}{c^2\Delta t^2}\sin^2\left(\frac{\omega\Delta t}{2}\right)=\frac{1}{\Delta z^2}\sin^2\left(\frac{k\Delta z}{2}\right)\qquad(\text{instead of }\omega^2=c^2k^2)$$

$c\Delta t \leq \Delta z$ $\rightarrow$ Numerical dispersion

For $c\Delta t \leq \Delta z$, the discrete dispersion relation

$$\frac{1}{c^2 \Delta t^2} \sin^2\left(\frac{\omega \Delta t}{2}\right) = \frac{1}{\Delta z^2} \sin^2\left(\frac{k \Delta z}{2}\right)$$

has real solutions ω, for any k:

$$\omega = \pm \frac{2}{\Delta t} \arcsin\left(\frac{c\Delta t}{\Delta z} \sin\left(\frac{k\Delta z}{2}\right)\right)$$

Thus, the phase velocity $v_\phi = \omega / k$ is:

$$v_\phi = \pm \frac{2}{k\Delta t} \arcsin\left(\frac{c\Delta t}{\Delta z} \sin\left(\frac{k\Delta z}{2}\right)\right)$$

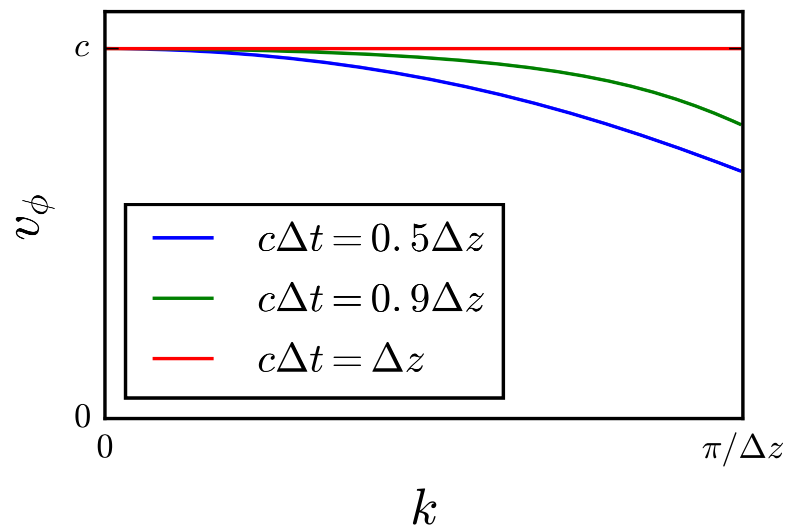
Numerical dispersion

In a PIC code, the **electromagnetic waves** propagate (in vacuum) at a **velocity which depends on** k (and on Δt, Δz),

instead of propagating at the speed of light: $v_\phi = \pm c$

$c\Delta t \leq \Delta z \rightarrow$ Numerical dispersion

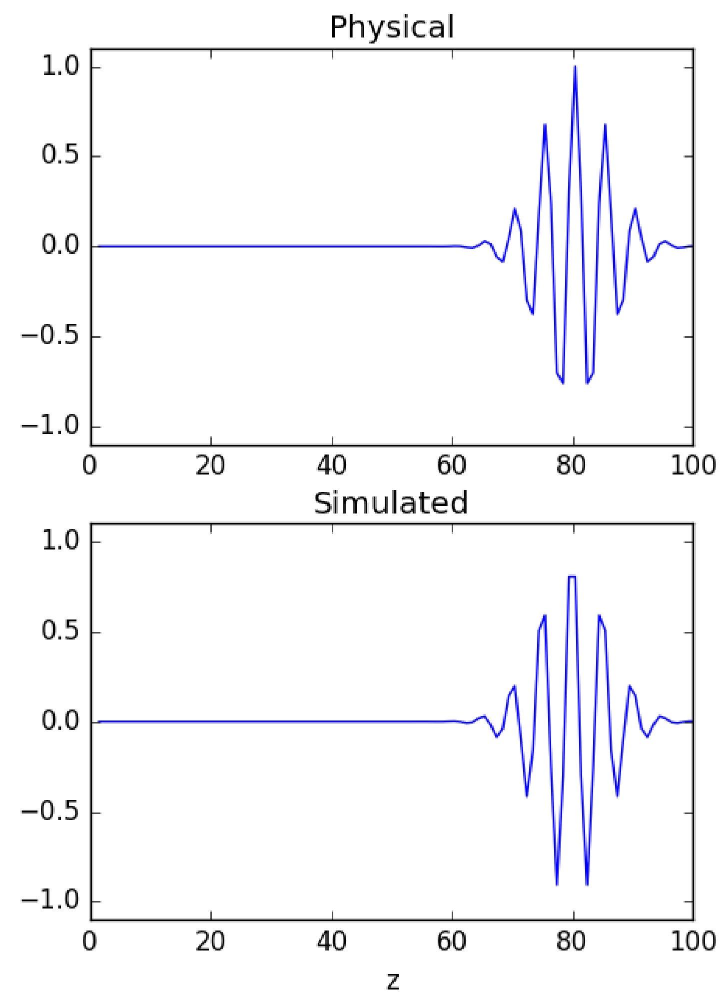
$$v_\phi = \frac{2}{k\Delta t} \arcsin\left(\frac{c\Delta t}{\Delta z} \sin\left(\frac{k\Delta z}{2}\right)\right)$$



NB: $k = \pi/\Delta z$, $\lambda = 2\Delta z$: shortest wavelength supported by the grid.

The shorter the wavelength, the slower the propagation.

Animation: $c\Delta t = 0.5\Delta z$



$c\Delta t > \Delta z \rightarrow$ Courant limit

For $c\Delta t > \Delta z$, the discrete dispersion relation

$$\frac{1}{c^2\Delta t^2}\sin^2\left(\frac{\omega\Delta t}{2}\right) = \frac{1}{\Delta z^2}\sin^2\left(\frac{k\Delta z}{2}\right)$$

has **no real solutions** ω, for k close to $\pi/\Delta z$. The solution ω is **imaginary** and the corresponding mode is **unstable**.

Courant limit (a.k.a. CFL limit)

Standard EM-PIC codes are **unstable** for $c\Delta t > \Delta z$ (in 1D).

- Thus, practical use of **electromagnetic PIC** codes is restricted to $\Delta t \leq \Delta z/c$.
- For a given spatial resolution Δz, this limits **how fast** a simulation can advance in time.
- **Electrostatic PIC codes** do not have this limitation
  $\rightarrow$ Can be much faster than EM-PIC codes to simulate a system over a given period of time, by taking **large timesteps** Δt.

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Dispersion and Courant limit in 3D

Derivation of dispersion relation

Combine discrete Maxwell equation → Discrete propagation equation → Von Neumann analysis → Numerical dispersion relation

Same process in 3D. The Von Neumann analysis assumes:

$$E = E_0 e^{ik_x x + ik_y y + ik_z z - i\omega t}$$

3D Numerical dispersion relation

$$\frac{\sin^2\left(\frac{\omega \Delta t}{2}\right)}{c^2 \Delta t^2} = \frac{\sin^2\left(\frac{k_x \Delta x}{2}\right)}{\Delta x^2} + \frac{\sin^2\left(\frac{k_y \Delta y}{2}\right)}{\Delta y^2} + \frac{\sin^2\left(\frac{k_z \Delta z}{2}\right)}{\Delta z^2}$$

instead of the physical dispersion $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2)$

Courant limit (a.k.a CFL limit) in 3D

$$c\Delta t \leq \frac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}$$

Numerical dispersion in 3D

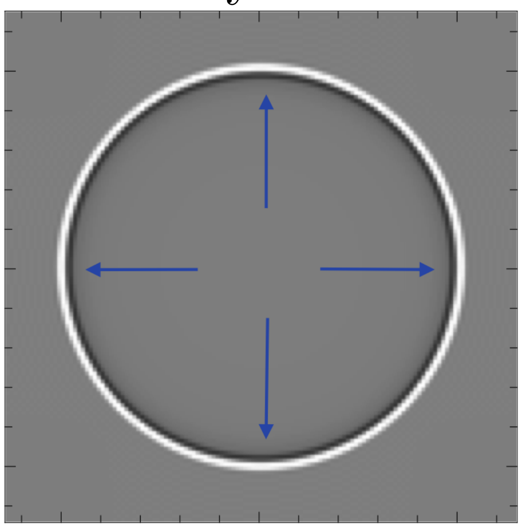
3D Discrete dispersion relation

$$\frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{c^2\Delta t^2} = \frac{\sin^2\left(\frac{k_x\Delta x}{2}\right)}{\Delta x^2} + \frac{\sin^2\left(\frac{k_y\Delta y}{2}\right)}{\Delta y^2} + \frac{\sin^2\left(\frac{k_z\Delta z}{2}\right)}{\Delta z^2}$$

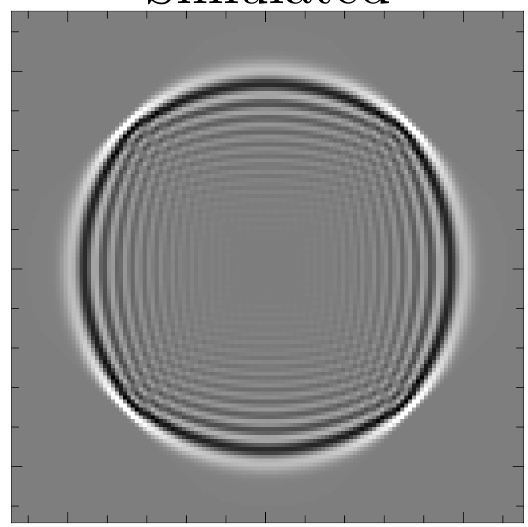
Velocity depends on the **wavelength and propagation direction**.

Example: expanding electromagnetic wave

Physical



Simulated

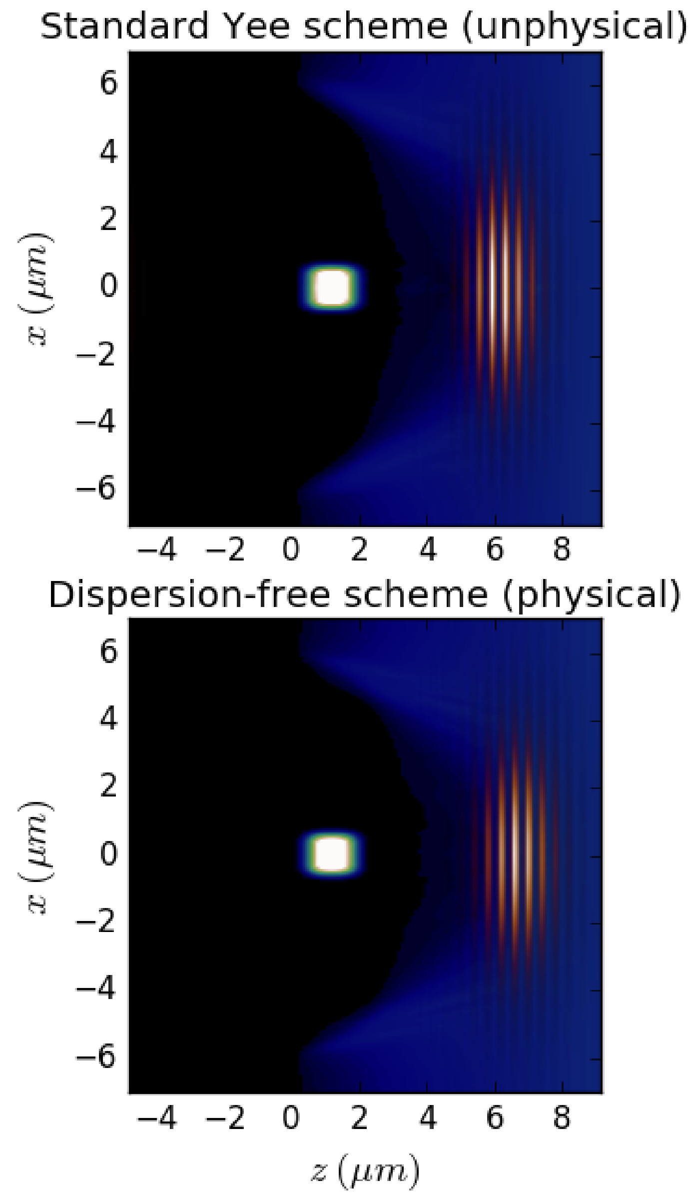


Even for $\Delta t = \Delta t_{CFL}$: waves are **slower than** c along the main axes.

Impact of numerical dispersion

Animation: laser-wakefield acceleration

- A short and intense **laser pulse**, followed by a relativistic **electron bunch**, enters a **plasma** (generated from a gas jet).
- The laser pulse generates a **wake** in the plasma, with **electric fields** that can **accelerate** the electron bunch.
- Simulation with the Yee scheme (and low resolution):
  - The laser is **artificially slow** (numerical dispersion)
  - Thus the electron bunch **catches up** with the laser very soon!



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Yee scheme

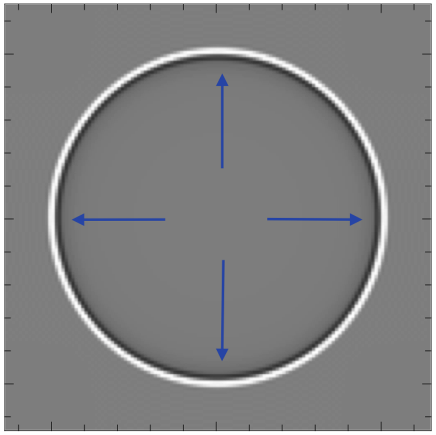
Finite-difference in space and time

e.g. continuous equation : $$\frac{\partial B_z}{\partial t} = -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

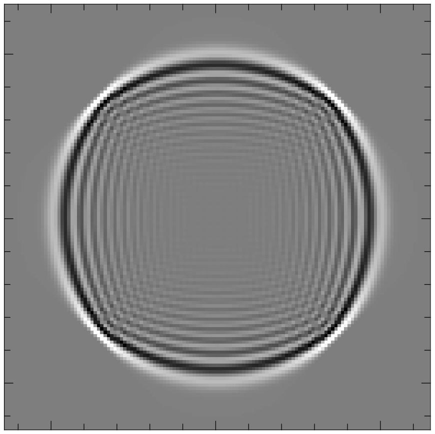
$\rightarrow$ discrete equation : $$B_z^{n+1/2} = B_z{}^{n-1/2} - \Delta t(\hat{\partial}_x E_y|^n - \hat{\partial}_y E_x|^n)$$

with $\hat{\partial}_x F|^n_{i,j,\ell} = \frac{F^n_{i+\frac{1}{2},j,\ell} - F^n_{i-\frac{1}{2},j,\ell}}{\Delta x}$

Physical



Simulated



- Anisotropic
- Waves propagate **slower** than c.

Pseudo-spectral solver

Fourier transform in space, finite-difference in time

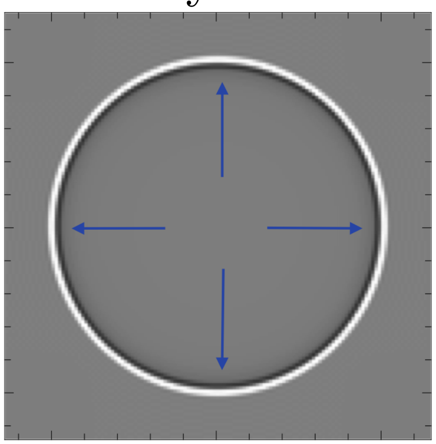
e.g. continuous equation : $$\frac{\partial B_z}{\partial t} = -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

$\rightarrow$ Fourier space : $$\frac{\partial \hat{\mathcal{B}}_z}{\partial t} = -\left(ik_x \hat{\mathcal{E}}_y - ik_y \hat{\mathcal{E}}_x \right)$$

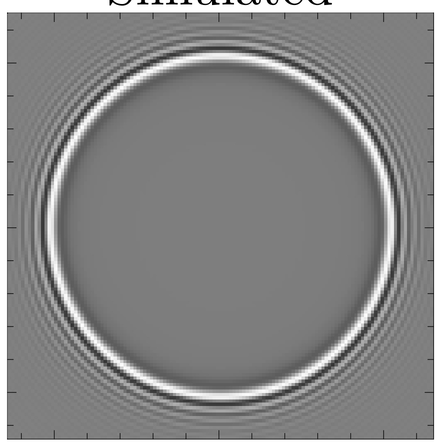
$\rightarrow$ Finite difference in time : $$\hat{\mathcal{B}}_z^{n+1/2} = \hat{\mathcal{B}}_z^{n-1/2} - \Delta t \left(ik_x \hat{\mathcal{E}}_y^n - ik_y \hat{\mathcal{E}}_x^n \right)$$

$\rightarrow$ Use backwards FFT to obtain $B_z^{n+1/2}$ from $\hat{\mathcal{B}}_z^{n+1/2}$

Physical



Simulated



- Isotropic
- Waves propagate **faster** than c.

Analytical pseudo-spectral solver (Haber et al., 1973)

Fourier transform in space, finite-difference in time

e.g. continuous equation : $$\frac{\partial B_z}{\partial t} = -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

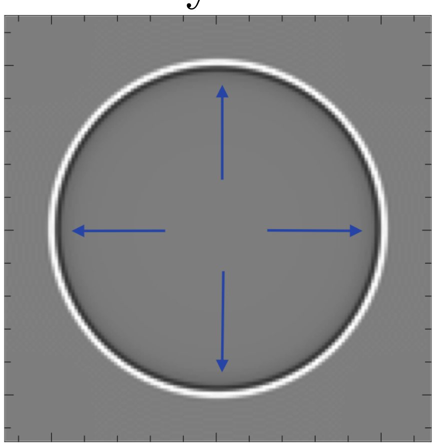
$\rightarrow$ Fourier space : $$\frac{\partial \hat{\mathcal{B}}_z}{\partial t} = -\left(ik_x \hat{\mathcal{E}}_y - ik_y \hat{\mathcal{E}}_x \right)$$

$\rightarrow$ Analytical integration of the coupled Maxwell equations in time:

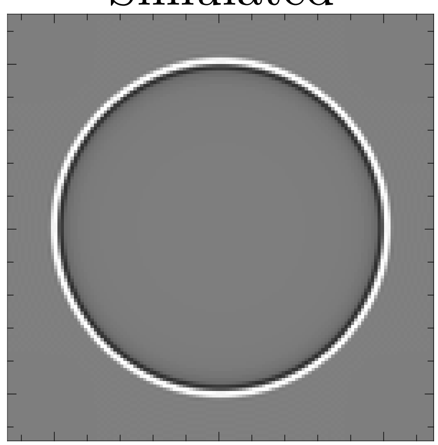
$$\hat{\mathcal{B}}_z^{n+1} = \cos(kc\Delta t)\hat{\mathcal{B}}_z^n - \frac{\sin(kc\Delta t)}{kc}\left(ik_x \hat{\mathcal{E}}_y^n - ik_y \hat{\mathcal{E}}_x^n \right) \qquad k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$\rightarrow$ Use backwards FFT to obtain B_z^{n+1} from $\hat{\mathcal{B}}_z^{n+1}$

Physical



Simulated



- Isotropic
- Waves propagate **exactly** at c.

Dispersion and Courant limit: conclusions

- Electromagnetic solvers have a **maximum value** for the timestep Δt (Courant limit), which depends on the dimension (and the method of discretization)
- Below the Courant limit, waves may propagate at speeds that **artificially differ** from c (numerical dipersion).
  This can have a strong impact in some physical situations.
- Spectral solvers can mitigate (or even eliminate) numerical dispersion.

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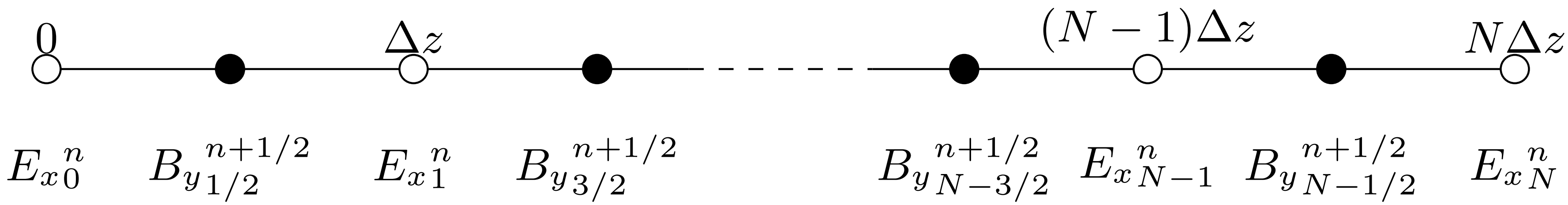
Boundary conditions and EM-PIC

Reminder: 1D discrete Maxwell equations in vacuum

$$\frac{{B_y}_{\ell+1/2}^{n+1/2} - {B_y}_{\ell+1/2}^{n-1/2}}{\Delta t} = -\frac{{E_x}_{\ell+1}^{n} - {E_x}_{\ell}^{n}}{\Delta z}$$

$$\frac{{E_x}_{\ell}^{n+1} - {E_x}_{\ell}^{n}}{\Delta t} = -c^2\frac{{B_y}_{\ell+1/2}^{n+1/2} - {B_y}_{\ell-1/2}^{n+1/2}}{\Delta z}$$

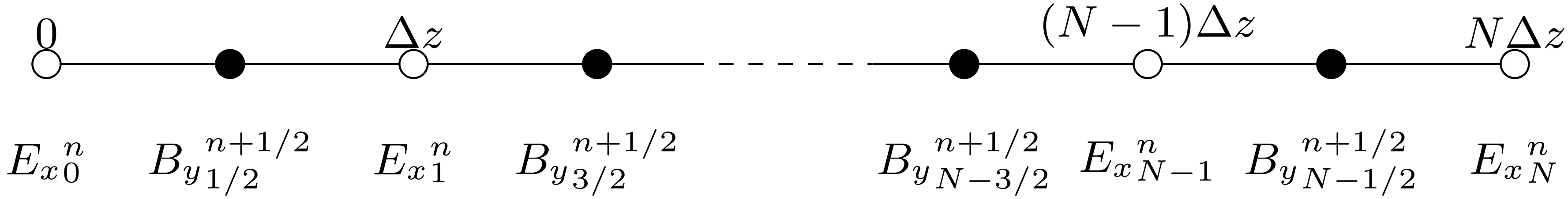


The grid is finite:

- For $\ell = 0$: ${B_y}_{\ell-1/2}^{n+1/2}$ is undefined.
- For $\ell = N$: ${B_y}_{\ell+1/2}^{n+1/2}$ is undefined.

→ **Assumptions** are needed, for the value of ${B_y}_{-1/2}^{n+1/2}$ and ${B_y}_{N+1/2}^{n+1/2}$.

Boundary conditions and EM-PIC



Typical assumptions

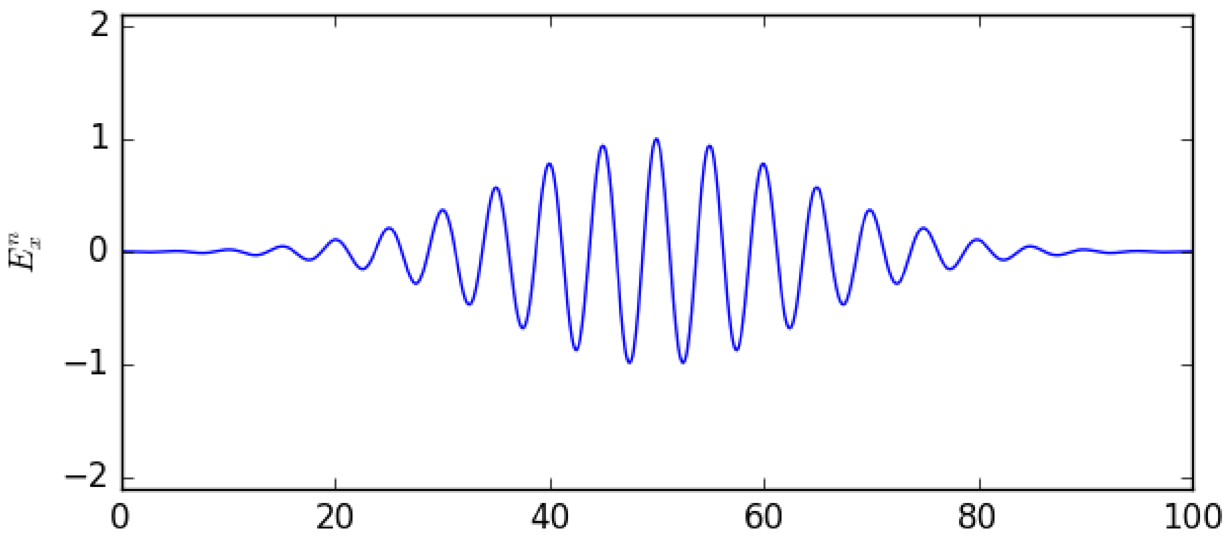
- Periodic: ${B_y}_{-1/2}^{n+1/2} = {B_y}_{N-1/2}^{n+1/2}$ and ${B_y}_{N+1/2}^{n+1/2} = {B_y}_{1/2}^{n+1/2}$
- Dirichlet: ${B_y}_{-1/2}^{n+1/2} = 0$ and ${B_y}_{N+1/2}^{n+1/2} = 0$
- Neumann: ${B_y}_{-1/2}^{n+1/2} = {B_y}_{1/2}^{n+1/2}$ and ${B_y}_{N+1/2}^{n+1/2} = {B_y}_{N-1/2}^{n+1/2}$
  (i.e. $\partial_z B_y|_0^{n+1/2} = 0$ and $\partial_z B_y|_N^{n+1/2} = 0$)

Boundary conditions and EM-PIC

Problem:

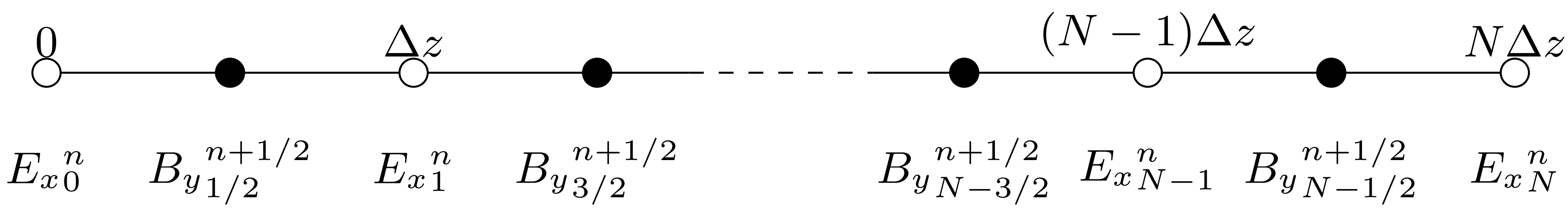
Dirichlet and Neumann boundary conditions **reflect** the EM waves.
For many physical problems, we need the boundaries to **absorb** the waves.

Animation: Neumann boundary conditions



This is because, physically, an **outgoing wave** does not satisfy $B_y(n\Delta z) = 0$ (Dirichlet) or $\partial_z B_y(n\Delta z) = 0$ (Neumann)

Silver-Müller absorbing boundary (right-hand side)



The value of ${B_y}_{N+1/2}^{n+1/2}$ should be chosen so as to be **consistent with an outgoing wave.**

Physically, for an outgoing wave propagating to the right (from Maxwell's equation):

$$B_y(z,t) = \frac{1}{c} E_x(z,t)$$

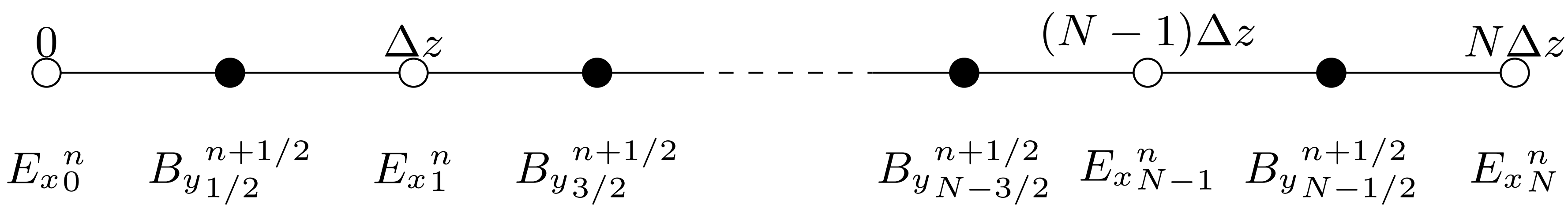
Numerically, we can express it as:

$$B_y|_N^{n+1/2} = \frac{1}{c} E_x|_N^{n+1/2}$$

Because of **staggering**:

$$\frac{{B_y}_{N+1/2}^{n+1/2} + {B_y}_{N-1/2}^{n+1/2}}{2} = \frac{1}{c}\frac{{E_x}_N^{n+1} + {E_x}_N^n}{2}$$

Silver-Müller absorbing boundary (right-hand side)



By combining the equations:

$$\frac{{B_y}_{N+1/2}^{n+1/2} + {B_y}_{N-1/2}^{n+1/2}}{2} = \frac{1}{c}\frac{{E_x}_N^{n+1} + {E_x}_N^n}{2} \quad \text{(right-propagating wave)}$$

$$\frac{{E_x}_N^{n+1} - {E_x}_N^n}{\Delta t} = -c^2 \frac{{B_y}_{N+1/2}^{n+1/2} - {B_y}_{N-1/2}^{n+1/2}}{\Delta z} \quad \text{(Maxwell equation)}$$

we obtain

Silver-Müller boundary condition (right-hand side)

$$ {E_x}_N^{n+1} = \left(1 - \frac{2c\Delta t}{c\Delta t + \Delta z}\right){E_x}_N^n + \frac{2c^2\Delta t}{c\Delta t + \Delta z}{B_y}_{N-1/2}^{n+1/2}$$

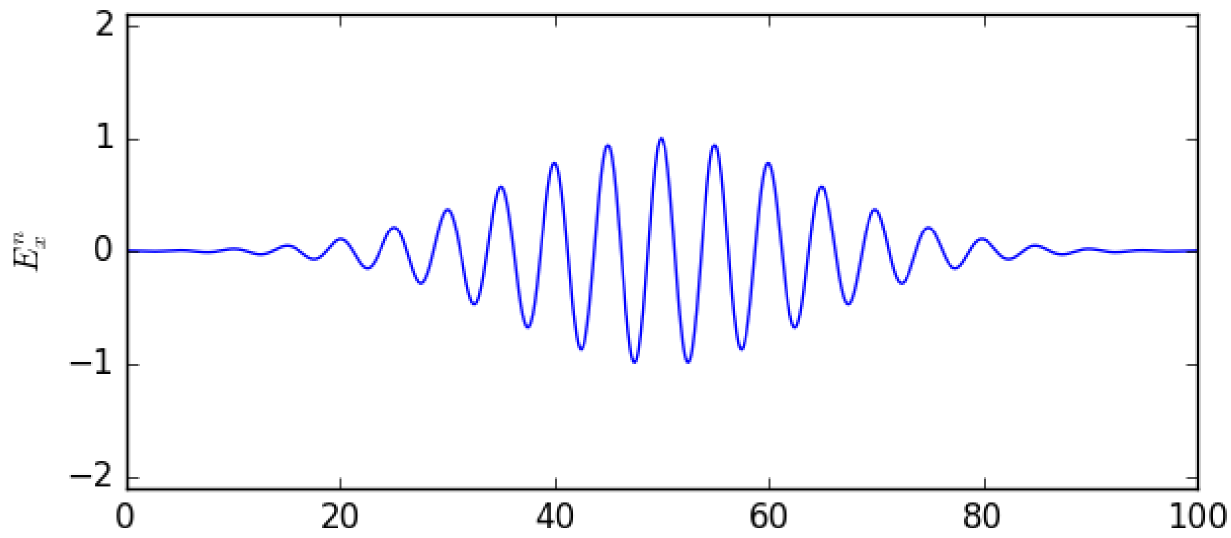
See e.g. Bjorn Engquist (1977)

Silver-Müller absorbing boundary (right-hand side)

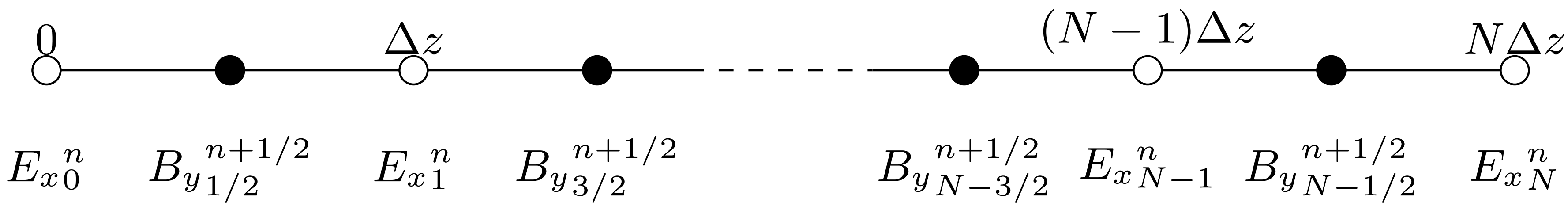
Silver-Müller boundary condition (right-hand side)

$$E_x{}_N^{n+1} = \left(1 - \frac{2c\Delta t}{c\Delta t + \Delta z}\right) E_x{}_N^{n} + \frac{2c^2\Delta t}{c\Delta t + \Delta z} B_y{}_{N-1/2}^{n+1/2}$$

Animation: Silver-Müller boundary conditions



Silver-Müller absorbing boundary (left-hand side)



By combining the equations:

$$\frac{{B_y}_{1/2}^{n+1/2} + {B_y}_{-1/2}^{n+1/2}}{2} = -\frac{1}{c}\,\frac{{E_x}_0^{n+1} + {E_x}_0^n}{2} \quad \text{(left-propagating wave)}$$

$$\frac{{E_x}_0^{n+1} - {E_x}_0^n}{\Delta t} = -c^2\,\frac{{B_y}_{1/2}^{n+1/2} - {B_y}_{-1/2}^{n+1/2}}{\Delta z} \quad \text{(Maxwell equation)}$$

we obtain

Silver-Müller boundary condition (left-hand side)

$${E_x}_0^{n+1} = \left(1 - \frac{2c\Delta t}{c\Delta t + \Delta z}\right){E_x}_0^n - \frac{2c^2\Delta t}{c\Delta t + \Delta z}{B_y}_{1/2}^{n+1/2}$$

Silver-Müller absorbing boundary in 3D

Maxwell equation:

$$\frac{{E_x}^{n+1}_{i+\frac{1}{2},j,\ell} - {E_x}^{n}_{i+\frac{1}{2},j,\ell}}{c^2\Delta t} = \frac{{B_z}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},0} - {B_z}^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2},0}}{\Delta y} - \frac{{B_y}^{n+\frac{1}{2}}_{i+\frac{1}{2},j,\ell+\frac{1}{2}} - {B_y}^{n+\frac{1}{2}}_{i+\frac{1}{2},j,\ell-\frac{1}{2}}}{\Delta z}$$

Silver-Müller boundary condition (left-hand side)

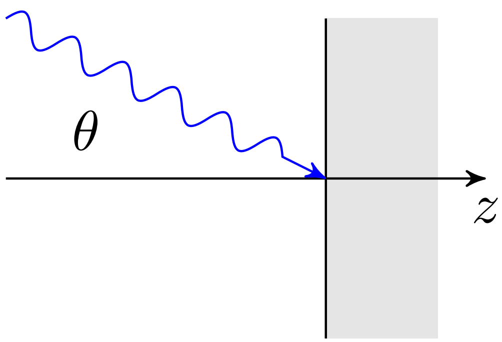
$$\begin{aligned}{E_x}^{n+1}_{i+\frac{1}{2},j,0} = \left(1 - \frac{2c\Delta t}{c\Delta t + \Delta z}\right) {E_x}^{n}_{i+\frac{1}{2},j,0} - \frac{2c^2\Delta t}{c\Delta t + \Delta z} {B_y}^{n+\frac{1}{2}}_{i+\frac{1}{2},j,\frac{1}{2}} \\ + c^2\Delta t \frac{{B_z}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},0} - {B_z}^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2},0}}{\Delta y}\end{aligned}$$

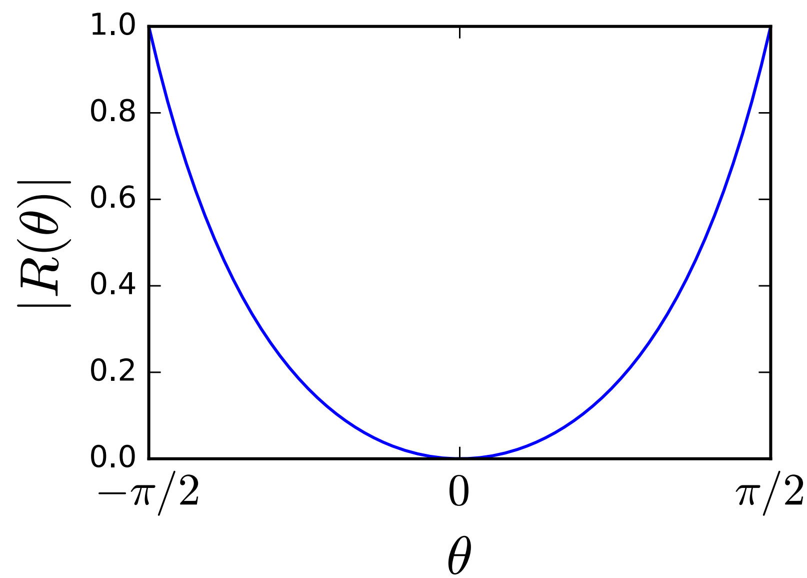
\+ Similar equations for the right-hand side
\+ Similar equations for B_x and E_y

Silver-Müller absorbing boundary in 3D

Limitation

In 3D, the Silver-Müller boundary conditions are only well-adapted for waves in **normal incidence**.
The reflection coefficient $R(\theta)$ quickly increases with the angle of incidence θ.





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2. Open boundaries conditions
   - Silver-Müller boundary conditions
   - Perfectly Matched Layers

Perfectly Matched Layers (in 2D)

Perfectly Matched Layers (Berenger, 1994)

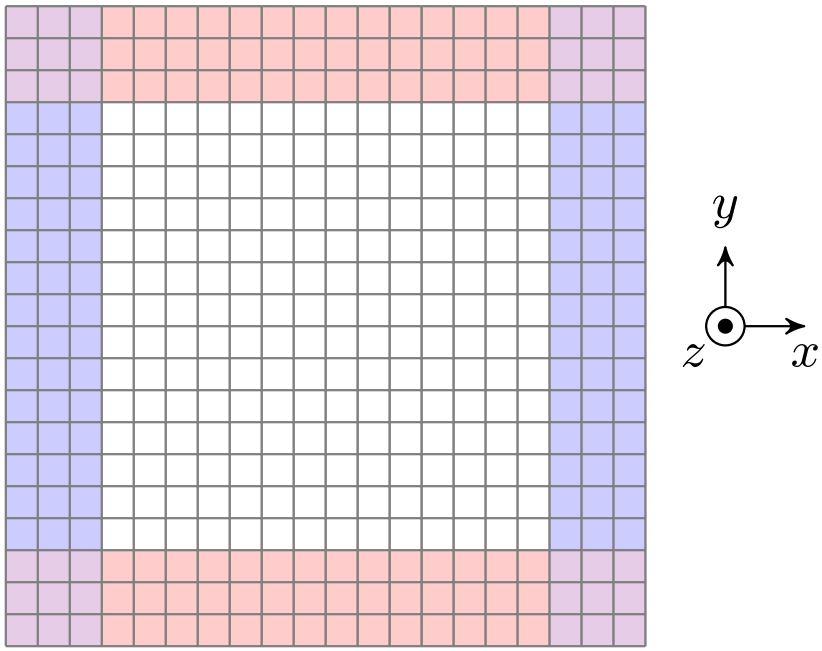
Surround the simulation box by **additional layers of cells**, where the Maxwell equations are **modified** so as to **progressively damp** the waves.

In the bulk:

$$\partial_t E_x = c^2 \partial_y B_z$$
$$\partial_t E_y = -c^2 \partial_x B_z$$
$$\partial_t B_z = -\partial_x E_y + \partial_y E_x$$

In e.g. the right-hand layer:

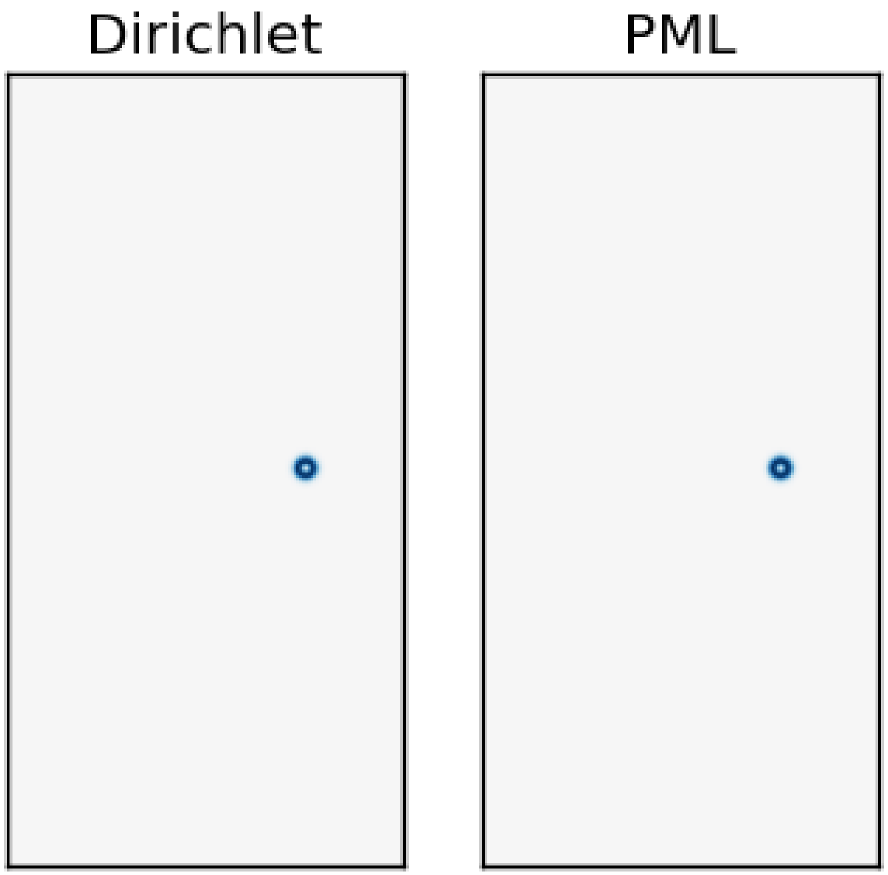
$$\partial_t E_x = c^2 \partial_y B_z$$
$$\partial_t E_y = -c^2 \partial_x B_z - \boldsymbol{\frac{\sigma}{\epsilon_0} E_y}$$
$$B_z = B_{zx} + B_{zy}$$
$$\partial_t B_{zx} = -\partial_x E_y - \boldsymbol{\frac{\sigma}{\epsilon_0} B_{zx}}$$
$$\partial_t B_{zy} = \partial_y E_x$$



Modified Maxwell equations:

- Artificial (unphysical) conductivity σ
- The B_z field is (artificially) split in two

Perfectly Matched Layers (in 2D)

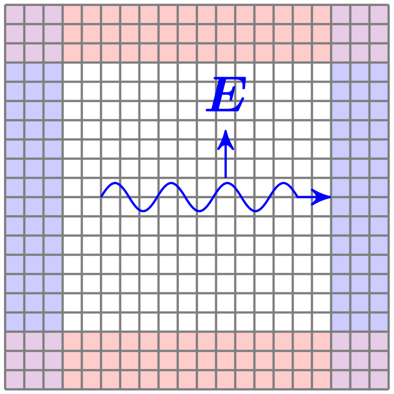


Animation with propagating waves:

- Waves in normal incidence are **absorbed**.
- Waves in grazing incidence **propagate** as if they did not “feel” the boundary.

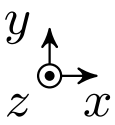
Perfectly Matched Layers: normal incidence

Explanation based on **continuous equations**



Transverse EM wave propagating along x

$$E_x = 0 \quad E_y \neq 0 \quad \rightarrow B_{zy} = 0 \quad B_z = B_{zx}$$



In the bulk:

$$\begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t E_y &= -c^2 \partial_x B_z \\ \partial_t B_z &= -\partial_x E_y + \partial_y E_x \end{aligned} \quad \rightarrow \quad \begin{aligned} \partial_t E_y &= -c^2 \partial_x B_z \\ \partial_t B_z &= -\partial_x E_y \end{aligned}$$

In the right-hand layer:

$$\begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t E_y &= -c^2 \partial_x B_z - \frac{\sigma}{\epsilon_0} E_y \\ B_z &= B_{zx} + B_{zy} \\ \partial_t B_{zx} &= -\partial_x E_y - \frac{\sigma}{\epsilon_0} B_{zx} \\ \partial_t B_{zy} &= \partial_y E_x \end{aligned} \quad \rightarrow \quad \begin{aligned} \partial_t E_y &= -c^2 \partial_x B_z - \frac{\sigma}{\epsilon_0} E_y \\ \partial_t B_z &= -\partial_x E_y - \frac{\sigma}{\epsilon_0} B_z \end{aligned}$$

Perfectly Matched Layers: normal incidence

There is a solution (continuous in E_y and B_z) with **no reflected wave**.

In the bulk ($x < 0$):

$$\begin{aligned}\partial_t E_y &= -c^2 \partial_x B_z \\ \partial_t B_z &= -\partial_x E_y\end{aligned}$$

Solution:

$$E_y = E_0 \cos(k(x - ct))$$

$$B_z = \frac{E_0}{c} \cos(k(x - ct))$$

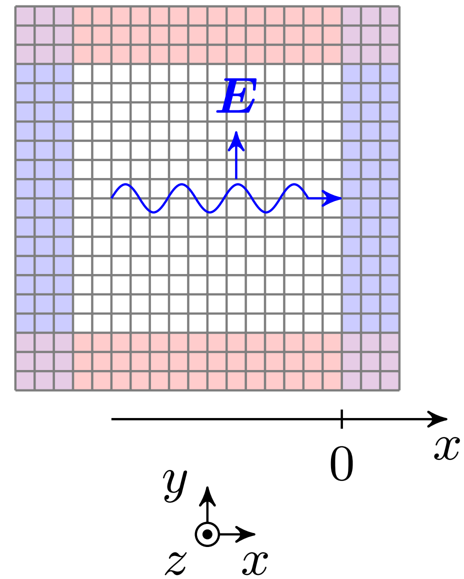
In the right-hand layer ($x > 0$):

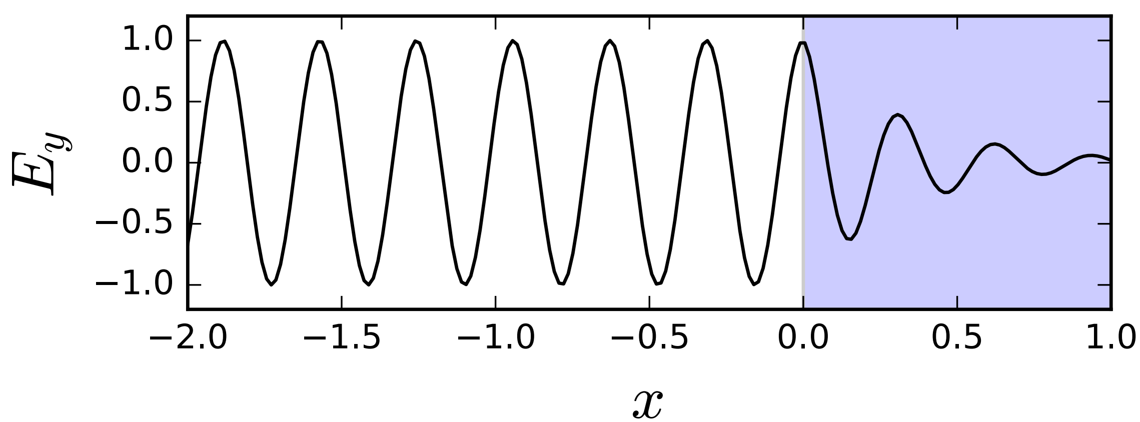
$$\begin{aligned}\partial_t E_y &= -c^2 \partial_x B_z - \frac{\sigma}{\epsilon_0} E_y \\ \partial_t B_z &= -\partial_x E_y - \frac{\sigma}{\epsilon_0} B_z\end{aligned}$$

Solution:

$$E_y = E_0 \cos(k(x - ct)) e^{-\frac{\sigma}{\epsilon_0 c} x}$$

$$B_z = \frac{E_0}{c} \cos(k(x - ct)) e^{-\frac{\sigma}{\epsilon_0 c} x}$$





The wave is damped before reaching the end of the outer layer.

Perfectly Matched Layers: grazing incidence

Transverse EM wave propagating along y

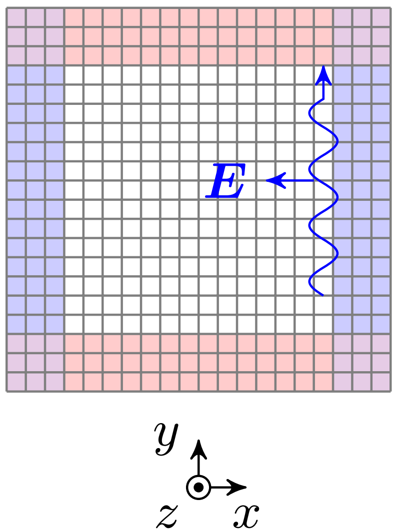
$$E_x \neq 0 \quad E_y = 0 \quad \rightarrow B_{zx} = 0 \quad B_z = B_{zy}$$

In the bulk:

$$\begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t E_y &= -c^2 \partial_x B_z \\ \partial_t B_z &= -\partial_x E_y + \partial_y E_x \end{aligned} \quad \rightarrow \quad \begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t B_z &= \partial_y E_x \end{aligned}$$

In the right-hand layer:

$$\begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t E_y &= -c^2 \partial_x B_z - \frac{\sigma}{\epsilon_0} E_y \\ B_z &= B_{zx} + B_{zy} \\ \partial_t B_{zx} &= -\partial_x E_y - \frac{\sigma}{\epsilon_0} B_{zx} \\ \partial_t B_{zy} &= \partial_y E_x \end{aligned} \quad \rightarrow \quad \begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t B_z &= \partial_y E_x \end{aligned}$$



The propagation equations are **identical** in the bulk and the outer layer.
A wave in **grazing indidence** does not "feel" the boundary.

Open boundary conditions: conclusion

- If no **special care** is taken at the boundary, it will **by default** produce a reflected wave.
- **Silver-Müller boundary conditions:**
  - Easy to implement
  - But only cancels reflection for waves at normal incidence
- **Perfectly Matched Layers**
  - Need extra layers of cells, where the Maxwell equations are artificially modified.
  - The anisotropic Maxwell equations lead to proper behavior for waves with any incidence angle.

References

Berenger, J.-P. (1994). A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*, 114(2):185–200.

Bjorn Engquist, A. M. (1977). Absorbing boundary conditions for the numerical simulation of waves. *Mathematics of Computation*, 31(139):629–651.

Haber, I., Lee, R., Klein, H., and Boris, J. (1973).