Physic 231 Lecture 16

Main points of last lecture:
Description of circular motion in cylindrical coordinates (r,θ):
\[ r = \sqrt{x^2 + y^2}; \theta = \tan^{-1}\left(\frac{y}{x}\right) \]
\[ x = r \cos(\theta); y = r \sin(\theta) \]
\[ \Delta s = \Delta \theta r \]

Main points of today’s lecture:
Rotational motion definitions:
\[ \bar{\omega} = \frac{\Delta \theta}{\Delta t}, \omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \]
\[ \bar{\alpha} = \frac{\Delta \omega}{\Delta t}, \alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} \]
\[ \Delta s = \Delta \theta r; \nu_t = \omega r; a_t = \alpha r \]
Rotational kinematics equations:
\[ \Delta \theta = \bar{\omega} t \]
\[ \omega = \omega_0 + \alpha t \]
\[ \bar{\omega} = \frac{1}{2} (\omega + \omega_0) \]
\[ \Delta \theta = \frac{1}{2} (\omega + \omega_0) t \]
\[ \Delta \theta = \omega_0 t + \frac{1}{2} \alpha t^2 \]
\[ \omega^2 = \omega_0^2 + 2\alpha \Delta \theta \]
Rolling motion:
\[ \Delta x = s = r \Delta \theta \]
\[ v = r \omega; \quad a = r \alpha \]
Example

A wheel has a radius of 4.1 m. How far (path length) does a point on the circumference travel if the wheel is rotated through angles of 30°, 30 rad, and 30 rev, respectively?

if $\theta = 30^\circ$, then we must first convert to radians:

$$\theta = 30^\circ \frac{1 \text{ rad}}{57.3^\circ} = 0.523 \text{ rad}$$

$$s = r\theta = 4.1m \cdot 0.523 = 2.15m$$

if $\theta = 30$ rad, then:

$$s = r\theta = 4.1m \cdot 30 = 123m$$

if $\theta = 30$ rev, then we must first convert to radians:

$$\theta = 30 \text{ rev} \frac{2\pi \text{ rad}}{\text{rev}} = 188.5 \text{ rad}$$

$$s = r\theta = 4.1m \cdot 188.5 = 723m$$
Angular velocity

- The rate of change of the angular displacement is defined to be the angular velocity $\omega$.
- In terms of the angular displacement, the average angular velocity over a time interval $\Delta t$ is:
  \[ \bar{\omega} = \frac{\Delta \theta}{\Delta t} \]
- The instantaneous angular velocity can be obtained by making the time interval very short:
  \[ \omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \]
- Note: if $\Delta \theta$ measured in radians, $\Delta s = r \Delta \theta$ is the arc length covered in time $\Delta t$. Thus:
  \[ \bar{v}_t = \frac{\Delta s}{\Delta t} = \frac{r \Delta \theta}{\Delta t} = r \bar{\omega} \]
- is the average tangential speed. And
  \[ v_t = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{r \Delta \theta}{\Delta t} = r \omega \]
  is the instantaneous tangential speed of the object.
Example

- Two people start at the same place and walk around a circular lake in opposite directions. The first has an angular speed of $1.7 \times 10^{-3}$ rad/s, while the second has an angular speed of $3.4 \times 10^{-3}$ rad/s. How long will it be before they meet?

  we want $\Delta \theta_1 - \Delta \theta_2 = 2\pi$:

  $\Delta \theta_1 = \omega_1 t$; $\Delta \theta_2 = \omega_2 t$

  $2\pi = \Delta \theta_1 - \Delta \theta_2 = \omega_1 t - \omega_2 t = (\omega_1 - \omega_2)t$

  $t = \frac{2\pi}{(\omega_1 - \omega_2)} = \frac{6.28}{1.7 \times 10^{-3} \text{ rad/s} - (-3.4 \times 10^{-3} \text{ rad/s})} = 1231 \text{s}$
Angular acceleration

• Many times, the angular velocity changes with time. We quantify this by the angular acceleration \( \alpha \).

• In term of the angular velocity, the average angular acceleration over a time interval \( \Delta t \) is:

\[
\bar{\alpha} = \frac{\Delta \omega}{\Delta t}
\]

• Note: if \( \Delta \omega \) measured in rad/s, \( \Delta v_t = r \Delta \omega \) is the change in tangential speed during time \( \Delta t \). Thus:

\[
\bar{a}_t = \frac{\Delta v_t}{\Delta t} = \frac{r \Delta \omega}{\Delta t} = r \bar{\alpha}
\]

• The instantaneous angular acceleration can be obtained by making the time interval very short:

\[
\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t}
\]

• is the average tangential acceleration. And \( a_t = r \alpha \) is the instantaneous tangential speed of the object.

\[
a_t = \lim_{\Delta t \to 0} \frac{\Delta v_t}{\Delta t} = \lim_{\Delta t \to 0} \frac{r \Delta \omega}{\Delta t} = r \alpha
\]
Analogy between linear and angular kinematics for constant acceleration

- If the acceleration is constant, we obtained a set of equation on the left side of the table relating $\Delta x$, $v$, $a$ and $t$. If the angular acceleration is constant and we go through the same steps, we obtain the analogous set of equation on the right side of the table relating $\Delta \theta$, $\omega$, $\alpha$, and $t$.

<table>
<thead>
<tr>
<th>Linear Kinematics</th>
<th>Angular Kinematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta x = \bar{v}t$</td>
<td>$\Delta \theta = \bar{\omega}t$</td>
</tr>
<tr>
<td>$v = v_0 + at$</td>
<td>$\omega = \omega_0 + \alpha t$</td>
</tr>
<tr>
<td>$\bar{v} = \frac{1}{2}(v + v_o)$</td>
<td>$\bar{\omega} = \frac{1}{2}(\omega + \omega_o)$</td>
</tr>
<tr>
<td>$\Delta x = \frac{1}{2}(v + v_o)t$</td>
<td>$\Delta \theta = \frac{1}{2}(\omega + \omega_o)t$</td>
</tr>
<tr>
<td>$\Delta x = v_0t + \frac{1}{2}at^2$</td>
<td>$\Delta \theta = \omega_0t + \frac{1}{2}\alpha t^2$</td>
</tr>
<tr>
<td>$v^2 - v_0^2 = 2a\Delta x$</td>
<td>$\omega^2 = \omega_0^2 + 2\alpha \Delta \theta$</td>
</tr>
</tbody>
</table>
Conceptual question

- A ladybug sits at the outer edge of a merry-go-round, and a gentleman bug sits halfway between her and the axis of rotation. The merry-go-round makes a complete revolution once each second. The gentleman bug’s angular speed is

  - a) half the ladybug’s.
  - b) the same as the ladybug’s.
  - c) twice the ladybug’s.
  - d) impossible to determine
Example

- After 10 s, a spinning roulette wheel has slowed to an angular speed of 1.88 rad/s. During this time, the wheel rotates through an angle of 44.0 rad. Determine the angular acceleration of the wheel.

\[
\Delta \theta = \omega t \\
\bar{\omega} = \frac{1}{2} (\omega + \omega_0) \\
\Delta \theta = \frac{1}{2} (\omega + \omega_0) t \\
\Delta \theta = \omega_0 t + \frac{1}{2} \alpha t^2 \\
\omega^2 = \omega_0^2 + 2\alpha \Delta \theta
\]

\[
\omega = \omega_o + \alpha t \Rightarrow \omega_o = \omega - \alpha t
\]

\[
\Delta \theta = \omega_o t + \frac{1}{2} \alpha t^2 = (\omega - \alpha t)t + \frac{1}{2} \alpha t^2
\]

\[
\Rightarrow \Delta \theta = \omega t - \frac{1}{2} \alpha t^2
\]

\[
\Rightarrow \alpha = \frac{\omega t - \Delta \theta}{\frac{1}{2} t^2} = \frac{1.88 \text{ rad} / \text{s} \cdot 10 \text{ s} - 44 \text{ rad}}{0.5(10 \text{ s})^2} = -0.5 \text{ rad} / \text{s}^2
\]
A centrifuge decelerates to rest over a time interval of 10 seconds from an initial angular velocity of 1000 rad/s. Over this time interval, the angular displacement is about:

- a) 5 rad
- b) 5000 rad
- c) 10 rad
- d) 10000 rad

\[
\omega = \omega_o + \alpha t \Rightarrow \alpha = \frac{\omega - \omega_o}{t} = \frac{-1000 \text{rad} / \text{s}}{10 \text{s}} = -100 \text{rad} / \text{s}^2
\]

\[
\Delta \theta = \omega_o t + \frac{1}{2} \alpha t^2 = (1000 \text{rad} / \text{s})(10 \text{s}) + \frac{1}{2}(-100 \text{rad} / \text{s}^2)(10 \text{s})^2
\]

\[
\Delta \theta = 5000 \text{rad}
\]
Rolling motion

- Consider a wheel of radius $r$ rolling on the ground. The relationship between the distance traveled and the angle is obtained by considering the arc length $s$:

$$d = s = r\theta$$

- Similarly there are corresponding relationships between $v$, $a$ and $\omega$, $\alpha$.

$$v = v_t = r\omega$$

$$a = a_t = r\alpha$$