Rotational motion and gravity

Angular displacement

For circular motion, one can determine the position by an angle

\[ \theta = \frac{\text{arclength}}{\text{radius}} \]

\[ \Delta \theta = \theta_f - \theta_i \]

Units \( 1 \text{ rad} = \frac{360}{2\pi} \text{ deg} \)
Angular speed

The average angular speed of a rotating rigid object is the ratio of the angular displacement and the time interval

$$\bar{\omega} = \frac{\Delta \theta}{\Delta t}$$

Units: 1 rad / s

Similar to the instantaneous velocity one can define the instantaneous angular speed:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}$$

Angular acceleration

The average angular acceleration of a rotating rigid object is the ratio of the angular speed and the time interval

$$\bar{\alpha} = \frac{\Delta \omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$$

Units: 1 rad / s²

Similar to the instantaneous velocity one can define the instantaneous angular acceleration:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t}$$
Angular variables: question time

A particle has circular motion. Initially it is located at 10 deg and after a few minutes is has 370 deg angular position. Which of the following is true?

a) It moved in the clockwise direction
b) It went round 2 times
c) It went round once

Two particles are moving in a circle, A performs a complete cycle in 10 s, whilst B takes 20 s. What is the relation between their average angular speeds:

a) The same
b) A is larger than B
c) B is larger than A

Motion under constant angular acceleration

<table>
<thead>
<tr>
<th>Rotational Motion About a Fixed Axis with Constant Acceleration</th>
<th>Linear Motion with Constant Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega = \omega_i + \alpha t )</td>
<td>( v = v_i + at )</td>
</tr>
<tr>
<td>( \Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2 )</td>
<td>( \Delta x = v_i t + \frac{1}{2} at^2 )</td>
</tr>
<tr>
<td>( \omega^2 = \omega_i^2 + 2\alpha \Delta \theta )</td>
<td>( v^2 = v_i^2 + 2a \Delta x )</td>
</tr>
</tbody>
</table>
Balls through the track: question time

Which path does the ball take a) b) c)

Constant angular acceleration: example

A wheel rotates with a constant angular acceleration of $3.50 \text{ rad/s}^2$.

a) If the angular speed of the wheel is initially $2.00 \text{ rad/s}$, through what angle does the wheel rotate in the first $t=2.00 \text{ s}$?

b) What is the angular speed of the wheel at $t=2.00 \text{ s}$?
Linear and angular quantities

\[ \Delta \theta = \frac{\Delta s}{r} \]

\[ \Delta \theta = \frac{1}{\Delta t} \frac{\Delta s}{r \Delta t} \]

\[ \omega = \frac{v}{r} \]

\[ \alpha = \frac{a}{r} \]

Constant angular acceleration: example

A compact disc player is designed such that the read head moves out from the center of the disc, the angular speed of the disc changes so that the linear speed at the position of the head will always be constant (about 1.3 m/s).

a) Find the angular speed of the disc when the read head is at a distance of 2.0 cm from the center of the disc.

b) Find the linear speed of an old-fashion record player at 2.0 cm from the center for a 45.0 rpm record.
Centripetal and tangential acceleration

When the trajectory is not in a straight line we can decompose the acceleration into centripetal and tangential.

- **centripetal**: in the radial direction, responsible for the change in direction of the velocity vector
  \[ a_c = \frac{v^2}{r} = \omega^2 r \]

- **tangential**: tangent to the trajectory, responsible for the change in magnitude of the velocity vector
  \[ a_t = \frac{\Delta v}{\Delta t} \]

Angular variables: question time

In a particular trial run, a driver traveled at a constant speed of 50 m/s for a couple of laps.

1) the tangential acceleration in A is larger (a), smaller (b) or equal (c) to that in B?
2) The centripetal acceleration in A is larger (a), smaller (b) or equal (c) to that in B?
3) The angular speed in A is larger (a), smaller (b) or equal (c) to that in B?
Chips on a rotating plate: question time

Which chip falls off first
a) Red  
b) Blue  
c) Black  
d) All at the same time

Strategy for rotational motion

- draw free-body diagram, showing forces
- choose a coordinate system where one axis is perpendicular to the circular path
- find the net force toward the center of the circular path
- write down Newton’s second law along the tangent and the centripetal directions.
- solve for the unknowns
Centripetal acceleration: example

A car travels at constant speed 13.4 m/s on a level circular turn of radius 50.0 m. What is the minimum coefficient of static friction between the tires and the roadway so that the car can make the circular turn without sliding?

Centripetal acceleration: example

Consider the same car traveling at 13.4 m/s now on a road without friction that curves with a radius of 50.0 m. What should be the angle of the curve so that the car can do the turn safely?
Centripetal acceleration: example

A child swings a yo-yo of weight $mg$ in a horizontal circle so that the cord makes a constant angle (30 deg) with the vertical axis. Find the centripetal acceleration of the yo-yo.

Fictitious forces

When the reference frame is being accelerated, through a centripetal force or a tangential force, one has the perception that there are additional forces acting on the particle, the so called fictitious forces

$$\vec{F}_{\text{fict}} = -m\vec{a}_{\text{ref}}$$

Example: centrifugal force
Newton’s law of universal gravitation

\[ F_g = G \frac{m_1 m_2}{r^2} \]

\[ G = 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \]

Calculate \( g_{\text{surface}} \):
\[
F_g = G \frac{M_E m_{\text{ball}}}{R_E^2}
\]
\[
F_g = m_{\text{ball}} \left( G \frac{M_E}{R_E^2} \right)
\]
\[
F_g = m_{\text{ball}} \quad g
\]

\( G = 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \)
\( M_E = 5.98 \times 10^{24} \text{ kg} \)
\( R_E = 6.38 \times 10^6 \text{ m} \)
Gravitational acceleration and height

The gravitational acceleration on Earth changes with the height of the body!

\[
g_{\text{surface}} = G \frac{M_E}{R_E^2}
\]

\[
g_h = G \frac{M_E}{(R_E + h)^2}
\]

Gravitational Potential Energy

At the surface of the earth we defined the gravitational potential energy as:

\[
PE_g = mgh
\]

At an arbitrary height this expression is not valid as it assumes constant acceleration. Instead we will define:

\[
PE_g = -G \frac{M_E m}{(R_E + h)} = -G \frac{M_E m}{r}
\]

Where zero potential energy is at infinity.
Gravity revisited: question time

Which one is correct: As a ball drops and falls to the ground,

a) the force that the ball exerts on the earth is equal in magnitude to the force that the earth exerts on the ball.
b) the ball undergoes the same acceleration as the earth.
c) Earth is much more massive than the ball so the earth pulls much harder on the ball than the ball on the earth. Thus the ball falls while the earth remains stationary.

A satellite circles around the earth at a height $h=R_E$. An identical satellite circles around the earth at a height $3h$. How does the gravitational force on the second satellite compare with the first?

a) twice as large.
b) four times as large.
c) one-fourth as large.
d) half as large.

Escape speed

The escape speed is the speed that needs to be given to an object to free it from the gravitational field

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$

Calculate the escape speed from the Earth for a 5000 kg space craft:

**G**=6.673x10^{-11} N m^2/kg^2

**M_E**=5.98x10^{24} kg

**R_E**=6.38x10^6 m
Escape speed for various planets

**TABLE 7.2**

<table>
<thead>
<tr>
<th>Planet</th>
<th>$v_{esc}$ (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>4.3</td>
</tr>
<tr>
<td>Venus</td>
<td>10.3</td>
</tr>
<tr>
<td>Earth</td>
<td>11.2</td>
</tr>
<tr>
<td>Moon</td>
<td>2.3</td>
</tr>
<tr>
<td>Mars</td>
<td>5.0</td>
</tr>
<tr>
<td>Jupiter</td>
<td>60.0</td>
</tr>
<tr>
<td>Saturn</td>
<td>36.0</td>
</tr>
<tr>
<td>Uranus</td>
<td>22.0</td>
</tr>
<tr>
<td>Neptune</td>
<td>24.0</td>
</tr>
<tr>
<td>Pluto</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Kepler’s laws

1. All planets move in elliptical orbits with the Sun at one of the focal points

2. A line drawn from the Sun to any planet sweeps out equal areas in equal time intervals

3. The square of the orbital period of any planet is proportional to the cube of the average distance from the planet to the Sun.
Kepler’s third law

\[ R^3 = K_s T^2 \]

\[ K_s = \frac{GM_s}{4\pi^2} \]

Gravitational orbits: example

Geosynchronous orbits are such that the satellite is always at the same location with respect to the surface of the Earth. For this to occur the satellite must be at a height such that its revolution period is the same as the Earth. Prove that this height is given by:

\[ h_{surface} = \left( \frac{GM_E}{4\pi^2} \right)^{1/3} - R_E \]