1) The nucleon is in an eigenstate \( \frac{1}{2} \), \( \frac{3}{2} \), \( \frac{1}{2} \), \( \frac{3}{2} \). Is \( |\ell \ell j \ell m_j \rangle \)?

The shift of the energy \( V \) in this state is given by

\[
\Delta E_{So} = \langle s \ell j \ell m_j | V_{So} | s \ell j \ell m_j \rangle \quad \text{i.e. the expectation value for } V_{So}.
\]

Now, \( V_{So} = \frac{\hbar^2}{\mu} (J^2 + \frac{3}{4}) \)

\[
\frac{3}{4} = \frac{3}{4} + \frac{1}{4} \quad J^2 = \frac{3}{4} + \frac{3}{4} = \frac{3}{2} + \frac{3}{2} = \frac{3}{2} + \frac{1}{2} \frac{3}{2} \]

Also, \( \frac{1}{2} |s \ell j \ell m_j \rangle = j (\ell + 1) \hbar^2 |s \ell j \ell m_j \rangle \)

\[
L^2 |s \ell j \ell m_j \rangle = \ell (\ell + 1) \hbar^2 |s \ell j \ell m_j \rangle
\]

\[
\frac{1}{2} L^2 |s \ell j \ell m_j \rangle = \frac{1}{2} (\ell (\ell + 1) - \ell (\ell - 1)) \hbar^2 |s \ell j \ell m_j \rangle
\]

\[
\Rightarrow \Delta E_{So} = \frac{\hbar^2}{2} \left( j (\ell + 1) - \ell (\ell - 1) - \frac{3}{2} \right) |s \ell j \ell m_j \rangle
\]

\[
\Rightarrow \Delta E_{So} = \frac{\hbar^2}{2} \left( j (\ell + 1) - \ell (\ell - 1) - \frac{3}{2} \right)
\]

For \( j = \ell + \frac{1}{2} \)

\[
\ell (\ell + 1) = (\ell + \frac{1}{2}) (\ell + \frac{3}{2}) = \ell^2 + 2 \ell + \frac{3}{4}
\]

For \( j = \ell - \frac{1}{2} \)

\[
\ell (\ell - 1) = (\ell - \frac{1}{2}) (\ell + \frac{1}{2}) = \ell^2 - \frac{1}{4}
\]

For \( j = \ell + \frac{3}{2} \)

\[
\Delta E_{So} = \frac{\hbar^2}{2} \left( \ell^2 + 2 \ell + \frac{3}{4} - \ell^2 - \frac{1}{4} - \frac{1}{2} \right) = \frac{\hbar^2}{2} \]

For \( j = \ell - \frac{3}{2} \)

\[
\Delta E_{So} = \frac{\hbar^2}{2} \left( \ell^2 + 2 \ell + \frac{3}{4} - \ell^2 - \frac{1}{4} - \frac{1}{2} \right) = -\frac{\hbar^2}{2} \]

For \( j = \ell + \frac{1}{2} \)

\[
\Delta E_{So} = \frac{\hbar^2}{2} \left( \ell^2 + 2 \ell + \frac{3}{4} - \ell^2 - \frac{1}{4} - \frac{1}{2} \right) = \frac{\hbar^2}{2} \]

For \( j = \ell - \frac{1}{2} \)

\[
\Delta E_{So} = \frac{\hbar^2}{2} \left( \ell^2 + 2 \ell + \frac{3}{4} - \ell^2 - \frac{1}{4} - \frac{1}{2} \right) = -\frac{\hbar^2}{2} \left( \ell + 1 \right)
\]
<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Nucleon</th>
<th>Spin-Parity</th>
<th>U</th>
<th>Unmeasured</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^3$He</td>
<td>$\pi 1s_{1/2}$</td>
<td>$1/2^+$</td>
<td>2.79</td>
<td>2.98</td>
</tr>
<tr>
<td>$^3$He</td>
<td>$\nu 1s_{1/2}$</td>
<td>$1/2^+$</td>
<td>-1.91</td>
<td>-2.13</td>
</tr>
<tr>
<td>$^7$Li</td>
<td>$\pi 1p_{3/2}$</td>
<td>$3/2^-$</td>
<td>3.71</td>
<td>3.26</td>
</tr>
<tr>
<td>$^6$Be</td>
<td>$\nu 1p_{3/2}$</td>
<td>$3/2^-$</td>
<td>-1.91</td>
<td>-1.13</td>
</tr>
<tr>
<td>$^8$B</td>
<td>$\pi 1p_{3/2}$</td>
<td>$3/2^-$</td>
<td>3.71</td>
<td>2.69</td>
</tr>
<tr>
<td>$^{10}$C</td>
<td>$\nu 1p_{3/2}$</td>
<td>$3/2^-$</td>
<td>-1.91</td>
<td>-1.03</td>
</tr>
<tr>
<td>$^{12}$C</td>
<td>$\pi 1p_{1/2}$</td>
<td>$1/2^-$</td>
<td>0.64</td>
<td>0.70</td>
</tr>
<tr>
<td>$^{13}$N</td>
<td>$\nu 1p_{1/2}$</td>
<td>$1/2^-$</td>
<td>-1.26</td>
<td>-1.32</td>
</tr>
<tr>
<td>$^{15}$N</td>
<td>$\pi 1p_{1/2}$</td>
<td>$1/2^-$</td>
<td>-1.26</td>
<td>-0.83</td>
</tr>
<tr>
<td>$^{16}$O</td>
<td>$\nu 1d_{5/2}$</td>
<td>$5/2^+$</td>
<td>4.79</td>
<td>4.72</td>
</tr>
<tr>
<td>$^{17}$F</td>
<td>$\pi 1d_{5/2}$</td>
<td>$5/2^+$</td>
<td>4.79</td>
<td>2.03</td>
</tr>
</tbody>
</table>

If we approximate the potential by a spherical Wood-Saxon potential, this can be understood. These numbers correspond to the maximum number of protons or neutrons that can be added before one starts pushing nucleons into the next major shell, which have significantly high single particle energies. This gives the magic numbers 2, 8, 20, 28, 50, 82, 126. Also shown in Fig. 8.6 are the levels one obtains by adding the spin-orbit potential the same procedure with this potential gives magic numbers 4, 6, 10, 20, 28, 50, 82, 126.
\( ^{20}\text{Ne} \) is even-even, all even-even nuclei have their neutron-odd angular momenta paired to zero total angular momentum and their proton-odd angular momenta paired to zero total angular momentum. This is due to pairing. There is an even number of all filled orbits so parity is preserved.

\[ I^\pi = 0^+ \text{ for } ^{20}\text{Ne} \]

\( ^{27}\text{Al} \) has an odd proton; thus the spin and parity of the nucleus is given by that of this odd proton's wavefunction, \( \frac{9}{2}^- \).

So:

\[ I^\pi = \frac{9}{2}^+ \text{ for } ^{27}\text{Al} \]

\( ^{61}\text{Sc} \) has an odd proton in the \( \frac{7}{2}^- \) orbit, thus

\[ I^\pi = \frac{7}{2}^- \text{ for } ^{61}\text{Sc} \]

8.6 In the rotational model, the excitation energy is given by

\[ E^* = \frac{\hbar^2 (I(I+1))}{2Jl} \]

\[ \Rightarrow Jl = \frac{\hbar^2 (I(I+1))}{2E^*} \]

If \( E^* \) is in keV, the units will be \( \frac{\hbar^2}{\text{keV}} \)

\[
\begin{array}{c|c}
I & \frac{\hbar^2 (I(I+1))}{2E^*} \\
2 & \frac{\hbar^2}{100} = 0.09 \\
4 & \frac{\hbar^2}{225} = 0.0623 \\
6 & \frac{\hbar^2}{441} = 0.0655 \\
8 & 0.0692 \\
\end{array}
\]

In converted units, \( Jl \) for a sphere of radius \( R \) is

\[ Jl = \frac{2}{5} MR^2 \Rightarrow \frac{\hbar^2}{Jl} = \frac{2}{5} \frac{MC^2}{\hbar^2} R^2 \]

\[ = \frac{\hbar^2}{Jl} = 0.19 \frac{(170)(93.5)}{(19.7)^2} \cdot (1.3 \cdot 170)^2 \text{ mev}^{-1} \]

\[ = 84.76 \text{ mev}^{-1} \]

\[ = 0.85 \text{ kev}^{-1} \]

\[ \Rightarrow Jl = \frac{0.85}{\text{kev}} \]
5. \( \pi^0 \rightarrow 2\gamma \)

\[
\rho^\mu(i) = (\mathbf{E}_\pi, \mathbf{P}_\pi)
\]

\[
\gamma^\mu = (\gamma^0, \gamma^i)
\]

\[
\gamma^\mu(2) = (\gamma^0(2), \gamma^i(2))
\]

\[
\rho^\mu(1) = (\gamma^\mu(1) + \gamma^\mu(2)) \cdot (\gamma^\mu(1) + \gamma^\mu(2))
\]

\[
\rho^\mu(1) = k^\mu(1) + k^\mu(2)
\]

\[
\rho^\mu(1) = 2k^\mu(1)k^\mu(2) - k^\mu(1)k^\mu(2)
\]

\[
2k^\mu = E_\pi \Rightarrow k^\mu = E_\pi/c
\]

\[
m_\pi^2 c^2 = 2(E_\pi c)^2 (1 - \cos(\theta))
\]

\[
\sin^2(\theta) = \frac{1}{2} (1 - \cos(\theta)) \Rightarrow (1 - \cos(\theta)) = 2\sin^2(\theta)
\]

\[
m_\pi^2 c^2 = 2E_\pi^2 (4c^2), 2\sin^2(\theta) = E_\pi^2/c^2 \sin^2(\theta)
\]

\[
\sin(\theta) = \frac{m_\pi c^2}{E_\pi}
\]

\[\theta = \sin^{-1}\left(\frac{m_\pi c^2}{E_\pi}\right)\]

b) \( E_\pi = 1 \text{ GeV} \Rightarrow 2\theta = \sin^{-1}\left(\frac{1.9}{1000}\right) = 17^\circ \)

\( E_\pi = 10 \text{ GeV} \Rightarrow 2\theta = 2\sin^{-1}\left(\frac{10}{10000}\right) = 1.6^\circ \)