PLEASE WAIT UNTIL YOU ARE TOLD TO BEGIN THE EXAM.

While waiting, carefully fill in the information requested below

Your Name: ___________ 
Your Student Number: ___________

POSSIBLY USEFUL CONSTANTS AND EQUATIONS

\[ \frac{\hbar c}{\gamma} = 197 \text{MeV} \cdot \text{fm} \]
\[ \frac{\hbar c}{\gamma} = 1240 \text{MeV} \cdot \text{fm} \]
\[ c = 3 \times 10^{22} \text{fm} / \text{c} \]
\[ 2R = \frac{1.22\lambda}{\sin \theta} \text{ for first minimum} \]
\[ \text{at } \theta \text{ for diffraction from an absorptive (black) sphere of radius } R. \]
\[ r_p = \frac{\kappa}{2T_{cm}} \left( 1 + \csc(\theta_{cm} / 2) \right) \]
\[ \kappa = \frac{Z_p Z_T e^2}{4\pi e_0} \]
\[ \frac{d\sigma}{d\Omega_{Ruth}} = \frac{\kappa}{4T_{cm}} \frac{1}{\sin^2 \left( \frac{\theta_{cm}}{2} \right)} \]
\[ \frac{d\sigma}{d\Omega_{Mott}} = \frac{d\sigma}{d\Omega_{Ruth}} \left( 1 - \frac{v^2}{c^2} \sin^2 \left( \frac{\theta_{cm}}{2} \right) \right) \]

\[ 1u = 1.66 \times 10^{-27} \text{kg} \]
\[ \mu c^2 = 931.5 \text{MeV} \]
\[ M_p c^2 = 938.3 \text{MeV} \]
\[ M_n c^2 = 939.6 \text{MeV} \]
\[ M_e c^2 = 0.511 \text{MeV} \]
\[ \frac{e^2}{4\pi e_0} = 1.44 \text{MeV} \]
\[ r = \frac{4\pi e_0 \hbar^2 n^2}{Z m e^2} \]
\[ r_B = 5.29 \times 10^4 \text{fm} \]
\[ \omega = \frac{V}{2R} \exp \left[ -G(Q) \right] \]
\[ G(Q)_{t=0} = \frac{4\pi Z\alpha}{\sqrt{2Q/\mu c^2}} - \frac{8}{\hbar c} \sqrt{\frac{\mu c^2 RZ\alpha}{\mu c^2 RZ\alpha}} \]
\[ G(Q)_{t=0} = G(Q)_{t=0} \left( 1 + \frac{V_{cen}}{2V_{coll}} \right) \]

DO NOT TURN THIS PAGE UNTIL THE EXAM STARTS
This exam is worth 75 points. Each problem on the exam has its approximate point value listed. Take a moment and look over the exam before you begin. To receive full credit for each answer, you must work neatly, show your work and simplify your answer to the extent possible.

1. The semi-empirical liquid-drop formula for the total binding energy of a nucleus can be written in the form:

\[ B(Z, A) = b_1 A + b_2 A^{2/3} + b_3 Z^2 / A^{1/3} + b_4 (N - Z)^2 / A + \delta(Z, A) \]

(Here, I have replaced the coefficients \(a_n\), \(a_c\), etc. with coefficients \(b_n\), in order that the meaning and sign of each term is not trivially obvious from the way the equation is written down.) The term proportional to \(b_3\), for example, represents the contribution to the binding energy coming from the Coulomb potential energy of the protons in the nucleus. The sign of \(b_3\) is negative, because the Coulomb potential energy is positive and therefore makes the nucleus less well bound.

a) (6 pts) What is the origin of the term proportional to \(b_2\)? What is the sign of \(b_2\) (positive or negative)? What is the reason for that sign?

\(b_2\) multiplies the surface energy. Nucleons in the surface are less bound because they interact with fewer neighboring nucleons via the attractive strong force. This reduces the binding energy in proportion to the surface area, hence \(b_2\) is negative.

b) (7 pts) What is the corresponding origin of the term proportional to \(b_4\)? What is the sign of \(b_4\) (positive or negative)? What is the reason for that sign?

\(b_4\) multiplies the symmetry term. Nuclei are better bound if there are equal numbers of neutrons and protons than if there are large excesses of either. This reflects an increase in \(b_4\) due to the Pauli principle and an increase in \(b_4\) due to the potential energy part of the symmetry energy. \(b_4\) is negative reflecting these considerations.
2. The actinide nucleus $^{223}$Ra can decay alternatively by alpha decay with a branching ratio of about 100% and by $^{14}$C decay with a branching ratio of $9 \times 10^{-3}$%.

a) (4 pts) Like many heavy nuclei, $^{223}$Ra is deformed. If the deformation were large as suggested in the drawing of the nuclear shape depicted below, would you expect these particles to be emitted from the ends (points A and B) or from the sides (points C and D)? Explain your reasoning.

Most particles emitted from A or B because the barrier is narrower there.

b) (8 pts.) Like alpha particles, $^{14}$C nuclei tunnel through the Coulomb barrier. The emission of $^{14}$C can be calculated using the WKB integral, just as we calculated alpha particle emission. Based on your understanding of alpha decay, give all of the reasons you can think why the $^{14}$C decay rate is much smaller than the alpha decay rate. (Please note that the Q-value ($Q_{14} = 37.5$ MeV) for $^{14}$C decay is much larger than the Q-value for alpha decay ($Q_{\alpha} = 5.98$ MeV).)

$$W \propto \exp \left( -2 \sqrt{\frac{2m}{\hbar^2}} \sqrt{V - Q} \right)$$

Reasons for reduced $^{14}$C B.R.

1. $V = \frac{Z_C Z_{\alpha} e^2}{4 \pi \epsilon_0 R}$ is roughly 3 x larger for $^{14}$C

2. $V$ is roughly 3 x larger for $^{14}$C

The barrier width = $V_{\text{outer}} - R$

$$\frac{V_{\text{outer}, C}}{V_{\text{outer}, \alpha}} = \frac{\frac{Z_C Z_{\alpha} e^2}{4 \pi \epsilon_0 Q_C}}{\frac{Z_{\alpha} Z_{\alpha} e^2}{4 \pi \epsilon_0 Q_{\alpha}}} = \frac{Z_C}{Z_{\alpha}} \frac{Q_{\alpha}}{Q_C} = 3 \times \frac{6}{38} < 1$$

so the barrier width is actually less for $^{14}$C.
3. The data below depicts the differential cross section for 200 MeV protons scattered elastically by $^{40}$Ca. At this energy, the Coulomb interaction has a negligible effect. Protons follow nearly straight line paths.

a) (7 pts.) Assume that all the protons that reach the half density radius $^{40}$Ca are absorbed and all others are not. Use the fact that the protons are diffracted from the $^{40}$Ca nucleus and the diffraction pattern in this figure to estimate the half density radius $R_{1/2}$ within which protons are essentially completely absorbed.

b) (5 pts.) Write down an expression for the absorption cross section in terms of $R_{1/2}$ and compute its value.

\[ R_{1/2} = \frac{0.61}{\sin \theta} \]

\[ \sin \theta = \sin 17^\circ = 0.292 \]

\[ \lambda = \sqrt{\frac{h c}{2 \cdot m c^2 \cdot 200}} = \frac{120 \text{ MeV} \cdot \text{fm}}{1.934 \cdot 700 \text{ MeV} \cdot \text{fm}} \approx 2.02 \]

\[ \Rightarrow R_{1/2} = \frac{0.61 (2.02 \text{ fm})}{0.292} = 4.29 \text{ fm} \]

\[ \sigma_{\text{abs}} = \pi R_{1/2}^2 = 58 \text{ fm}^2 \]
4. Particles scatter off a potential $U(r)$. Suppose the impact parameter $b$ for a particle is related to the center of mass scattering angle $\theta_{cm}$ of the particle by

$$b = R \cos (\theta_{cm}/2)$$

a) (8 pts) Derive an expression for the differential cross section $\frac{d\sigma}{d\Omega}$.

\[
\frac{d\sigma}{d\Omega} = 2\pi b \left| \frac{db}{d\theta} \right|
\]

\[
dR = 2\pi \sin \theta d\theta.
\]

\[
= \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| = \frac{R \cos \left( \theta_{cm}/2 \right)}{\sin \theta} \frac{R \sin \left( \theta_{cm}/2 \right)}{2}
\]

using $\sin \theta = 2 \sin \theta_1 \cos \theta_2$

\[
\frac{d\sigma}{d\Omega} = \frac{R^2}{4}
\]

b) (4 pt) Using your expression in part a), calculate the total cross section.

\[
\sigma = \int d\Omega \frac{R^2}{4} = \pi R^2
\]
5. Consider a sample of $^{238}\text{U}$ ore in which the yields of the nuclei in the decay chain have reached their secular equilibrium values. The decay chain for $^{238}\text{U}$ is shown below; the columns denote the Z and the rows denote the A of the nuclei. The numbers in the box indicate the nuclear half-lives (y means years, and d means days).

a) (5 pts) Using the radioactive decay law derive mathematically the secular equilibrium relationship between the numbers $N_{226}$ and $N_{222}$ of $^{226}\text{Ra}$ and $^{222}\text{Rn}$ nuclei and their mean lives $\tau_{226}$ and $\tau_{222}$.

\[ \frac{dN_{226}}{dt} = 0 = \omega_{226} N_{226} - \omega_{222} N_{222} \]

\[ \Rightarrow \frac{N_{226}}{\tau_{226}} = \frac{N_{222}}{\tau_{222}} \]

\[ \tau_{226} = \tau_{222} \]

b) (4 pts) If one observes $^{222}\text{Rn}$ decays in the ore sample at a rate of $4 \times 10^3$ per year, how many $^{226}\text{Ra}$ nuclei are in this sample of ore?

\[ N_{226} = \int \omega_{226} \frac{N_{222}}{\tau_{226}} dt \]

\[ = 4 \times 10^3 \times 1 \text{ y} \]

\[ = 9.5 \times 10^3 \text{ N}_{226} \]
6. a) (10 pts.) For $\beta^+$ emission and electron capture, describe how the charge and nucleon numbers of the parent and daughter nuclei differ. Indicate what particles are destroyed in the process and what new particles are created.

- **Capture:** $e^+ (Z, N) \rightarrow (Z-1, N) + \nu_e$
  - $Z$ decreases by 1, $A$ stays the same, one electron is destroyed, one neutrino is created, one proton is changed into a neutron.

- **Beta emission:** $(Z, N) \rightarrow (Z-1, N) + e^+ + \nu_e$
  - $Z$ decreases by 1, $A$ stays the same, one position is created, one neutrino is created, one proton is changed into a neutron.

b) (7 pts) The radioactive noble gas $^{37}$Ar decays to $^{37}$Cl by electron capture with a Q-value of 814 keV. What is the Q-value for the $\beta^+$ decay of $^{37}$Ar? Is $\beta^+$ decay allowed for this nucleus?

In electron capture, the initial state has $M_e c^2$ more rest energy. In $\beta^+$ emission the final state has $M_e c^2$ more rest energy.

$$Q_{\beta^+} = Q_{ec} - 2M_e c^2 = 814 \text{ keV} - 2(511 \text{ keV})$$

$$= -208 \text{ keV}$$

$\beta^+$ decay is not allowed by energy conservation.