Main points of today’s lecture:

- **Moment of inertia:**
  \[ I = \sum_i m_i r_i^2 \]

- **Parallel axis theorem:**
  \[ I = I_{CM} + MR_{CM}^2 \]

- Analogies between rotational and translational motion

- Solutions involving:
  \[ I; \tau = I\alpha \]

\[ KE_{rot} = \frac{1}{2} I\omega^2; \quad L = I\omega \]

- Rolling motion:
  \[ KE_{total} = \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 \]
Moment of inertia for collections of masses

- Let’s consider the tangential acceleration of a single mass \( m_1 \) on attached to a mass rod of length \( r_1 \) and fixed to rotate about axis O:

\[
F_{1t} = m_1 a_{1t} = m_1 (r_1 \ddot{\alpha}) \Rightarrow \tau_1 = F_{1t}r_1 = m_1 r_1^2 \ddot{\alpha} = I_1 \ddot{\alpha}
\]

- If we attach a second mass \( m_2 \) on attached to a mass rod of length \( r_2 \) which is rigidly attached to the pivot so its angular acceleration is the same:

\[
F_{2t} = m_2 a_{2t} = m_2 (r_2 \ddot{\alpha}) \Rightarrow \tau_2 = F_{2t}r_2 = m_2 r_2^2 \ddot{\alpha}
\]

\[
\tau_{net} = \tau_1 + \tau_2 = m_1 r_1^2 \ddot{\alpha} + m_2 r_2^2 \ddot{\alpha} = (m_1 r_1^2 + m_2 r_2^2) \ddot{\alpha} = I \ddot{\alpha}
\]

- Similarly for three particles:

\[
\tau_{net} = \tau_1 + \tau_2 + \tau_3 = (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2) \ddot{\alpha} = I \ddot{\alpha}
\]

- In general:

\[
I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2 + \ldots \equiv \sum_i m_i r_i^2
\]
Moments of inertia of extended objects

- To get the moments of inertia, one calculates $I = \sum m_i r_i^2$ as well. Some values are given in the table to the right.

- Last row is an illustration of the “parallel axis theorem”:

$$I = I_{CM} + MR_{CM}^2$$

- Here $I_{CM}$ is the moment of inertia about an axis going through the center of mass. $I$ is the moment of inertia about another axis which is parallel to and displaced a distance $R_{CM}$ from the center of mass.
Analyses between straight line and rotational motion

- We already introduced the following analogous equations and quantities:

<table>
<thead>
<tr>
<th>$\Delta x = \bar{v}t$</th>
<th>$\Delta \theta = \bar{\omega}t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v = v_0 + at$</td>
<td>$\omega = \omega_0 + \alpha t$</td>
</tr>
<tr>
<td>$\bar{v} = \frac{1}{2}(v + v_o)$</td>
<td>$\bar{\omega} = \frac{1}{2}(\omega + \omega_0)$</td>
</tr>
<tr>
<td>$\Delta x = \frac{1}{2}(v + v_o)t$</td>
<td>$\Delta \theta = \frac{1}{2}(\omega + \omega_0)t$</td>
</tr>
<tr>
<td>$\Delta x = v_0t + \frac{1}{2}at^2$</td>
<td>$\Delta \theta = \omega_0t + \frac{1}{2}\alpha t^2$</td>
</tr>
<tr>
<td>$v^2 - v_0^2 = 2a\Delta x$</td>
<td>$\omega^2 = \omega_0^2 + 2\alpha\Delta \theta$</td>
</tr>
</tbody>
</table>

- New analogies:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F = ma$</td>
<td>$\tau = I\alpha$</td>
</tr>
<tr>
<td>$KE_{trans} = \frac{1}{2}mv^2$</td>
<td>$KE_{rot} = \frac{1}{2}I\omega^2$ rotational kinetic energy</td>
</tr>
<tr>
<td>$p = mv$</td>
<td>$L = I\omega$ angular momentum</td>
</tr>
</tbody>
</table>
Example

- A turntable platter has a radius of 0.150 m and is rotating at 3.49 rad/s. When the power is shut off, the platter slows down and comes to rest in 15.0 s, due to a net retarding torque of 6.2x10^{-3} N\cdot m. Assume the platter to be a uniform solid disk. Determine the mass of the platter. (Hint: calculate \( \alpha \), plug into Newton’s second law to find \( M \).)

  - a) 1.37 kg
  - b) 2.37 kg
  - c) 3.37 kg
  - d) 4.37 kg

\[ I = \frac{1}{2} MR^2 \]

\[ \omega = \omega_0 + \alpha t \Rightarrow \alpha = \frac{\omega - \omega_0}{t} = \frac{-3.49 \text{ rad} / \text{s}}{15 \text{s}} = -0.233 \text{ rad} / \text{s}^2 \]

\[ \tau = I \alpha = \frac{1}{2} MR^2 \alpha \]

\[ \Rightarrow M = \frac{\tau}{\frac{1}{2} R^2 \alpha} = \frac{-6.2 \times 10^{-3} \text{ Nm}}{\frac{1}{2} (0.15 \text{ m})^2 (-0.233 \text{ rad} / \text{s}^2)} = 2.37 \text{ kg} \]
Example

- A car is designed to get its energy from a rotating flywheel with a radius of 2.00 m and a mass of 500 kg. Before a trip, the flywheel is attached to an electric motor, which brings the flywheel’s rotational speed up to 5000 rev/min. (a) Find the kinetic energy stored in the flywheel. (b) If the flywheel is to supply energy to the car as would a 10.0-hp motor, find the length of time the car could run before the flywheel would have to be brought back up to speed.

\[ KE_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} I_i \omega^2 \]

\[ KE_{total} = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} I_i \omega^2 = \frac{1}{2} I_{total} \omega^2 \]

\( a) \ \omega = \frac{5000}{60} \cdot \frac{2\pi}{rad}{s} = 5.23 \times 10^2 \frac{rad}{s} \)

\[ I = \frac{1}{2} MR^2 = \frac{1}{2} \cdot 500kg \cdot (2m)^2 = 1000kg \cdot m^2 \]

\[ KE = \frac{1}{2} I \omega^2 = \frac{1}{2} (1000kg \cdot m^2)(5.23 \times 10^2 \frac{rad}{s})^2 \]

\[ KE = 1.37 \times 10^8 J \]

\( b) \ KE = \frac{P \Delta t}{P} \Rightarrow \Delta t = \frac{KE}{P} = \frac{1.37 \times 10^8 J}{(10)(746W/hp)} = 1.8 \times 10^4 s \approx 5h \)
Which one wins?

- Which object will be fastest?
  - a) hoop
  - b) cylinder
  - c) sphere
Rolling down the hill

- Starting from rest, an object with moment of mass $M$, moment of inertia $I$, and radius $R$ rolls down an incline of height $h$. What is the velocity at the bottom? (Hint use conservation of energy.)

![Diagram of an object rolling down a hill]

Conservation of energy:

$$KE_{total} = KE_{trans} + KE_{rot} = \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2$$

Rolling motion: $v = \omega R \Rightarrow \omega = \frac{v}{R}$

$$Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} I \left(\frac{v}{R}\right)^2 = \left[\frac{1}{2} M + \frac{1}{2} \frac{I}{R^2}\right] v^2$$

$$v = \sqrt{\frac{Mgh}{\left[\frac{1}{2} M + \frac{1}{2} \frac{I}{R^2}\right]}}$$

- If the object slides without rolling? ($I=0$)

a) $v = \sqrt{gh}$

b) $v = \sqrt{\frac{4}{3} gh}$

c) $v = \sqrt{\frac{10}{7} gh}$

d) $v = \sqrt{2gh}$
Conceptual question

- A force $F$ is applied to a dumbbell for a time interval $\Delta t$, first as in (a) and then as in (b). In which case does the dumbbell acquire the greater center-of-mass speed?

- a) (a)
- b) (b)
- c) no difference
- d) The answer depends on the rotational inertia of the dumbbell.
Angular momentum

- In analogy to the linear momentum $\vec{p} = m\vec{v}$ we have the angular momentum $\vec{L} = I\vec{\omega}$. For a single particle moving a circle, the magnitude of $L$ is:

$$L = I\omega = mr^2\omega = mrv_t$$

- Recall that if there are no external forces, the linear momentum will be conserved. Similarly, if there no external torques, the angular momentum will be conserved.

- For example, this means that an isolated object will keep the same value of the angular momentum throughout a problem.
  - In linear motion, conservation of $\vec{p} = m\vec{v}$ for an isolated object meant that the velocity $\vec{v}$ must remain constant.
  - In angular motion, conservation of $\vec{L} = I\vec{\omega}$ for an isolated object does not mean $\vec{\omega}$ must remain constant, because $I$ can change!

$$L_f = L_0 \quad \Rightarrow \quad I_f\omega_f = I_0\omega_0 \quad \Rightarrow \quad \omega_f = \frac{I_0}{I_f}\omega_0$$
Example

- A woman stands at the center of platform. The woman and the platform rotate with an angular speed of 5.00 rad/s. Friction is negligible. Her arms are outstretched and she is holding a dumbbell in each hand. In this position, the total moment of inertia of the rotating system (platform, woman and dumbbells) is 5.4 kg⋅m². By pulling in her arms, the moment of inertia is reduced to 3.8 kg⋅m². Find her new angular speed.

\[ \omega_f = \frac{I_0}{I_f} \omega_0 \]

\[ \omega_f = \frac{5.4}{3.8} \text{5rad / s} = 7.1 \text{rad / s} \]