Physics 231 Lecture 13

- Main points of today’s lecture:
- Elastic collisions in one dimension:
  \[ v_{1f} = \frac{(m_1 - m_2)}{m_1 + m_2} v_{10} + \frac{2m_2}{m_1 + m_2} v_{20} \]
  \[ v_{2f} = \frac{(m_2 - m_1)}{m_1 + m_2} v_{20} + \frac{2m_1}{m_1 + m_2} v_{10} \]
- Multiple impulses and rocket propulsion.
  \[ F_{\text{propellant}} = \Delta m \cdot v_f - \Delta m \cdot v_0 \]
  \[ F_{\text{propellant}} = \frac{\Delta m}{\Delta t} (v_f - v_0) = -\text{Thrust} \]
  \[ \text{Thrust} = \frac{\Delta m}{\Delta t} (v_0 - v_f) \]
- Center of Mass
  \[ M_{\text{tot}} X_{\text{cm}} = \sum_i m_i x_i \quad M_{\text{tot}} Y_{\text{cm}} = \sum_i m_i y_i \]
  \[ M_{\text{tot}} \vec{V}_{\text{cm}} = \vec{P}_{\text{tot}} = \sum_i m_i \vec{v}_i \]
1st Midterm Exam

• Exam is Wednesday October 5th.
• If you will be away on a University sponsored trip, you need to make alternative testing arrangements with me by the end of today (Friday).
• Problems will be similar to Lon-Capa, but somewhat simpler on the average. Answers will be Multiple Choice.
• You should prepare one 8.5” by 11” sheet of formulae. You can use both sides of the sheet.
• We will have a review on Monday. There are a set of practice problems on course web site. I have also distributed by email some suggestions about how you can prepare for the exam.
• You should review the homework, lectures, book and then prepare your formula sheet. You should then attempt the practice problems with only the formula sheet as a reference.
• Homework is not due next week. Instead, you should the do the corrections set, which will be available on LONCAPA as a regular homework assignment on Thursday evening next week. The corrections set will consist of the same problems as on the midterm exam. I have already sent an email about how the corrections set can improve your midterm score.
Conceptual quiz

• A compact car and a large truck collide head on and stick together. Which undergoes a larger change in the magnitude of the momentum? Assume this system of car plus truck to be isolated.
  - a) car
  - b) truck
  - c) The magnitude of the momentum change is the same for both vehicles.
  - d) Can’t tell without knowing the final velocity of combined mass.

The system of car and truck is isolated.

\[
\begin{align*}
\vec{P}_{\text{car,f}} + \vec{P}_{\text{truck,f}} &= \vec{P}_{\text{car,i}} + \vec{P}_{\text{truck,i}} \\
\Rightarrow \vec{P}_{\text{car,f}} - \vec{P}_{\text{car,i}} &= \vec{P}_{\text{truck,i}} - \vec{P}_{\text{truck,f}} \\
\Rightarrow \Delta \vec{P}_{\text{car}} &= -\Delta \vec{P}_{\text{truck}}
\end{align*}
\]
Energy loss in totally inelastic collisions

- Example: A 40 kg skater, sliding to the right without friction with a velocity of 1.5 m/s, suffers a head on collisions with a 30 kg skater who is initially at rest. How much energy is lost in this collision?
  
  - a) 19 J
  - b) 22 J
  - c) 25 J
  - d) 28 J

\[
\begin{align*}
\text{Initial momentum:} & \\
(40 \text{ kg}) \cdot 1.5 \text{ m/s} & = 60 \text{ kg \cdot m/s} \quad (1) \\
(30 \text{ kg}) \cdot 0 \text{ m/s} & = 0 \text{ kg \cdot m/s} \quad (2)
\end{align*}
\]

The two skaters stick together after the collision.

\[
\text{Final velocity:} \quad v_f = \frac{m_1 \cdot v_1}{m_1 + m_2} = \frac{40 \text{ kg} \cdot 1.5 \text{ m/s}}{70 \text{ kg}} = 0.86 \text{ m/s}
\]

\[
\begin{align*}
\text{Kinetic energy lost:} & \\
\text{KE}_{\text{loss}} & = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{1}{2} (m_1 + m_2) v_f^2 \\
& = \frac{1}{2} (40 \text{ kg}) (1.5 \text{ m/s})^2 + \frac{1}{2} (30 \text{ kg}) (0 \text{ m/s})^2 - \frac{1}{2} (40 \text{ kg} + 30 \text{ kg}) (0.86 \text{ m/s})^2 \\
& = 19.1 \text{ J}
\end{align*}
\]

<table>
<thead>
<tr>
<th>m₁</th>
<th>40 kg</th>
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<tbody>
<tr>
<td>m₂</td>
<td>30 kg</td>
</tr>
<tr>
<td>v₁₀</td>
<td>1.5 m/s</td>
</tr>
<tr>
<td>v₂₀</td>
<td>0</td>
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<tr>
<td>KEₗoss</td>
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Expressing Kinetic energy in terms of momentum

- A car has momentum $\vec{p}$ and mass $m$. In terms of these two quantities, express the kinetic energy.

\[ K.E. = \frac{1}{2}mv^2 \]

\[ p = mv \]

\[ v = \frac{p}{m} \]

\[ K.E. = \frac{1}{2}m \left( \frac{p}{m} \right)^2 = \frac{1}{2} \frac{p^2}{m^2} \]

\[ K.E. = \frac{1}{2} \frac{p^2}{2m} = \frac{p^2}{2m} \]
Review Conceptual Question

In the demonstration, one car is heavier than the other, but both experience the same force and both start from rest and run until they achieve the same displacement in the positive direction. Which car has the greater final momentum? Hint: express momentum in terms of work

A. The lighter car.

**B. The heavier car.**

C. They have the same momentum.

Force does the same work on each car. By the work energy theorem:

\[ W = KE_f - KE_i = KE_f \]

\[ W = KE_f = \frac{p^2}{2m} \]

\[ p^2 = 2mW \Rightarrow p = \sqrt{2mW} \]
Additional Questions

In the demonstration, two cars start from rest. One car is heavier than the other, but both experience the same force and both run for the same time. Which car has the greater final momentum?

A. The lighter car.
B. The heavier car.
C. They have the same momentum.

\[ \Rightarrow \bar{F}_{ave} \Delta t = \Delta \bar{p} = \bar{p}_f - \bar{p}_0 \]
\[ \bar{p}_0 = 0 \]
\[ \Rightarrow \bar{F}_{ave} \Delta t = \bar{p}_f \]
Totally elastic collisions

- Proof: To calculate the result of an elastic collision in one dimension, we considered the constraints of total momentum and energy conservation:

\[
\text{Eq. 1: } p_{tot} = m_1 v_{1,0} + m_2 v_{2,0} = m_1 v_{1,f} + m_2 v_{2,f} \\
\text{Eq. 2: } E_{tot} = \frac{1}{2} m_1 v_{1,0}^2 + \frac{1}{2} m_2 v_{2,0}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2
\]

- We rearrange both equations to get object 1 on the left and object 2 on the right:

Rearranged Eq. 1: \( m_1 (v_{1,0} - v_{1,f}) = m_2 (v_{2,f} - v_{2,0}) \)

Rearranged Eq. 2: \( \frac{1}{2} m_1 (v_{1,0}^2 - v_{1,f}^2) = \frac{1}{2} m_2 (v_{2,f}^2 - v_{2,0}^2) \)

\[\Rightarrow \frac{1}{2} m_1 (v_{1,0} - v_{1,f})(v_{1,0} + v_{1,f}) = \frac{1}{2} m_2 (v_{2,f} - v_{2,0})(v_{2,f} + v_{2,0})\]

New Eq. 2: \( (v_{1,0} + v_{1,f}) = (v_{2,f} + v_{2,0}) \)

- Combining the last equation and the rearranged Equation 1, we have two equations and 2 unknowns which we can solve to get \( v_{1,f} \) and \( v_{2,f} \):

In one dimension the result for the collisions of two masses is:

\[
\begin{align*}
v_{1,f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1,0} + \frac{2m_2}{m_1 + m_2} v_{2,0} \\
v_{2,f} &= \frac{m_2 - m_1}{m_1 + m_2} v_{2,0} + \frac{2m_1}{m_1 + m_2} v_{1,0}
\end{align*}
\]
Example

- A 2 kg cart moves with a velocity of 3 m/s to the right on a frictionless track. It collides elastically with a stationary 1 kg cart.

- a) What are the final velocities of the two carts?

\[
v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,0} + \frac{2m_2}{m_1 + m_2} v_{2,0}
\]

\[
v_{2,f} = \frac{m_2 - m_1}{m_1 + m_2} v_{2,0} + \frac{2m_1}{m_1 + m_2} v_{1,0}
\]

\[
a) \quad v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,0} = \frac{2\text{kg} - 1\text{kg}}{3\text{kg}} 3\text{m/s} = 1\text{m/s}
\]

\[
v_{2,f} = \frac{2m_1}{m_1 + m_2} v_{1,0} = \frac{2 \cdot 2\text{kg}}{3\text{kg}} 3\text{m/s} = 4\text{m/s}
\]

<table>
<thead>
<tr>
<th>(m_1)</th>
<th>2 kg</th>
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<tbody>
<tr>
<td>(m_2)</td>
<td>1 kg</td>
</tr>
<tr>
<td>(v_{1,0})</td>
<td>3 m/s</td>
</tr>
<tr>
<td>(v_{2,0})</td>
<td>0</td>
</tr>
<tr>
<td>(v_{1,f})</td>
<td>?</td>
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<tr>
<td>(v_{2,f})</td>
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Conceptual question

- A golf ball (mass 1) is fired at a bowling ball (mass 2) initially at rest and bounces back elastically. Compared to the bowling ball, the golf ball after the collision has
  - a) more momentum but less kinetic energy.
  - b) more momentum and more kinetic energy.
  - c) less momentum and less kinetic energy.
  - d) less momentum but more kinetic energy.
  - e) none of the above

\[ v_{2,0} = 0 \]
\[ \Rightarrow v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,0} \]
\[ \Rightarrow v_{2,f} = \frac{2m_1}{m_1 + m_2} v_{1,0} \]

If \( m_2 \gg m_1 \)
\[ v_{1,f} \approx \frac{-m_2}{m_2} v_{1,0} \approx -v_{1,0} \]
\[ v_{2,f} \approx \frac{2m_1}{m_2} v_{1,0} \]

\[ p_{1,f} = m_1 v_{1,f} \approx -m_1 v_{1,0} \]
\[ p_{2,f} = m_2 v_{2,f} \approx m_2 \frac{2m_1}{m_2} v_{1,0} \approx 2m_1 v_{1,0} \]
\[ KE_{1,f} = \frac{1}{2} m_1 \left( v_{1,f} \right)^2 \approx \frac{1}{2} m_1 \left( v_{1,0} \right)^2 \approx KE_{1,0} \]
\[ KE_{2,f} = \frac{1}{2} m_2 \left( v_{2,f} \right)^2 \approx \frac{1}{2} m_2 \left( \frac{2m_1}{m_2} v_{1,0} \right)^2 \]
\[ = \frac{4m_1}{m_2} KE_{1,0} \]
Example

- A 3 kg cart (cart 1) moving with a velocity of +2 m/s collides with a 3 kg cart (cart 2) moving with a velocity of -3 m/s. What are the final velocities of cart 1 and cart 2?
  - a) $v_1 = 2 \text{ m/s}$, $v_2 = -3 \text{ m/s}$
  - b) $v_1 = -3 \text{ m/s}$, $v_2 = 2 \text{ m/s}$
  - c) $v_1 = 1 \text{ m/s}$, $v_2 = -1.5 \text{ m/s}$
  - d) $v_1 = -1.5 \text{ m/s}$, $v_2 = 1 \text{ m/s}$

<table>
<thead>
<tr>
<th>$m_1$</th>
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<tbody>
<tr>
<td>$m_2$</td>
<td>3 kg</td>
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<tr>
<td>$v_{1,0}$</td>
<td>2 m/s</td>
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<tr>
<td>$v_{2,0}$</td>
<td>-3 m/s</td>
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<td>$v_{1,f}$</td>
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<tr>
<td>$v_{2,f}$</td>
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The two carts exchange velocities if they have equal masses.
Conceptual problem

Suppose you are on a cart, initially at rest on a track with very little friction. You throw balls at a partition that is rigidly mounted on the cart. If the balls bounce straight back as shown in the figure, is the cart put in motion?

- a) Yes, it moves to the right.
- b) Yes, it moves to the left.
- c) No, it remains in place.

The system of balls, man and car is isolated.

\[ \vec{P}_{\text{tot},f} = \vec{P}_{\text{balls}} + \vec{P}_{\text{man+cart}} = \vec{P}_{\text{tot},i} = 0 \]

\[ \Rightarrow \vec{P}_{\text{man+cart}} = -\vec{P}_{\text{balls}} = -\text{number of balls thrown} \quad \vec{m}_{\text{ball}} \vec{v}_{\text{ball}} \]
Center of Mass

- The center of mass of a system is a mass weighted average over the positions of the various masses.

\[
X_{\text{cm}} = \frac{1}{M_{\text{total}}} \sum_i m_i x_i \quad Y_{\text{cm}} = \frac{1}{M_{\text{total}}} \sum_i m_i y_i \quad Z_{\text{cm}} = \frac{1}{M_{\text{total}}} \sum_i m_i z_i
\]

- Example Three masses are lined up along the x axis, with y=0, and z=0 for all three masses. Mass 1 has \( m_1 = 2 \) kg is at \( x = 1 \) m. Mass 2 has \( m_2 = 0.5 \) kg and is at \( x = 3 \) m. Mass 3 has \( m_3 = 1.5 \) kg and is at \( x = 4 \) m. Where is the center of mass?

\[
X_{\text{cm}} = \frac{1}{M_{\text{total}}} \sum_i m_i x_i \quad Y_{\text{cm}} = Z_{\text{cm}} = 0
\]

\[
= \frac{1}{(2 + 0.5 + 1.5) \text{ kg}} (2\text{ kg} \times 1\text{ m} + 0.5\text{ kg} \times 3\text{ m} + 1.5\text{ kg} \times 4\text{ m}) = \frac{9.5}{4} \text{ m} \approx 2.4\text{ m}
\]
Properties of the center of mass

- The velocity of the center of mass is given by the total momentum divided by the total mass. Therefore, the center of mass velocity of an isolated system is constant. Movie

\[ \vec{V}_{cm} = \frac{\vec{P}_{tot}}{M_{total}} = \frac{1}{M_{total}} \sum_i m_i \vec{v}_i \]

- If an external force acts on a system of particles, the center of mass follows a trajectory that this the same as would be followed by a single particle with mass \( M_{total} \). Movie