3

Scattering theory

3.1 Introduction

3.1.1 Cross sections and scattering amplitudes

The scattering of one particle on another may be reduced, as seen in Chap. 1, to a dynamical problem involving only their relative coordinate $R$. If the energy of relative motion is $E$ and the reduced mass is $\mu$, then the corresponding wave number is related by $E = \hbar^2 k^2 / 2\mu$, or $k = \sqrt{2\mu E / \hbar^2}$. The relative velocity is $v = p/\mu = \hbar k/\mu$.

Let us choose a coordinate system so that the beam is a plane wave $Ae^{ikz}$ in the $+z$ direction, and that the target is centred at the origin. The scattered wave will be a spherical outgoing wave from the target, so will be asymptotically a function of radius $R$ only. The scattered wave will be different in various directions $(\theta, \phi)$, and these amplitudes will decay as $R^{-1}$ so the flux integrated over all angles remains constant. The combined incident and scattering asymptotic form is then

$$\psi_{\text{asym}} = A \left[ e^{ikz} + f(\theta, \phi)e^{ikR}/R \right]$$

(3.1)

The description ‘asymptotic’ here implies that this is the form in free space when outside the range of the interaction potential between the particles. The function $f(\theta, \phi)$ is called the scattering amplitude, and in general is complex-valued.

In order to calculate the scattering cross sections $\sigma$, we need first to know the fluxes of incident and scattered particles, which is the number of particles per second per unit area. In general, for a wave function $\psi$ and velocity $v$, the value of the flux is

$$j = v|\psi|^2.$$  

(3.2)

This means that the magnitude of the incident flux is $j_i = v|A|^2$ with $v =$
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Fig. 3.1. A plane wave in the $+z$ direction incident on a spherical target, giving rise to spherically-outgoing scattering waves.

$\hbar k/\mu$, and of the scattered flux is

\[
j_f = v|A|^2|f(\theta, \phi)|^2/R^2 \text{ particles/area/sec} \quad (3.3)
\]
\[
j_f = v|A|^2|f(\theta, \phi)|^2 \text{ particles/steradian/sec}, \quad (3.4)
\]

where the solid angle subtended by area $a$ at radius $R$ is $a/R^2$ steradian (sr). In Chap. 1, the cross section was defined as the ratio of scattered to incident flux, so here

\[
\sigma(\theta, \phi) = |f(\theta, \phi)|^2, \quad (3.5)
\]

which is independent of $A$, and has units of area/sr.

### 3.1.2 Partial wave scattering from a finite spherical potential

To find the scattering amplitude $f(\theta, \phi)$ in terms of the projectile target potential $V(R)$, and hence the cross section, we consider first the case of a spherical potential of finite range. That is the potential $V(R)$ only depends on the magnitude $R$, and that $V(R) = 0$ for $R \geq R_m$, where $R_m$ is the finite range of the potential. Keep the $z$-axis as the beam direction as in Fig. 3.1.

The time-independent Schrödinger equation for the relative motion with c.m. energy $E$ is

\[
[T + V - E]u(R, \theta, \phi) = 0, \quad (3.6)
\]

where the kinetic energy operator is $T = -\frac{\hbar^2}{2\mu} \nabla^2$. For spherical potentials as in Fig. 3.1, there is complete cylindrical symmetry for the potentials about the $z$-axis, and hence also for the wave function and the scattering amplitude, so both of these are no longer functions of $\phi$. 
3.1 Introduction

Partial wave expansions

The wave function \( u(R, \theta) \) is now expanded using Legendre polynomials

\[
\begin{align*}
    u(R, \theta) &= \sum_{L=0}^{\infty} \frac{(2L+1)}{2L+1} i^L P_L(\cos \theta) \frac{1}{R} \chi_L(R) . \\
    \text{(3.7)}
\end{align*}
\]

We do this since the \( P_L(\cos \theta) \) for integer \( L = 0, 1, \ldots \) are a orthonormal and complete set according to

\[
\int_0^\pi P_L(\cos \theta) P_L'(\cos \theta) \sin \theta d\theta = \frac{2}{2L+1} \delta_{LL'} .
\]

(3.8)

The further coefficients in Eq. (3.7) are arbitrary at this stage, and are chosen for later convenience. The \( 1/R \) factor implies that \( \chi_L(0) = 0 \) always.

Using the differential properties of the Legendre polynomials, Eq. (3.7) implies that for each \( L \) value we have to solve separately a partial wave equation

\[
\begin{align*}
    \left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} \right] + V(R) - E \right] \chi_L(R) = 0 .
\end{align*}
\]

(3.9)

One boundary condition already known is that \( \chi_L(0) = 0 \). The other is fixed by the large \( R \) behaviour, so that it reproduces the asymptotic form of Eq. (3.1) for some \( A, \sqrt{2\mu E/\hbar^2} \). We now fix \( A = 1 \), since the cross section is independent of \( A \), thus the other boundary condition is

\[
\begin{align*}
    u(R, \theta) \to_{R>R_m} e^{ikz} + f(\theta) \frac{e^{ikR}}{R} .
\end{align*}
\]

(3.10)

As usual in quantum mechanical matching, both the functions and their derivatives must agree continuously.

Plane wave solutions

For \( R > R_m \) where \( V(R) = 0 \), the partial wave equation Eq. (3.9) can be rewritten by a change of variable from \( R \) to \( \rho = kR \), giving

\[
\begin{align*}
    \left[ \frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} + 1 \right] \chi_L^{\text{asym}}(R) = 0 .
\end{align*}
\]

(3.19)

This equation for the asymptotic form \( \chi_L^{\text{asym}}(R) \) is a special case for \( \eta = 0 \) of the more general Coulomb wave equation

\[
\begin{align*}
    \left[ \frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2\eta}{\rho} + 1 \right] F_L(\eta, \rho) = 0
\end{align*}
\]

(3.20)
Table 3.1. Coulomb Functions

That $F_L(\eta, \rho)$ is regular means $F_L(\eta, \rho = 0) = 0$ and $G_L(\eta, 0) \neq 0$ implies irregularity. They are related by the Wronskian

$$G_L(\eta, \rho) \frac{dF_L(\eta, \rho)}{d\rho} - F_L(\eta, \rho) \frac{dG_L(\eta, \rho)}{d\rho} = 1$$

or $W(G, F) \equiv GF' - G'F = k$. (3.11)

Note that mathematics texts such as [1] usually define $G'$ as $dG/d\rho$, which we will denote if necessary by $\dot{G}$. We will henceforth use the prime to refer to derivatives with respect to $R$, so $G' = k\dot{G}$, etc. The Wronskian is $G\dot{F} - \dot{G}F = 1$.

It will often be more convenient to use the combinations of $F$ and $G$ that look like outgoing $e^{ikR}$ and ingoing $e^{-ikR}$ exponentials,

$$H_+^L(\eta, \rho) = G_L(\eta, \rho) \pm iF_L(\eta, \rho),$$

called Hankel Functions.

Coulomb Functions for $\eta = 0$
The $\eta = 0$ functions are more directly known in terms of Bessel functions:

$$F_L(0, \rho) = \rho j_L(\rho) = (\pi \rho/2)^{1/2}J_{L+1/2}(\rho)$$

$$G_L(0, \rho) = -\rho y_L(\rho) = -(\pi \rho/2)^{1/2}Y_{L+1/2}(\rho),$$

where the irregular spherical Bessel function $y_L(\rho)$ is sometimes written as $n_L(\rho)$ (the Neumann function). The functions for the first few $L$ values are

$$F_0(0, \rho) = \sin \rho,$$

$$G_0(0, \rho) = \cos \rho;$$

$$F_1(0, \rho) = (\sin \rho - \rho \cos \rho) / \rho,$$

$$G_1(0, \rho) = (\cos \rho + \rho \sin \rho) / \rho;$$

$$F_2(0, \rho) = ((3 - \rho^2) \sin \rho - 3\rho \cos \rho) / \rho^2,$$

$$G_2(0, \rho) = ((3 - \rho^2) \cos \rho + 3\rho \sin \rho) / \rho^2.$$ (3.14)

Their behaviour near the origin is

$$F_L(0, \rho) \sim \frac{1}{(2L+1)(2L-1) \cdots 3.1} \rho^{L+1}$$

$$G_L(0, \rho) \sim (2L - 1) \cdots 3.1 \rho^{-L},$$

and their asymptotic behaviour when $\rho \gg L$ is

$$F_L(0, \rho) \to_{\rho \to \infty} \sin(\rho - L\pi/2)$$

$$G_L(0, \rho) \to_{\rho \to \infty} \cos(\rho - L\pi/2)$$

$$H_+^L(0, \rho) \to_{\rho \to \infty} e^{\pm i(\rho - L\pi/2)} = i^{L+1} e^{\pm i \rho}.$$ (3.17)

The $H_+^L$ is therefore an outgoing wave, and $H_-^L$ an incoming wave.

Coulomb functions for $\eta \neq 0$ are discussed on p. 67.
3.1 Introduction

for $\eta = 0$. This second-order equation has two linearly independent solutions: the regular function $F_L(\eta, \rho)$ and the irregular function $G_L(\eta, \rho)$ as defined in Abramowitz and Stegun [1, ch. 14].

In this section we need just the $\eta = 0$ special cases (see Table 3.1), but we begin here with a definition of the complete Coulomb functions, as the $2\eta/\rho$ term will reappear in the next section when Coulomb potentials are introduced.

The regular solution $F_L(\eta = 0, kR)$ is used in the partial wave expansion of a plane wave. That is, a linear combination of the $F_L(\eta = 0, kR)$ gives $e^{ikz}$, namely

$$e^{ikz} = \sum_{L=0}^{\infty} (2L + 1) i^L P_L(\cos \theta) \frac{1}{kR} F_L(0, kR). \quad (3.21)$$

In terms of the $H^\pm_L$, this plane wave expansion can be written

$$e^{ikz} = \sum_{L=0}^{\infty} (2L + 1) i^L P_L(\cos \theta) \frac{1}{kR} i^2 (H^-_L(0, kR) - H^+_L(0, kR)). \quad (3.22)$$

**Radial solutions with a potential**

When $V(R) \neq 0$, we may integrate outwards a trial solution $\chi_L(R)$ of Eq. (3.9) starting with $\chi_L(0) = 0$ and some finite $\chi'_L(0) \neq 0$. In the asymptotic region, $\chi_L(R)$ will necessarily be a linear combination of some two linearly independent solutions of Eq. (3.19), such as $H^+_L(0, kR) \sim e^{ikR}$ and $H^-_L(0, kR) \sim e^{-ikR}$:

$$\chi_{asym}^L(R) \overset{R > R_m}{=} A_L \left[ H^-_L(0, kR) - S_L H^+_L(0, kR) \right], \quad (3.23)$$

for some complex constants $A_L$ and $S_L$. The $S_L$ is called the partial wave $S$ matrix element, and will be unity for zero potential $V(R)$, as the solution must then be proportional to $F_L(0, kR)$ only. The $S_L$ is determined from $\chi_L(R)$ by matching both sides of Eq. (3.23) and their derivatives at some radius $R \geq R_m$. This is most conveniently done by means of the inverse logarithmic derivative, which is called the $R$ matrix element:

$$R_L = \frac{1}{R_m} \chi_L(R_m), \quad (3.24)$$

where the $R_m^{-1}$ factor follows [2] and keeps the $R$ matrix dimensionless.

Given a trial solution $\chi_L(R)$ in the interior region, although its absolute normalisation is not yet known, its logarithmic derivative unambiguously determines $R_L$. The $R_L$ then determines the $S$ matrix element uniquely by
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matching with the inverse logarithmic derivative of the right side of Eq. (3.23):

\[ R_L = \frac{1}{R_m} \frac{H_L^- - S_L H_L^+}{H_L^- - S_L H_L^+}, \quad (3.25) \]

so

\[ S_L = \frac{H_L^- - R_m R_L H_L^-}{H_L^- - R_m R_L H_L^+}, \quad (3.26) \]

where the derivatives are with respect to \( R \). The \( S \) matrix elements \( S_L \) are thereby uniquely determined by the potential, and next we use them to find the scattering amplitude \( f(\theta) \).

Equation (3.23) implies that the full scattering wave function has the partial wave asymptotic form

\[ u(R, \theta) \rightarrow \frac{1}{R} \sum_{L=0}^{\infty} (2L + 1) i^L P_L(\cos \theta) \times A_L \left[ H_L^-(0, kR) - S_L H_L^+(0, kR) \right], \quad (3.27) \]

which we now have to match with Eq. (3.10) in order to determine \( f(\theta) \) in terms of the \( S_L \).

Substituting Eq. (3.22) in Eq. (3.10), equating to the right hand side of Eq. (3.27), multiplying by \( R \) and using the asymptotic forms of Eq. (3.18) of the \( H_L^\pm \) functions, we have

\[ \sum_{L=0}^{\infty} (2L + 1) i^L P_L(\cos \theta) A_L \left[ i^L e^{-ikR} - S_L i^{-L} e^{ikR} \right] = \sum_{L=0}^{\infty} (2L + 1) i^L P_L(\cos \theta) \frac{i}{2k} (i^L e^{-ikR} - i^{-L} e^{ikR}) + f(\theta) e^{ikR} \quad (3.28) \]

Collecting together the separate terms with \( e^{ikR} \) and \( e^{-ikR} \) factors, we find

\[ e^{ikR} \left[ \sum_{L=0}^{\infty} (2L + 1) i^L P_L(\cos \theta) \left\{ A_L S_L i^{-L} - \frac{i}{2k} i^{-L} \right\} \right] \]

\[ = e^{-ikR} \left[ \sum_{L=0}^{\infty} (2L + 1) i^L P_L(\cos \theta) \left\{ A_L i^L - \frac{i}{2k} i^L \right\} \right] \quad (3.29) \]

Since the \( e^{\pm ikR} \) are linearly independent, and the two \([..]\) expressions in this equation are independent of \( R \), they must each be identically zero. Furthermore, using the orthogonality of the Legendre polynomials, the second \{..\}
expression must also be zero, which implies $A_L = i/2k$. Substituting this result into the first zero expression, we derive

$$f(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L + 1) P_L(\cos \theta) (S_L - 1).$$ (3.30)

This important equation (3.30) constructs the scattering amplitude in terms of the $S$ matrix elements. The resulting full scattering wave function is

$$u(R, \theta) = \sum_{L=0}^{\infty} (2L + 1) i^L P_L(\cos \theta) \frac{1}{R} \chi_L(R),$$ (3.31)

where the boundary condition for $R > R_m$ is in detail

$$k \chi_{\text{asym}}^L (R) = \frac{i}{2} \left[ H_L^- (0, kR) - S_L H_L^+ (0, kR) \right].$$ (3.32)

**Phase shifts**

From each $S$ matrix element $S_L$ we may construct a phase shift $\delta_L$ for each partial wave by $S_L = e^{2i\delta_L}$, or $\delta_L = \frac{1}{2i} \ln S_L$. In terms of this phase shift, the asymptotic form of Eq. (3.32) of the scattering wave function may be rewritten as

$$k \chi_{\text{asym}}^L (R) = e^{i\delta_L} \left[ \cos \delta_L F_L(0, kR) + \sin \delta_L G_L(0, kR) \right],$$ (3.33)

and the scattering amplitude now appears as

$$f(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L + 1) P_L(\cos \theta) e^{i\delta_L} \sin \delta_L.$$ (3.34)

The solution can be also be written as

$$k \chi_{\text{asym}}^L (R) = F_L(0, kR) + T_L H_L^+ (0, kR),$$ (3.35)

where $T_L = e^{i\delta_L} \sin \delta_L$ is called the partial wave $T$ matrix element. It is the coefficient of $H_L^+ (0, kR)$, an outgoing wave, and is related to the $S$ matrix element by $S_L = 1 + 2i T_L$. The scattering amplitude in terms of the $T_L$ is

$$f(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L + 1) P_L(\cos \theta) T_L.$$ (3.36)

For zero potential, $\delta_L = T_L = 0$, and $k \chi_{\text{asym}}^L (R) = F_L(0, kR)$ only, the regular partial wave component of the incident plane wave.

A third asymptotic form is

$$k \chi_{\text{asym}}^L (R) = e^{i\delta_L} \cos \delta_L \left[ F_L(0, kR) + K_L G_L(0, kR) \right],$$ (3.37)
Table 3.2. Relations between the wave functions and the phase shifts, $K$, $T$ and $S$ matrix elements, for a specific partial wave. The last two lines list the consequences of zero and real potentials.

<table>
<thead>
<tr>
<th>K</th>
<th>T</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{i\delta}[F \cos \delta + G \sin \delta]$</td>
<td>$\frac{1}{1-iK}$</td>
<td>$[F + K G]$</td>
</tr>
<tr>
<td>$\delta = \delta$</td>
<td>$\arctan K$</td>
<td>$\arctan \frac{T}{1+iT}$</td>
</tr>
<tr>
<td>$K = \tan \delta$</td>
<td>$K$</td>
<td>$\frac{T}{1+iT}$</td>
</tr>
<tr>
<td>$T = e^{i\delta} \sin \delta$</td>
<td>$\frac{1-iK}{1+iK}$</td>
<td>$T$</td>
</tr>
<tr>
<td>$S = e^{2i\delta}$</td>
<td>$\frac{1+iK}{1-iK}$</td>
<td>$1+2iT$</td>
</tr>
</tbody>
</table>

with $K_L = \tan \delta_L$ called the partial wave $K$ matrix element. The $S$ matrix element

$$ S_L = \frac{1+iK_L}{1-iK_L}. \quad (3.38) $$

For zero potential, $K_L = 0$. The $K$ matrix element may be directly found from the $R$ matrix element $R_L$ of the interior solution at some radius $R_m$ as

$$ K_L = -\frac{F_L - R_m R_L F'_L}{G_L - R_m R_L G'_L}. \quad (3.39) $$

All of the above scattering phase shifts $\delta_L$ and partial wave elements $T_L$, $S_L$ and $K_L$ are independent of $R_m$ provided that it is larger than the range of the potential.

From Eq. (3.39) the consequences of $V(R)$ being real are clear. In this case, the trial function $\chi_L(R)$ will be real, and hence also $R_L$ by Eq. (3.24), $K_L$ by Eq. (3.39), and hence $\delta_L$ will be real since $\tan \delta_L = K_L$. It is for these reasons that scattering from a real potential is most often described by a (real) phase shift. The $S$ matrix $S_L = e^{2i\delta_L}$ then has unit modulus $|S_L| = 1$. The relations between the phase shifts and the $K$, $T$ and $S$ matrix elements are summarised in Table 3.2.
3.1 Introduction

Total cross sections

The cross section $\sigma(\theta) = |f(\theta)|^2$ integrated over the entire sphere gives

$$\sigma_T = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \sigma(\theta)$$

$$= 2\pi \int_0^\pi d\theta \sin \theta |f(\theta)|^2$$

$$= \frac{4\pi}{k^2} \sum_{L=0}^\infty (2L + 1) \sin^2 \delta_L$$

$$= \frac{\pi}{k^2} \sum_{L=0}^\infty (2L + 1)|1 - S_L|^2,$$

(3.40)

using the orthogonality and normalisation Eq. (3.8) of the Legendre polynomials.

The Optical Theorem relates the total cross section $\sigma_T$ to the zero-angle scattering amplitude. Using $P_L(1) = 1$, we have :

$$f(0) = \frac{1}{2ik} \sum_{L=0}^\infty (2L + 1)(e^{2i\delta_L} - 1),$$

(3.41)

so

$$\Im f(0) = \frac{1}{k} \sum_{L=0}^\infty (2L + 1) \sin^2 \delta_L$$

$$= \frac{k\sigma_T}{4\pi}.$$  

(3.42)

This relation exists because the incident and scattered waves at zero scattering angle have a fixed relative phase, and interfere in a manner that portrays the total loss of flux from the incident wave to the scattered waves.

Scattering from an arbitrary incident direction

To find the scattering from an incident beam in direction $k$ to direction $k'$, the Legendre polynomial $P_L(\cos \theta) = P_L(\hat{k} \cdot \hat{k}')$ in Eq. (3.30) may be replaced using the addition theorem for spherical harmonics,

$$P_L(\cos \theta) = \frac{4\pi}{2L + 1} \sum_{M} Y_L^M(\hat{k})Y_L^M(\hat{k}')^*,$$

(3.43)

giving a generalisation of Eq. (3.21):

$$e^{ik \cdot R} = 4\pi \sum_{LM} i^L Y_L^M(\hat{R})Y_L^M(\hat{k})^* \frac{1}{kR} F_L(0, kR).$$

(3.44)
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Table 3.3. Classical Coulomb scattering

The classical scattering orbit in a Coulomb potential is a hyperbola, in which the distance of the projectile in polar coordinates \((R, \alpha)\), starting at \(\alpha = \pi\), is

\[
\frac{1}{R(\alpha)} = \frac{1}{b} \sin \alpha - \frac{D}{2b^2}(1 + \cos \alpha)
\]

(3.47)

where \(D = Z_1 Z_2 e^2 / E = 2\eta/k\) is the distance of closest approach in a head-on collision, and \(b\) is the impact parameter for a general collision. As \(R \to \infty\) after the collision, \(\alpha\) becomes the scattering angle \(\theta\), which from Eq. (3.47) is

\[
\tan \frac{\theta}{2} = \frac{\eta}{bk}.
\]

(3.48)

The distance of closest approach for arbitrary scattering angle \(\theta\) is

\[
R_{\text{near}}(\theta) = \frac{\eta}{k} \left[ 1 + \csc \frac{\theta}{2} \right],
\]

(3.49)

and the classical differential cross section, using Eq. (3.48) is

\[
\sigma(\theta) \equiv \frac{b(\theta)}{\sin \theta} \frac{db}{d\theta} = \frac{\eta^2}{4k^2 \sin^2(\theta/2)}.
\]

(3.50)

Thus the general scattering amplitude is

\[
f(k' \leftarrow k) = \frac{2\pi}{ik} \sum_{LM} Y^M_L(\hat{k}')Y^M_L(\hat{k})^* (e^{2i\delta_L} - 1),
\]

(3.45)

and the full scattering wave function is

\[
\psi(R; k) = 4\pi \sum_{LM} i^L Y^M_L(\hat{R})Y^M_L(\hat{k})^* \frac{1}{R} \chi_L(R).
\]

(3.46)

Note that a spherical harmonic in the \(+z\) direction is \(Y^M_L(0, 0) = \delta_{M0} \sqrt{\frac{2L+1}{4\pi}}\).

3.1.3 Coulomb and nuclear potentials

Pure point Coulomb scattering

If we consider only the Coulomb potential between two particles with charges \(Z_1\) and \(Z_2\), each assumed to be pointlike, then it is

\[
V_c(R) = \frac{Z_1 Z_2 e^2}{R}.
\]

(3.60)
Table 3.4. Coulomb Functions for $\eta \neq 0$

The functions $F_L(\eta, \rho)$, $G_L(\eta, \rho)$ and $H^\pm_L(\eta, \rho)$ are the solutions of Eq. (3.20) for $\eta \neq 0$. In terms of a ‘Coulomb constant’

$$C_L(\eta) = \frac{2^L e^{-\pi \eta/2} |\Gamma(1 + L + i\eta)|}{(2L + 1)!}$$

(3.51)

and the confluent hypergeometric function $\mathrm{I}_1(a; b; z) \equiv M(a, b, z)$, the regular function is defined as

$$F_L(\eta, \rho) = C_L(\eta) \rho^{L+1} e^{\mp i\rho} \mathrm{I}_1(L+1 \mp i\eta; 2L+2; \pm 2i\rho),$$

(3.52)

where either the upper or lower signs may be taken for the same result. The irregular functions have the corresponding definition

$$H^\pm_L(\eta, \rho) = e^{\pm i\theta} (\mp 2i\rho)^{1+L+\eta} M(1+L \pm i\eta, 2L+2, \mp 2i\rho),$$

(3.53)

where $\theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta \ln(2\rho)$ and

$$\sigma_L(\eta) = \arg \Gamma(1 + L + i\eta)$$

(3.54)

is called the Coulomb phase shift. The functions may easily be calculated numerically [4], also for complex arguments [5].

Their behaviour near the origin is thus

$$F_L(\eta, \rho) \sim C_L(\eta) \rho^{L+1}, \quad G_L(\eta, \rho) \sim [(2L+1)C_L(\eta) \rho^L]^{-1},$$

(3.55)

noting that

$$C_0(\eta) = \sqrt{\frac{2\pi\eta}{e^{2\pi\eta} - 1}} \quad \text{and} \quad C_L(\eta) = \frac{\sqrt{L^2 + \eta^2}}{L(2L+1)} C_{L-1}(\eta).$$

(3.56)

A transition from small-$\rho$ power law behaviour to large-$\rho$ oscillatory behaviour occurs outside the classical turning point. This point is where $1 = 2\eta/\rho + L(L + 1)/\rho^2$, namely

$$\rho_{tp} = \eta \pm \sqrt{\eta^2 + L(L + 1)}. \quad \text{(3.57)}$$

In nuclear reactions $\eta$ is usually positive, so with purely Coulomb and centrifugal potentials there is only one turning point. Classically the turning point is at the distance of closest approach, $R_{\text{near}}$ of Eq. (3.49), and these quantities are related by $\rho_{tp} = kR_{\text{near}}$ if the classical impact parameter $b$ is related to the quantum mechanical partial wave $L$ according to

$$k b = \sqrt{L(L + 1)} \approx L + \frac{1}{2}. \quad \text{(3.58)}$$

The asymptotic behaviour of the Coulomb functions outside the turning point ($\rho \gg \rho_{tp}$) is

$$F_L(\eta, \rho) \sim \sin(\theta), \quad G_L(\eta, \rho) \sim \cos(\theta) \quad \text{and} \quad H^\pm_L(\eta, \rho) \sim e^{\pm i\theta}. \quad \text{(3.59)}$$
For scattering with relative energy $E$ (velocity $v$), we define a dimensionless Sommerfeld parameter, as in Chapter 1,

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v} = \frac{Z_1 Z_2 e^2 \mu}{\hbar^2 k} .$$

(3.61)

A plane wave is no longer $e^{ik \cdot R}$, but has logarithmic terms. The general Coulomb distorted wave for asymptotic momentum $k$ is [3]

$$\psi_C(k, R) = e^{ik \cdot R} e^{-\pi \eta/2} \Gamma(1 + i\eta) \frac{1}{\Gamma(1 + i\eta)} F_1(-i\eta; 1; i(kR - k \cdot R)) ,$$

(3.62)

with Gamma function $\Gamma(x)$ and confluent hypergeometric function $1 F_1(a; b; z)$. In partial wave form, the generalisation of the standard partial wave expansion of Eq. (3.21) is, with $k$ in the $+\hat{z}$ direction,

$$\psi_C(k, R) = \sum_{L=0}^{\infty} (2L + 1) \frac{P_L(\cos \theta)(e^{2i\sigma_L(\eta)} - 1)}{k R^L} F_L(\eta, kR) ,$$

(3.63)

where the Coulomb phase shift $\sigma_L(\eta)$ is defined in Eq. (3.54). The asymptotic forms for small and large $\rho$ are given in Table 3.4. The asymptotic outgoing part in $\psi_C(k, R)$ has an amplitude which, following the pattern of Eq. (3.30), is formally the partial wave sum

$$f_c(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L + 1) P_L(\cos \theta)(e^{2i\sigma_L(\eta)} - 1) ,$$

(3.64)

again using the Coulomb phase shift of Eq. (3.54). However, this series expression does not converge. It only has meaning if a screened Coulomb potential is used, and then the screening radius let tend to infinity [6, §14-a]. If Eq. (3.62) is used directly, the asymptotic amplitude is found [6, §14-a] to be

$$f_c(\theta) = -\frac{\eta}{2k \sin^2(\theta/2)} \exp \left( -i\eta \ln(\sin^2(\theta/2)) + 2i\sigma_0(\eta) \right) ,$$

(3.65)

which is called the point Coulomb scattering amplitude. The point-Coulomb cross section is therefore

$$\sigma_R(\theta) = |f_c(\theta)|^2 = \frac{\eta^2}{4k^2 \sin^4(\theta/2)} ,$$

(3.66)

and is called the Rutherford cross section, because it happens to be the same as in classical scattering theory (see box).
Coulomb+nuclear scattering

When the scattering potential is \( V(R) = V_c(R) + V_n(R) \) for some potential \( V_n(R) \) in addition to the point Coulomb potential of Eq. (3.60), the phase shift will also be changed. Define the additional nuclear phase shift \( \delta_n^L \) by

\[
\delta_n^L = \sigma_L(\eta) + \delta_n^L.
\] (3.67)

The combined scattering amplitude \( f_{nc}(\theta) \) will, using Eq. (3.30), will have a factor

\[
e^{2i\delta_n^L} - 1 = (e^{2i\sigma_L(\eta)} - 1) + e^{2i\sigma_L(\eta)}(e^{2i\delta_n^L} - 1),
\] (3.68)

so that the partial wave sums will be a combination of the point Coulomb amplitude of Eq. (3.64) and an additional ‘Coulomb-distorted nuclear amplitude’ \( f_n(\theta) \):

\[
f_{nc}(\theta) = f_c(\theta) + f_n(\theta).
\] (3.69)

To find the Coulomb-distorted nuclear phase shift \( \delta_n^L \), or equivalently the \( S \) matrix element \( S_n^L = e^{2i\delta_n^L} \), we use a formula derived from Eq. (3.10), namely

\[
k\chi_{L,\text{sym}}(R) = i \left( H_L^- (\eta, kR) - S_n^L H_L^+ (\eta, kR) \right),
\] (3.70)

which simply to restore the first argument \( \eta \) for the Coulomb functions to the value given in Eq. (3.61). The new scattering amplitude for the nuclear potential on top of the point Coulomb scattering is found by using the second term in Eq. (3.68):

\[
f_n(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L + 1) P_L(\cos \theta) e^{2i\sigma_L(\eta)}(S_n^L - 1),
\] (3.71)

Note that we can optionally multiply both \( f_c \) of Eq. (3.65) and \( f_n \) of Eq. (3.71) by a phase factor such as \( \exp(-2i\sigma_0(\eta)) \), as only relative phase makes any difference to the Coulomb+nuclear cross section

\[
\sigma_{nc}(\theta) = |f_c(\theta) + f_n(\theta)|^2 = |f_{nc}(\theta)|^2.
\] (3.72)

Because this cross section diverges to infinity at small angles, very often elastic scattering cross sections are presented numerically in terms of their ratio to Rutherford

\[
\sigma / \sigma_R = \sigma_{nc}(\theta) / \sigma_R(\theta),
\] (3.73)

which becomes unity at small angles.
When cross sections are plotted as excitation functions of energy, there are often a range of narrower and wider peaks. Many of these peaks are caused by resonances, when the interacting particles are trapped together inside a potential barrier for some period of time $\tau$. This gives rise to variations in the excitation function with widths $\Gamma \sim \hbar/\tau$. These variations can be peaks, interference dips, or a peak next to a dip: all these patterns may result from a single resonance interfering with other scattering mechanisms.

A resonance is described by its total spin and parity, called $J^\pi_T$ in the next section §3.2.1, along with its energy $E_R$ and width $\Gamma$ in units of energy. It will typically show up in all partial waves coupled to the total $J^\pi_T$, characteristically as a rapid rise of the scattering phase shift $\delta(E)$ as seen in Fig. 3.2. In the one channel case, this is with the form

$$\delta(E) = \delta_{bg}(E) + \delta_{\text{res}}(E)$$

where

$$\delta_{\text{res}}(E) = \arctan \left( \frac{\Gamma/2}{E_R - E} \right),$$

for $\Gamma > 0$, and for some background phase shift $\delta_{bg}(E)$ that varies only slowly over the interval $E_R \pm \Gamma$. A resonance with this pattern is called
3.1 Introduction

Fig. 3.3. Possible Breit-Wigner resonances. The upper panel shows resonant phase shifts with several $\delta_{bg} = 0, \pi/4, \pi/2$ and $3\pi/4$, and the lower panel gives the corresponding contributions to the total elastic scattering cross section from that partial wave.

a pure Breit-Wigner resonance. If the background phase shift $\delta_{bg}(E) \sim 0$, then by Eq. (3.40) it produces a contribution to the total cross section of

$$\sigma_T^{\text{res}} \simeq \frac{4\pi}{k^2} (2L + 1) \sin^2 \delta_{\text{res}}(E)$$

$$= \frac{4\pi}{k^2} (2L + 1) \frac{\Gamma^2}{4(E - E_R)^2 + \Gamma^2}, \quad (3.75)$$

which shows a clear peak at $E \sim E_R$. If $\delta_{bg}(E) \neq 0$ then Fig. 3.3 shows some of other patterns that may be produced. If moreover $\delta_{bg}(E)$ varies with energy, a resonance may still exist even though the phase shift does not pass $\pi/2$. In Ch. 7, we will see in Fig. 7.1 that this is the case with the $p_{1/2}$ resonance in Fig. 3.2, for which a complex resonance pole is present at $1.9 - 6.1i/2$ MeV. A wide resonance (large $\Gamma$) such as this one has a broad effect on scattering over a large energy range.

An analysis of the propagation of a wave packet in the presence of a resonance shows that there is a time delay of $\tau \sim d\delta_{\text{res}}(E)/dE$, so the form of Eq. (3.74) implies that $\tau \sim 1/\Gamma$ as earlier expected.

If the S matrix is calculated for a Breit-Wigner resonance using the parametrisation of Eq. (3.74), then, using $K = \tan \delta_{\text{res}}(E) = \frac{1}{2}\Gamma/(E_R - E)$
and Eq. (3.38),

\[ S(E) = e^{2i\delta(E)} \frac{E - E_R - i\Gamma/2}{E - E_R + i\Gamma/2}. \]  (3.76)

This expression shows that if the function \( S(E) \) were continued to complex energies \( E \), it would have pole, where the denominator is zero, at \( E_p = E_R - i\Gamma/2 \). This is a pole in the fourth quadrant of the complex energy plane, near but below the real energy axis. The existence of such a complex pole leads most theorists to define resonance by its pole position, by the parameters \( J^\pi \), \( E \), and \( \Gamma \). This definition is applicable to multichannel as well as one-channel scattering.

Mathematically, a resonance is a pole of the S matrix in the fourth complex energy quadrant, with \( \Re E_p > 0 \) and \( \Im E_p < 0 \). In this complex plane, bound states are also poles, on the negative real energy axis. If the binding potential becomes weaker, bound states move toward zero energy, and then become resonances if there is a reflecting barrier potential. This reflecting barrier could be a repulsive Coulomb barrier for proton–nucleus scattering, or a centrifugal barrier for either neutrons or protons in angular momentum \( L > 0 \) states. The width of the resonance is extremely sensitive to the height of these barriers.

The case of neutral scattering in \( L = 0 \) partial waves deserves special attention, since here there is no repulsive barrier to trap e.g. a s-wave neutron. There is no Breit-Wigner form now, and mathematically the S matrix pole \( S_p \) is found to be on the negative imaginary \( k \) axis. This corresponds to a negative real pole energy, but this is not a bound state, for which the poles are always on the positive imaginary \( k \) axis. The neutral unbound poles are called virtual states, to be distinguished from both bound states and resonances. The dependence on the sign of \( k_p = \pm \sqrt{2\mu E_p/h^2} \) means we should write the S matrix as a function of \( k \) not \( E \). A pure virtual state has pole at \( k_p = i/a \) on the negative imaginary axis, described by a negative value of \( a \) called the scattering length. This corresponds to the analytic form

\[ S(k) = \frac{k + i/a}{k - i/a}, \]  (3.77)

giving \( \delta(k) = -\arctan ak \), or \( k \cot \delta(k) = -1/a \). These formulae describe the phase shift behaviour close to the pole, in this case for low momenta where \( k \) not too much larger than \( 1/|a| \). For more discussion see for example [6, §13-b].
3.1 Introduction

Table 3.5. Typical nuclear scattering potentials

The interaction potential between a nucleon and a nucleus is usually well described by an attractive nuclear well of the form

\[
V(R) = - \frac{V_{ws}}{1 + \exp \left( \frac{R - R_{ws}}{a_{ws}} \right)}
\]

(3.78)

which is called a ‘Woods-Saxon’ (or ‘Saxon-Woods’ or ‘Fermi’) shape. The central depth \(V_{ws}\) is typically between 40 and 50 MeV, and the diffuseness \(a_{ws}\) about 0.6 fm. The radius \(R_{ws}\) is proportional to the size of the nucleus, and is commonly around \(R_{ws} = r_{ws}A^{1/3}\) for a nucleus of \(A\) nucleons, with \(r_{ws} \approx 1.2\) fm. Similar potentials can be used for the interaction between two nuclei with mass numbers \(A_1\) and \(A_2\), if the radii are scaled instead as \(R_{ws} = r_{ws}(A_1^{1/3} + A_2^{1/3})\), since this is proportional to the sum of the individual radii.

This potential is usually combined with an imaginary and a spin-orbit part. The imaginary part, which is present at higher scattering energies as discussed in §3.1.5, is also often given by a Woods-Saxon form

\[
W(R) = \frac{V_i}{1 + \exp \left( \frac{R - R_i}{a_i} \right)}
\]

(3.79)

for a similar geometry \(R_i \approx R_{ws}\) and \(a_i \approx a_{ws}\), and a depth \(V_i\) fitted to experiments giving \(V_i \sim 10 - 20\) MeV depending on energy. Sometimes a surface-peaked imaginary contribution is also included, with a shape like the derivative of Eq. (3.79).

The spin-orbit potentials will be described in §4.1.3.

3.1.5 Complex potentials

Often, the effective interactions between two nuclei are best if they are allowed to have negative imaginary as well as real components. This is particularly true when there are open reaction channels not explicitly included in the coupled-channels model, for example at higher incident energies, or sometimes at low energies where there are strong exothermic \(Q > 0\) reaction channels.

If the potential is \(V(R) + iW(R)\) then it is no longer Hermitian, and the \(S\) matrix is no longer unitary. If the imaginary part \(W < 0\), then we have absorptive potentials, and a loss of flux. This can be made to approximate the flux leaving in the exit channels that are not explicitly in the multichannel model.

If the Schrödinger equation \([T + V + iW]\psi = i\hbar \partial \psi / \partial t\) is solved with an imaginary potential, then we can calculate the rate of loss of flux from the
continuity equation $\text{div} \ j + \partial \rho / \partial t = 0$ with density $\rho = |\psi|^2$. The current $j$ is

$$j = \frac{\hbar}{2\mu} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (3.80)$$

so, using the Schrödinger equation, we find

$$\text{div} \ j = -\frac{2}{\hbar} W \psi^* \psi . \quad (3.81)$$

If $W < 0$ then the imaginary potential acts as a sink of particles, which are then removed from the incident beam with a rate

$$\frac{\partial \rho}{\partial t} = -\frac{2}{\hbar} W \rho \quad (3.82)$$

so $\rho(t) \propto e^{2Wt/\hbar} . \quad (3.83)$

All the previous scattering theory remains valid, but now the phase shifts $\delta_L$ become complex, and the moduli $|S_L| \neq 1$. For absorptive potentials we have $|S_L| < 1$.

There is now an absorptive cross section $\sigma_A$ of flux disappearing from the elastic channel, which can be calculated as an integral over $W(R)$

$$\sigma_A = -\frac{2}{\hbar v} \sum_L \int_0^\infty W(R)|\psi_L(R)|^2 dR . \quad (3.84)$$

Since the difference between the intensities in the $L$'th partial wave of the incoming and outgoing waves in Eq. (3.32) is $1 - |S_L|^2$, this absorption must be equal to the sum over all the partial waves

$$\sigma_A = \frac{\pi}{k^2} \sum_L (2L + 1)(1 - |S_L|^2) . \quad (3.85)$$

### 3.2 Multi-channel scattering

#### 3.2.1 Multiple channels

Definitions

In the reaction between two nuclei such as $^{12}C + d$, a variety of mass re-arrangements may be possible, such as also $^{13}C + p$, $^{14}N + \gamma$, etc. Define each of these as a mass partition, and label them according to variable $\kappa$. We deal in this chapter only with two-body partitions, leaving three-body breakup and structure until later.

Each partition $\kappa$ will therefore consist of two bodies, one like the projectile, and the other like a target. Let $p$ and $t$ label the state (the energy level) of each body, to distinguish their ground from any excited states. Each state will have a definite spin and parity, so we label their spins by $J_p$ and $J_t$, etc.
denote their nuclear internal coordinates by $\xi_p$ and $\xi_t$, and therefore use $\phi_{J_p}^{\kappa_p}(\xi_p)$ and $\phi_{J_t}^{\kappa_t}(\xi_t)$ to refer to the whole quantum states of the projectile and target, respectively.

Let $L$ by their relative angular momentum, as in the previous section. Now, however, we have to couple together $L$, $J_p$, and $J_t$ to make some total angular momentum $J_T$. This can be done in two ways:

<table>
<thead>
<tr>
<th>'S basis'</th>
<th>Channel spin $S$</th>
<th>$J_p + J_t = S$</th>
<th>$L + S = J_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>'J basis'</td>
<td>Projectile $J$</td>
<td>$L + J_p = J$</td>
<td>$J + J_t = J_T$</td>
</tr>
</tbody>
</table>

Sometimes these schemes will be called ‘LS’ and ‘jj’ respectively.

These are two complete and orthonormal basis schemes. The S basis has the advantage that compound nucleus resonances typically have little mixing of channel spin quantum number, whereas the J basis has the advantage that projectile spin-orbit forces are diagonal in this basis. FRESCO uses the J basis for both bound and scattering states. Later we give the conversion formulae between the two basis schemes.

The set of all quantum numbers for a given total $J_T$, in the J basis $\{\kappa pt, LJ_p, J_t\}$, will be abbreviated by $\alpha$. Each $\kappa$ denotes a mass partition, each $\kappa pt$ denotes an excited state pair, and each $\alpha$ denotes a partial wave channel. The unqualified noun ‘channel’ will refer to one of these according to context.

The relative coordinate will be $R_\kappa$ depending on partition, so the wave function of relative motion can be written $\psi_\alpha(R_\kappa)$. Thus, in each partition, the two state of two nuclei in relative motion with total angular momentum $J_T$ and projection $M_T$ is in the J basis

$$\Psi_{\kappa J_T}^{M_T}(R_\kappa, \xi_p, \xi_t) = \sum_{L, J_p J_t M_T \mu_p \mu_t} \phi_{J_p}^{\kappa_p}(\xi_p) \phi_{J_t}^{\kappa_t}(\xi_t) i^L Y_L^M(\hat{R}_\kappa) \frac{1}{R_\kappa} \psi_\alpha(R_\kappa)$$

$$= \sum_{\alpha} \left[ i^L Y_L^M(\hat{R}_\kappa) \otimes \phi_{J_p}^{\kappa_p}(\xi_p) \right] \otimes \phi_{J_t}^{\kappa_t}(\xi_t) J_{J_T M_T} \frac{1}{R_\kappa} \psi_\alpha(R_\kappa)$$

$$= \sum_{\alpha} |(LJ_p) J, J_t; J_T \kappa \rangle \psi_\alpha(R_\kappa)/R_\kappa$$

$$= \sum_{\alpha} |\alpha; J_T \rangle \psi_\alpha(R_\kappa)/R_\kappa$$.
In the S (channel spin) basis, the total wave function is
\[
\Psi_{M_T}^{J_T}(R_\kappa, \xi_p, \xi_t) = \left[ i^L \mathcal{Y}_M^L (\hat{R}_\kappa) \otimes \left[ \phi_{J_p}^{\kappa p}(\xi_p) \otimes \phi_{J_t}^{\kappa t}(\xi_t) \right] J_{M_T} \right] \frac{1}{R_\kappa} \psi_\beta(R_\kappa) ,
\]
where \( \beta \) is the set of quantum numbers \( \{ \kappa pt, LJ \} \). Partial waves in the channel spin basis are often labelled by \( 2S+1LJ \), for example \( ^3P_2 \) for \( S=1, L=1 \) and \( J=2 \).

There is a unitary transformation between the S and J bases,
\[
\psi_\beta = \sum_\alpha \langle \beta | \alpha \rangle \psi_\alpha \quad \text{and} \quad \psi_\alpha = \sum_\beta \langle \alpha | \beta \rangle \psi_\beta
\]
where
\[
\langle \alpha | \beta \rangle = \sqrt{(2S+1)(2J+1)} W(LJ_pJ_TJ_t; JS)
\]

The total wave function for superpositions of all partitions is
\[
\Psi_{J_T}^{M_T} = \sum_\kappa \Psi_{M_T}^{J_T}(R_\kappa, \xi_p, \xi_t)
\]

### Parity

Nuclear and Coulomb interactions do not change parity, so that each projectile and target state has a specific parity \( \pi_\kappa \) and \( \pi_t \). The parity for a partial wave \( L \) is \( (-1)^L \), so the total parity of a partial wave channel is \( \pi = (-1)^L \pi_\kappa \pi_t \).

Each parity \( \pi = \pm 1 \) in a scattering calculation can be calculated separately, since parities are not mixed by Coulomb or nuclear couplings. This means we will label coupled channels sets not just by \( J_T \) but by \( J_T \pi \), and sums over \( J_T \) become over \( J_T \pi \).

### Multichannel S matrix

The S matrix element of Section 3.1.2 is now generalised to a full matrix \( S_{\alpha \alpha_i}^{J_T \pi} \) for each total angular momentum and parity \( J_T \pi \), where \( \alpha_i \) is the partial wave channel with the incoming plane wave, and \( \alpha \) is an outgoing channel.

This means that Eqs. (3.10, 3.70) are generalised to depend on the entrance channel \( \alpha_i \),
\[
\psi_{\alpha_i}^{(\alpha)}(R_\kappa) = \frac{i}{2} \left( H_{L_i}^- (\eta_\kappa, k_\alpha R_\kappa) \delta_{\alpha \alpha_i} - H_{L_i}^+ (\eta_\kappa, k_\alpha R_\kappa) S_{\alpha \alpha_i}^{J_T \pi} \right)
\]

The S matrix \( S_{\alpha \alpha_i}^{J_T \pi} \) gives the amplitude of an outgoing wave in channel \( \alpha \) that arises from a incoming plane wave in channel \( \alpha_i \), in addition to the
scattering from a diagonal point Coulomb potential. The cross sections, we saw in Section 3.1.1, depend on the channel velocity multiplying the square modulus of an amplitude. It is therefore convenient to combine these velocity factors with the \( S \) matrix, by defining

\[
\tilde{S}_{\alpha\alpha} = \frac{\sqrt{v_\alpha}}{v_\alpha} S_{\alpha\alpha},
\]

where the velocities satisfy \( \mu_\alpha v_\alpha = \hbar k_\alpha \). The combination \( \tilde{S}_{\alpha\alpha} \) may now be used to find the multichannel cross sections, and its matrix elements may be more directly found from revised boundary conditions

\[
\psi^{(\alpha)}_\alpha(R_\kappa) = \frac{i}{2} \left( H^- (\eta_\alpha, k_\alpha R_\kappa) \delta_{\alpha\alpha} - H^+ (\eta_\alpha, k_\alpha R_\kappa) \frac{\sqrt{v_\alpha}}{v_\alpha} \tilde{S}_{\alpha\alpha} \right),
\]

Both \( S_{\alpha\alpha} \) and \( \tilde{S}_{\alpha\alpha} \) can be regarded as complex numbers in matrices \( S \) and \( \tilde{S} \). The column index in these matrices refers to the incoming channel, and the row index the exit channel.

**Multichannel cross section**

The scattering amplitude from an incoming elastic channel \( (\kappa, p\gamma, l\iota) \) to \( (\kappa p, \mu\lambda, \mu\iota) \) depends on the \( m \)-substates of the initial nuclei \( \mu_p, \mu_\gamma, \mu_\iota \), and the final nuclei \( \mu_p, \mu_\lambda, \mu_\iota \), as well as on the scattering angle \( \theta \), so the cross section for an unpolarised beam is found by summing over final \( m \)-states, and averaging over initial states:

\[
\sigma_{\kappa \nu \lambda} (\theta) = \frac{1}{(2J_\gamma + 1)(2J_\iota + 1)} \sum_{\mu_p \mu_\gamma \mu_\iota} |f^{\kappa \nu \lambda}_{\mu_p \mu_\gamma \mu_\iota} (\theta)|^2.
\]

(3.94)

The scattering amplitude may be found in terms of the \( S \) matrix in the \( J \) basis by

\[
f^{\kappa \nu \lambda}_{\mu_p \mu_\gamma \mu_\iota} (\theta) = \delta^{\mu_p \mu_p} \delta^{\mu_\iota \mu_\iota} f_c (\theta) + \frac{4\pi}{k_\iota} \sum_{L, L, M, J, J} \langle L, J, \gamma, \mu_p | J, M, \mu_\gamma, \mu_\iota \rangle \\
\langle J, M, J, \mu_\iota | J_T M_T \rangle \langle J, \mu_p | J \rangle \langle J, M, J_T | J_T M_T \rangle \\
Y_L^M (k') Y_L^M (k) \frac{i}{2} \delta_{\alpha\alpha} \tilde{S}_{\alpha\alpha} e^{2i(\sigma_L (\eta_\alpha) + \sigma_L (\eta_\iota))},
\]

(3.95)

and in the \( S \) (channel spin) basis by

\[
f^{\kappa \nu \lambda}_{\mu_p \mu_\gamma \mu_\iota} (\theta) = \delta^{\mu_p \mu_p} \delta^{\mu_\iota \mu_\iota} f_c (\theta) + \frac{4\pi}{k_\iota} \sum_{L, L, s, s, M, J} \langle L, J, \gamma, \mu_p | J, \mu_\iota, s \rangle \\
\langle L, J, s, s | J_T M_T \rangle \langle J, \mu_p | J \rangle \langle J, \mu_\iota | s \rangle \langle L, s, s | J_T M_T \rangle \\
Y_L^M (k') Y_L^M (k) \frac{i}{2} \delta_{\beta\beta} \tilde{S}_{\beta\beta} e^{2i(\sigma_L (\eta_\beta) + \sigma_L (\eta_\iota))}.
\]

(3.96)
Scattering theory

Polarised beams

The dependence of these cross sections on the polarisation of the beam, namely a non-uniform distribution over initial $m$-state $\mu_{pi}$, is described by the tensor analysing powers $T_{kq}$ for this reaction. If the spherical tensor $\tau_{kq}$ is an operator with matrix elements

$$(\tau_{kq})_{mm'} = \hat{k}\langle J_{p}m, kq|J_{p}m'\rangle,$$  \hspace{1cm} (3.97)

with $\hat{k} \equiv \sqrt{2k+1}$, we define, for $k = 1, 2, ...$ and $0 \leq q \leq k$,

$$T_{kq}^{\text{opt}}(\theta) = \frac{\text{Tr}(f_{kq}^{+}f^{+})}{\text{Tr}(f^{+}f^{+})} \hspace{1cm} (3.98)$$

Integrated cross sections

The integrated outgoing cross section in a non-elastic excited state pair $\kappa pt$ is

$$\sigma_{\kappa pt} = 2\pi \int_{0}^{\pi} d\theta \sin \theta \sigma_{\kappa pt}(\theta)$$

$$= \frac{\pi}{k^{2}_i} \frac{1}{(2J_{p_{i}}+1)(2J_{t_{i}}+1)} \sum_{J_{T}L\alpha_{i}} (2J_{T}+1)|\tilde{S}_{J_{T}^{\pi}L\alpha_{i}}|^{2}. \hspace{1cm} (3.99)$$

The reaction cross section $\sigma_{R}$ is defined as the flux leaving the elastic channel, and depends only on the elastic S matrix element $S_{\alpha_{i}\alpha_{i}}$ as

$$\sigma_{R} = \frac{\pi}{k^{2}_i} \frac{1}{(2J_{p_{i}}+1)(2J_{t_{i}}+1)} \sum_{J_{T}\pi\alpha_{i}} (2J_{T}+1)(1 - |S_{\alpha_{i}\alpha_{i}}|^{2}). \hspace{1cm} (3.100)$$

The total elastic cross section is defined only for neutral scattering ($\eta = 0$), and is a generalisation of Eq. (3.40):

$$\sigma_{\text{el}} = \frac{\pi}{k^{2}_i} \frac{1}{(2J_{p_{i}}+1)(2J_{t_{i}}+1)} \sum_{J_{T}\pi\alpha_{i}} (2J_{T}+1)(1 - |S_{\alpha_{i}\alpha_{i}}|^{2}). \hspace{1cm} (3.101)$$

The total cross section is the sum of the reaction and elastic cross sections

$$\sigma_{\text{tot}} = \sigma_{R} + \sigma_{\text{el}}$$

$$= \frac{\pi}{k^{2}_i} \frac{1}{(2J_{p_{i}}+1)(2J_{t_{i}}+1)} \sum_{J_{T}\pi\alpha_{i}} (2J_{T}+1)2[1 - \Re S_{\alpha_{i}\alpha_{i}}]. \hspace{1cm} (3.102)$$
3.2 Multi-channel scattering

The absorption cross section is defined as the reaction cross section minus all the explicit outgoing cross sections:

\[ \sigma_A = \sigma_R - \sum_{\kappa pt \neq \kappa, p, t_i} \sigma_{\kappa pt} . \]  

(3.103)

This may arise from imaginary parts of optical potentials, or otherwise from an inaccurate, for example perturbative, solution of the multichannel equations.

If the coupled equations are solved precisely with complex potentials, then \( \sigma_A \) is non-zero, and can also be calculated as an integral over \( W(R) \):

\[ \sigma_A = -\sum_{\alpha_i} \frac{2}{\hbar v_i} \int_0^\infty W_{\alpha}(R_\kappa)|\psi_{\alpha_i}(R_\kappa)|^2 dR_\kappa . \]  

(3.104)

3.2.2 Coupled equations

For total energy \( E \), including now the projectile and target excitation energies, and Hamiltonian \( H \), we have to solve

\[ [H - E]\Psi_{J_T}^{M_T} = 0 , \]  

(3.105)

for which the results will be independent of \( M_T \) when \( H \) is rotationally invariant, and does not depend on any particular direction in space.

For each partition \( \kappa \), the total Hamiltonian \( H \) can be written in terms of internal eigenenergies, kinetic energies, and interaction potentials:

\[ H = H_{np}(\xi_p) + H_{nt}(\xi_t) + T_\kappa(R_\kappa) + V(R_\kappa, \xi_p, \xi_t) , \]  

(3.106)

where \( V(R_\kappa, \xi_p, \xi_t) \to 0 \) as \( R_\kappa \to \infty \).

The internal nuclear states \( \phi_{J_p}^{np}(\xi_p) \) and \( \phi_{J_t}^{nt}(\xi_t) \) satisfy energy-eigen-equations for their Hamiltonians

\[ H_{np}(\xi_p)\phi_{J_p}^{np}(\xi_p) = \epsilon_{np}\phi_{J_p}^{np}(\xi_p) , \]

\[ H_{nt}(\xi_t)\phi_{J_t}^{nt}(\xi_t) = \epsilon_{nt}\phi_{J_t}^{nt}(\xi_t) , \]  

(3.107)

and the kinetic energy operator depends on the masses \( m_{np} \) and \( m_{nt} \) via the partition’s reduced mass \( \mu = m_{np}m_{nt}/(m_{np}m_{nt}) \) as

\[ T_\kappa(R_\kappa) = -\frac{\hbar^2}{2\mu_\kappa} \nabla^2_{R_\kappa} . \]  

(3.108)

For convenience, we define a joint projectile and target Hamiltonian as \( H_\kappa = H_{np} + H_{nt} \), a joint eigenstate as \( \phi^{\kappa pt} = \phi_{J_p}^{np}\phi_{J_t}^{nt} \), and \( V_\kappa = V(R_\kappa, \xi_p, \xi_t) \), so \( [H_\kappa \phi^{\kappa pt} = (\epsilon_{np} + \epsilon_{nt})\phi^{\kappa pt} \) for the combined internal structures. Then \( H = T_\kappa + H_\kappa + V_\kappa \).
The coupled equations are now found by expanding the total wave function \( \Psi_{J_T \pi} \) in either the S or J partial wave basis. In the J basis, for example, \( \Psi_{J_T \pi} = \sum_\alpha |\alpha; J_T \pi \kappa\rangle \psi_\alpha(R_\kappa) / R_\kappa \), so from Eq. (3.105) for a given \( J_T \pi \) value (now omitted from the \( |\alpha\rangle \) basis states), we find

\[
\sum_\alpha R_\kappa' \langle \alpha' | H - E | \alpha \rangle R_\kappa^{-1} \psi_\alpha(R_\kappa) = 0 ,
\]

abbreviated as

\[
\sum_\alpha H_{\alpha' \alpha} \psi_\alpha(R_\kappa) = 0 ,
\]

which gives a separate equation for each \( \alpha' \) set of quantum numbers, and the Hamiltonian matrix element has been called \( H_{\alpha' \alpha} \).

Now

\[
[H - E]|\alpha\rangle = [H - E]|(LJ_p)J, J; J_T \pi \kappa\rangle
\]

\[
= [T_\kappa + H_\kappa + V_\kappa - E]|(LJ_p)J, J; J_T \pi \kappa\rangle
\]

\[
= [T_\kappa + \epsilon_{kp} + \epsilon_{kp} + V_\kappa - E]|(LJ_p)J, J; J_T \pi \kappa\rangle
\]

\[
= [T_\kappa + V_\kappa - E_{\kappa pt}]|(LJ_p)J, J; J_T \pi \kappa\rangle ,
\]

where \( E_{\kappa pt} = E - \epsilon_{kp} - \epsilon_{kp} \) is the asymptotic kinetic energy for a given excited-state pair \( \kappa pt \). This means that the coupled equations (3.109) may be simplified in two ways, by replacing \( H \) either by \( T_\kappa + H_\kappa + V_\kappa \) from the right hand side, or by \( T_\kappa' + H_\kappa' + V_\kappa' \) from the left side. The first option is called the prior form of the matrix element, and the second the post form. Ideally both choices should give the same results, but approximations may vary, as will be discussed in Chapter 6.

The prior form of the matrix element is therefore

\[
H_{\alpha' \alpha} = R_\kappa' \langle \alpha' | T_\kappa + V_\kappa - E_{\kappa pt} | \alpha \rangle R_\kappa^{-1}
\]

\[
= N_{\alpha' \alpha} [T_\kappa L(R_\kappa) - E_{\kappa pt}] + \hat{V}_{\alpha' \alpha}^\kappa ,
\]

where the partial-wave kinetic energy operator, the same as the one-channel operator of Eq. (3.9), is

\[
T_{\kappa L}(R_\kappa) = -\frac{\hbar^2}{2\mu_\kappa} \left[ \frac{d^2}{dR^2} - \frac{L(L + 1)}{R^2} \right] ,
\]

the coupling interactions between channels are the operators

\[
\hat{V}_{\alpha' \alpha}^\kappa = R_\kappa' \langle \alpha' | V_\kappa | \alpha \rangle R_\kappa^{-1} ,
\]

and the norm overlaps between the partial wave basis states are

\[
N_{\alpha' \alpha} = R_\kappa' \langle \alpha' | \alpha \rangle R_\kappa^{-1} .
\]
within the same partition, \( \kappa' = \kappa \), the norm overlaps are diagonal, \( N_{\alpha' \alpha} = \delta_{\alpha' \alpha} \). This suggests we separate the matrix elements of \( T_{\kappa L} - E_{\kappa pt} \) within a partition from those between partitions.

This gives from Eq. (3.109) the coupled channels equation set

\[
\begin{align*}
[T_{\kappa' L}(R'_\kappa) - E_{\kappa' pt'}] \psi_{\alpha'}(R'_\kappa) + \sum_\alpha \hat{V}_{\alpha' \alpha}^\kappa \psi_{\alpha}(R'_\kappa) \\
+ \sum_{\alpha, \kappa \neq \kappa'} N_{\alpha' \alpha}[T_{\kappa L} - E_{\kappa pt}] \psi_{\alpha}(R_\kappa) = 0 ,
\end{align*}
\]

which, on interchanging primes and the unprimed, gives a more natural

\[
\begin{align*}
[T_{\kappa L}(R_\kappa) - E_{\kappa pt}] \psi_{\alpha}(R_\kappa) + \sum_{\alpha'} \hat{V}_{\alpha' \alpha}^{\kappa'} \psi_{\alpha'}(R_\kappa') \\
+ \sum_{\alpha', \kappa' \neq \kappa} N_{\alpha \alpha'}[T_{\kappa' L'} - E_{\kappa' pt'}] \psi_{\alpha'}(R_\kappa') = 0 .
\end{align*}
\]

The third terms in these equations are called non-orthogonality terms because they involve the overlap of basis functions \( \langle \alpha' | \alpha \rangle \) between different mass partitions, and arise particularly in transfer reactions. We will see in Ch. 6 that they may be neglected in some circumstances, which would allow the coupled equations to be written in the more familiar form

\[
\begin{align*}
[T_{\kappa L}(R_\kappa) - E_{\kappa pt}] \psi_{\alpha}(R_\kappa) + \sum_{\alpha'} \hat{V}_{\alpha' \alpha}^{\kappa'} \psi_{\alpha'}(R_\kappa') = 0 .
\end{align*}
\]

The equations (3.116) use the prior form. This may be discerned from the fact that the superscript \( \kappa' \) in the coupling potential \( \hat{V}_{\alpha' \alpha}^{\kappa'} \) refers to the initial channel \( \alpha' \) rather than the final channel \( \alpha \). The same solutions should also obtained if the converse post form matrix elements, where Eq. (3.112) is replaced by

\[
H_{\alpha' \alpha} = R_{\alpha'} \langle \alpha' | T_{\kappa'} + V_{\kappa'} - E_{\kappa' pt'} | \alpha \rangle R^{-1}_\kappa
= [T_{\kappa' L'} - E_{\kappa' pt'}] N_{\alpha' \alpha} + \hat{V}_{\alpha' \alpha}^{\kappa'} .
\]

so the Eq. (3.116) (after swapping primes) would have couplings like \( \hat{V}_{\alpha' \alpha}^{\kappa} \).

The detailed construction of all the coupling potentials \( \hat{V}_{\alpha' \alpha}^{\kappa} \) is the subject of Ch. 4.

### 3.2.3 Properties of the multi-channel \( S \) matrix

**Unitarity**

It may be that the couplings \( \hat{V}_{\alpha' \alpha} \) are Hermitian, that is if \( \hat{V}_{\alpha' \alpha} = \hat{V}_{\alpha' \alpha}^* \). This is true if the coupling matrix is real and symmetric, but also hold
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more generally for self-adjoint or Hermitian couplings. Hermitian operators should be familiar in quantum mechanics since they have real eigenvalues and their eigenvectors form an orthogonal set.

For scattering, the consequence of Hermiticity is that the $S$ matrix $\tilde{S}_{\alpha\alpha}$ of Eq. (3.92) is unitary:

$$\tilde{S}^{-1} = \tilde{S}^\dagger = (\tilde{S}^*)^T$$

so $\tilde{S}^\dagger \tilde{S} = 1$, or, for $\alpha_i, \alpha_i'$ as two incoming channels,

$$\sum_\alpha \tilde{S}_{\alpha\alpha}^* \tilde{S}_{\alpha\alpha'} = \delta_{\alpha\alpha'}$$

and, in particular

$$\sum_\alpha |\tilde{S}_{\alpha\alpha}|^2 = 1$$

Each row and each column of the $S$ matrix is therefore a vector with unit norm.

Unitarity implies that the absorption cross section of Eq. (3.103) is $\sigma_A = 0$, so that the reaction cross section (the flux leaving the entrance channel) is precisely equal to the sum of all the outgoing cross sections.

Reciprocity

Another important case to consider is when the interactions $\hat{V}_{\alpha\alpha'}$ obey reciprocity, which is that the Hamiltonian is invariant under time reversal. In the theory of coupled channels, this means that the forward and reverse couplings have the same magnitude, and that the coupling matrix $\hat{V}_{\alpha\alpha'} = \hat{V}_{\alpha'\alpha}$ is symmetric. In this case, the $S$ matrix is also symmetric:

$$\tilde{S} = \tilde{S}^T$$

The reciprocity condition is distinct from unitarity, but real symmetric coupling matrices can lead to both unitarity and symmetry of the $S$ matrix. Hamiltonians as commonly used with complex potentials cannot be unitary, but can almost always be made symmetric.

For more details about unitarity, time reversal and reciprocity, see Taylor [6, §§6-e, 17-e] or Satchler [7, §9.5].

3.2.4 Detailed balance

From the symmetry of the $S$ matrix that follows from reciprocity in the case of symmetric coupling potentials, we may derive a direct connection between
the cross sections for the forward and reverse reactions. From Eq. (3.99),
\[
\sigma_{\kappa p_{i} t_{i} \rightarrow \kappa p_{i} t_{i}} = \frac{\pi}{k_{i}^{2}} \frac{1}{(2J_{p_{i}} + 1)(2J_{t_{i}} + 1)} \sum_{J_{T \pi \alpha \alpha_{i}}} (2J_{T} + 1)|\tilde{S}_{J_{T \pi \alpha \alpha_{i}}}^{\alpha_{i}}|^{2} \tag{3.123}
\]
The symmetry \(\tilde{S}_{\alpha \alpha_{i}} = \tilde{S}_{\alpha_{i} \alpha}\) implies that the equivalent expression for \(\sigma_{\kappa p_{i} t_{i} \rightarrow \kappa p_{i} t_{i}}\) is
\[
k_{i}^{2} (2J_{p_{i}} + 1) (2J_{t_{i}} + 1) \sigma_{\kappa p_{i} t_{i} \rightarrow \kappa p_{i} t_{i}} = k_{i}^{2} (2J_{p_{i}} + 1) (2J_{t_{i}} + 1) \sigma_{\kappa p_{i} t_{i} \rightarrow \kappa p_{i} t_{i}} \tag{3.124}
\]
so
\[
\sigma_{\kappa p_{i} t_{i} \rightarrow \kappa p_{i} t_{i}} = \frac{k_{i}^{2} (2J_{p_{i}} + 1) (2J_{t_{i}} + 1)}{k_{i}^{2} (2J_{p_{i}} + 1) (2J_{t_{i}} + 1)} \sigma_{\kappa p_{i} t_{i} \rightarrow \kappa p_{i} t_{i}} \tag{3.125}
\]
This is called the principle of detailed balance. The Hermiticity of the Hamiltonian leads to unitarity \(S\) matrices, but that by itself is not sufficient for detailed balance between the cross sections.

A slightly different expression holds for photon channels. Although we will see in §3.5.1 that they can be considered as spin 1 objects, the gauge condition implies that there are only two independent polarisation projections, and hence \((2s_{\gamma} + 1)\) should have the value of 2.

### 3.3 Integral forms

#### 3.3.1 Green function methods

**Integral solutions of inhomogeneous equations**

Up to now we have solved only homogeneous Schrödinger equations like \([H - E]\Psi = 0\). Sometimes we may need to solve inhomogeneous equations like \([H - E]\Psi = S\) with outgoing boundary conditions, for some \(S\) called a source term, as such equations arise as part of a coupled channels set. The inhomogenous equation may still be solved by differential methods as discussed in Ch. 6, but often it is useful to give an integral expression for its solution, and it is especially useful that there exist simple integrals giving the asymptotic outgoing part of the solution, namely its \(T\) matrix element.

This section shows how to use Green function methods to solve the inhomogeneous differential equation.

Consider the general problem of solving the coupled equations like Eq. (3.117)
\[
[T_{\kappa L}(R) + U_{\kappa}(R) - E_{\kappa p_{i}}]^{\Psi}_{\alpha}(R) + \sum_{\alpha'}\langle\alpha| V_{\kappa}| \alpha'\rangle^{\Psi}_{\alpha'}(R) = 0 \tag{3.126}
\]
where all the couplings (local or non-local, apart from the \(U_{\kappa}(R)\)) appear in the \(V_{\kappa}\) matrix elements. The solutions must satisfy the standard outgoing
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boundary conditions of Eq. (3.93). Suppose that all the $\psi_{\alpha'}(R)$ are known for $\alpha' \neq \alpha$, in which case we may solve the inhomogeneous equation (3.126) for the wave function $\psi_\alpha(R)$ in a non-elastic channel ($\alpha \neq \alpha_i$), using the source term

$$S_\alpha(R) = \sum_{\alpha'} V_{\alpha \alpha'}^{\kappa} \psi_{\alpha'}(R).$$

(3.127)

This is to solve the inhomogeneous equation

$$[T_{\kappa L}(R) + U_\kappa(R) - E_{\kappa pt}]\psi_\alpha(R) + S_\alpha(R) = 0.$$  

(3.128)

The outgoing-wave boundary conditions from Eq. (3.93) may be summarised in $T$ matrix form, following Eq. (3.35), as

$$\psi_\alpha(R') = F_\alpha(R) \delta_{\alpha \alpha_i} + H_\alpha^+(R) T_{\alpha \alpha_i}.$$  

(3.129)

where $\alpha_i$ is the incoming elastic channel, in which the solution is $\psi_{\alpha_i}(R) = F_{\alpha_i}(R)$ in the limit of $S_\alpha(R) = 0$.

**Definition of $G^+(R, R')$**

Let us use Green function methods to find the outgoing solution of the linear equation

$$\left[ \frac{d^2}{dR^2} - \tilde{U}(R) + k^2 \right] \psi_\alpha(R) = \frac{2\mu_k}{\hbar^2} S_\alpha(R),$$

(3.130)

where $\tilde{U}(R) \equiv \frac{2\mu_k}{\hbar^2} U_\kappa(R)$. The general source term $S_\alpha(R)$ can always be written as a superposition of solutions for $\delta$-function sources $\delta(R - R')$ at $R'$, since

$$S_\alpha(R) = \int \delta(R - R') S_\alpha(R') dR'.$$

(3.131)

Thus all we need is the solution function of $R$ of Eq. (3.130) for a $\delta$-function source at $R'$. We denote this solution by $G^+(R, R')$, since it depends on both $R$ and $R'$, and it is precisely the Green function $G^+(R, R')$ satisfying

$$\left[ \frac{d^2}{dR^2} - \tilde{U}(R) + k^2 \right] G^+(R, R') = \delta(R - R').$$

(3.132)

The desired solution is therefore a superposition of all the $G^+(R, R')$ with amplitudes corresponding to the magnitude of the source term at $R'$, namely $\frac{2\mu_k}{\hbar^2} S_\alpha(R')$. This gives the wave function in terms of the integral expression

$$\psi_\alpha(R) = \delta_{\alpha \alpha_i} F_{\alpha_i}(R) + \frac{2\mu_k}{\hbar^2} \int G^+(R, R') S_\alpha(R') dR'.$$

(3.133)
where we have added in the homogeneous solution $F_\alpha(R)$ present in the elastic channel $\alpha = \alpha_i$. This equation is often written more compactly in operator notation

$$\psi_\alpha = F_\alpha + \hat{G}^+ S_\alpha ,$$  \hspace{1cm} (3.134)

where $\hat{G}^+$ is defined as the Green integral operator that has the kernel function $2\mu_r/\hbar^2 G^+(R,R')$, and where $F_\alpha$ is understood as the homogeneous solution in only the elastic channel. Furthermore, because $\hat{G}^+ S_\alpha$ is the solution $\psi$ of the differential equation $[E - T] \psi = S_\alpha$ with $T$ the kinetic energy operator, the Green operator with the ‘distorting’ potential $U$ can be written as

$$\hat{G}^+ = [E - T - U]^{-1}$$  \hspace{1cm} (3.135)

with the specified outgoing boundary conditions, as in

$$\psi_\alpha = F_\alpha + [E - T - U]^{-1} S_\alpha .$$  \hspace{1cm} (3.136)

**To find $G^+(R,R')$**

For fixed $R'$, when $R \neq R'$ we have from Eq. (3.132) that

$$\left[\frac{d^2}{dR^2} - \tilde{U}(R) + k^2\right] G^+(R,R') = 0 .$$  \hspace{1cm} (3.137)

Since this is a second-order linear differential equation, any solution must be a linear combination of two fixed linearly-independent solutions

$$G^+(R,R') = a(R')\tilde{F}(R) + b(R')\tilde{H}^+(R) ,$$  \hspace{1cm} (3.138)

where the coefficients are as yet unknown functions of $R'$.

Let us choose $\tilde{F}(R)$ and $\tilde{H}^+(R)$ as the the regular and an irregular solutions of

$$\left[\frac{d^2}{dR^2} - \tilde{U}(R) + k^2\right] \begin{cases} \tilde{F}(R) \\ \tilde{H}^+(R) \end{cases} = 0 .$$  \hspace{1cm} (3.139)

where $\tilde{F}(R)$ being regular means that $\tilde{F}(0) = 0$. We chose the regular solution that for $R > R_m$ becomes $\tilde{F}(R) = F_\alpha(R)$ (asymptotically $\sin kR$), and the irregular solution that becomes $\tilde{H}^+(R) = H_\alpha^+(R)$ (asymptotically $\sim e^{ikR}$), for $R_m$ outside the radius of the potential.

(If $U(R) = 0$ then $\tilde{F}(R) = F_L(0,kR)$; if $U(R)$ is a Coulomb potential $V_c(R)$ then $\tilde{F}(R) = F_L(\eta,kR)$; many other solutions are possible for various potentials.)

From the $R = 0$ boundary condition, $G^+(0,R') = 0$, we conclude from Eq.
(3.138) that \( b(R') = 0 \), since \( R < R' \). The \( R \to \infty \) boundary condition of Eq. (3.129) implies that for \( R > R' \), \( G^+(R, R') = H^+_R(R) \to e^{ikR} \). Therefore

\[
G^+(R, R') = \begin{cases} 
  a(R') \hat{F}(R) \text{ for } R < R' \\
  b(R') \hat{H}^+(R) \text{ for } R > R'
\end{cases} \quad (3.140)
\]

To fix \( a(R') \), \( b(R') \) we use the differential equation (3.132). Integrating this with respect to \( R \) over the range from just below \( R' \) to just above, we find

\[
\left. \frac{d}{dR} G^+(R, R') \right|_{R=R'} = 1 \quad (3.141)
\]

so from Eq. (3.140) we have, using derivatives with respect to \( R \),

\[
b(R) \hat{H}^+(R) - a(R) \hat{F}'(R) = 1 \quad (3.142)
\]

Now \( G(R, R') \) is itself continuous over \( R \sim R' \), so

\[
b(R) \hat{H}^+(R) - a(R) \hat{F}(R) = 0 \quad (3.143)
\]

and, solving Eqs. (3.142, 3.143) simultaneously, we have

\[
b(R) = \frac{\hat{F}}{W(\hat{F}, \hat{H}^+)} \quad a(R) = \frac{\hat{H}^+}{W(\hat{F}, \hat{H}^+)} \quad (3.144)
\]

where \( W(f, g) = fg' - f'g \) is the Wronskian for two functions \( f(R) \) and \( g(R) \). For our chosen \( \hat{F}, \hat{H}^+ \), the Wronskian \( W(\hat{F}, \hat{H}^+) = -k \) from Eq. (3.11).

The full Green function is therefore

\[
G^+(R, R') = -\frac{1}{k} \begin{cases} 
  \hat{H}^+(R') \hat{F}(R) \text{ for } R < R' \\
  \hat{F}(R') \hat{H}^+(R) \text{ for } R > R'
\end{cases} \quad (3.145)
\]

\[
= -\frac{1}{k} \hat{F}(R_<) \hat{H}^+(R_>) \quad (3.146)
\]

where \( R_< = \min(R, R') \) and \( R_> = \max(R, R') \).

The solution of the original inhomogeneous equation (3.128) is therefore

\[
\psi_\alpha(R) = -\frac{2\mu}{\hbar^2 k} \int \hat{F}(R_<) \hat{H}^+(R_>) S_\alpha(R')dR' \quad (3.147)
\]

At large distances \( R > R' \), by construction \( \psi_\alpha(R) \to T_\alpha H^+(R) \), because of Eq. (3.129), so

\[
T_\alpha H^+(R) = -\frac{2\mu}{\hbar^2 k} \hat{H}^+(R) \int \hat{F}(R') S_\alpha(R')dR' \quad (3.148)
\]

and we arrive at a very useful integral expression for the partial wave \( T \).
3.3 Integral forms

Integral form:

$$T_{\alpha} = -\frac{2\mu}{\hbar^2 k} \int \tilde{F}(R') S_{\alpha}(R') dR'.$$  \hfill (3.149)

This may be rewritten in Dirac bra-ket notation as

$$T_{\alpha} = -\frac{2\mu}{\hbar^2 k} \langle \tilde{F}^{\star} | S_{\alpha} \rangle$$  \hfill (3.150) \\
$$= -\frac{2\mu}{\hbar^2 k} \langle \tilde{F}^{(-)} | S_{\alpha} \rangle .$$  \hfill (3.151)

The complex conjugation in Eq. (3.150) is necessary to cancel the conjugation implicit in the matrix elements. The \(^{(-)}\) superscript in the second form Eq. (3.151) is used because the conjugate wave function \(\tilde{F}^{\star}\) satisfies a boundary condition with an \textit{incoming} boundary conditions:

$$\tilde{F} \rightarrow F + \tilde{T} H^+ \rightarrow \sin(kR) + \tilde{T} e^{ikR},$$  \hfill (3.152) \\
so \(\tilde{F}^{\star} \rightarrow F^{\star} + \tilde{T}^{\star} H^{\star+} \rightarrow \sin(kR) + \tilde{T}^{\star} e^{-ikR} .$$  \hfill (3.153)

where \(\tilde{T}\) is the scattering from the potential \(U_{\alpha}(R)\) that defines the homogeneous functions \(\tilde{F}(R)\).

When \(U_{\alpha}(R) = 0\), the integral expression is called a \textit{plane wave T matrix element}, and when \(U_{\alpha}(R)\) is the some nuclear+Coulomb distorting potential in the exit channel \(\alpha\), the integral becomes a \textit{distorted wave T matrix element}.

An \textit{operator notation} is often used, so Eq. (3.133) can be written more compactly as

$$\psi = \phi + \hat{G}^+ S$$  \hfill (3.154) \\
$$= \phi + \hat{G}^+ V \psi ,$$  \hfill (3.155)

using \(\phi\) to refer to the homogeneous solution present only in the elastic channel, and using the operator definitions of Eq. (3.136), \(\hat{G}^+ = [E - T]^{-1}\), with the + sign indicating outgoing boundary conditions of Eq. (3.129), and \(T\) the kinetic energy operator. The equation (3.155) is called a partial-wave \textit{Lippmann-Schwinger equation}, and in this notation the T-matrix (3.151) is the integral

$$T = -\frac{2\mu}{\hbar^2 k} \langle \phi^{(-)} | V | \psi \rangle \equiv -\frac{2\mu}{\hbar^2 k} \int \phi(R)V(R)\psi(R)dR .$$  \hfill (3.156)

In a multichannel formulation, \(\psi\) and \(\phi\) are interpreted as vectors (\(\phi\) being only non-zero in the elastic channel), \(V\) as a matrix, and \(\hat{G}^+\) is a matrix of integral operators.
3.3.2 Two potential formula

If a channel potential is composed of two parts $U(R) = U_1(R) + U_2(R)$, then it is possible to use $U_1$ as the distorting potential, $U_2$ as the coupling interaction, and derive a $T$ matrix integral expression for the scattering from their combined $U(R)$ potential. This will be useful when $U_1$ can be thought of as ‘strong but easy’ and $U_2$ as ‘small but difficult’, but first we derive an exact two-potential formula.

For each partial wave, let us define solutions $\phi$ for the free field, $\chi$ for $U_1$ only, and $\psi$ for the full case, and use Eq. (3.155) to write down the corresponding Lippmann-Schwinger equations. Their asymptotic amplitudes will be derived from Eq. (3.156):

Free: $[E - T] \phi = 0 \quad \hat{G}_0 = [E - T]^{-1} \quad \phi \rightarrow F$

Distorted: $[E - T - U_1] \chi = 0 \quad \chi = \phi + \hat{G}_0 U_1 \chi \quad \chi \rightarrow \phi + T_1 H^+$

Full: $[E - T - U_1 - U_2] \psi = 0 \quad \psi = \phi + \hat{G}_0 (U_1 + U_2) \psi \quad \psi \rightarrow \phi + T_{12} H^+ . \quad (3.156)$

From Eq. (3.156), the distorted wave $T$-matrix is $T_1 = -\frac{2\mu}{\hbar k^2} \langle \phi(-)|U_1|\psi \rangle$, and similarly the $T$ matrix from the combined potentials, $T_{12}$, satisfies

$$-\frac{\hbar^2 k}{2\mu} T_{12} = \int \phi(U_1 + U_2) \psi \, dR$$
$$= \int (\chi - \hat{G}_0 U_1 \chi)(U_1 + U_2) \psi \, dR$$
$$= \int \chi(U_1 + U_2) \psi - (\hat{G}_0 U_1 \chi)(U_1 + U_2) \psi \, dR . \quad (3.157)$$

Because the kernel function of Eq. (3.145) for the operator $\hat{G}_0$ is symmetric under $R \leftrightarrow R'$ interchange,

$$-\frac{\hbar^2 k}{2\mu} T_{12} = \int \chi(U_1 + U_2) \psi - \chi U_1 \hat{G}_0 (U_1 + U_2) \psi \, dR$$
$$= \int \chi(U_1 + U_2) \psi - \chi U_1 [\psi - \phi] \, dR$$
$$= \int \phi U_1 \chi + \chi U_2 \psi \, dR \quad (3.158)$$
$$= \langle \phi(-)|U_1|\chi \rangle + \langle \chi(-)|U_2|\psi \rangle . \quad (3.159)$$

Note that in Eq. (3.158) the two terms in the integrand are products of three complex functions, and the product order is unimportant. In Eq. (3.159), by contrast, the functions are no longer interchangeable.
Thus \( T_{12} = T_1 + T^{(1)}_2 \), defining an additional term
\[
T^{(1)}_2 = -\frac{2\mu}{\hbar^2 k} \int \chi U_2 \psi \, dR
\]
(3.160)
as the scattering \( T \) matrix contribution from coupling \( U_2 \), with \( U_1 \) appearing as a distorting potential in \( \chi \). The equation (3.159) is called the two potential formula, and is valid for both real and complex potentials \( U_1, U_2 \). It uses the \((-)\) superscript to indicate an additional complex conjugation for the left-hand wave function, as in Eq. (3.151). It is an exact equation.

The exact wave function \( \psi \) is the solution of the implicit equation
\[
\psi = \chi + \hat{G}_1 U_2 \psi,
\]
(3.161)
using \( \hat{G}_1 = [E - T - U_1]^{-1} \) with outgoing wave boundary conditions. The first term \( \chi \) represents the contribution present if \( U_2 = 0 \).

### 3.3.3 Born series and approximations

**One-potential scattering**

For a potential \( U(R) \), solving the Lippmann-Schwinger equation \( \chi = \phi + \hat{G}_0 U \chi \) provides an exact solution for the wave function \( \chi \) for potential \( U \). This however is an implicit equation, as \( \chi \) appears on both the left and right sides. To find it explicitly, we would have to sum the iterated Born series:
\[
\chi = \phi + \hat{G}_0 U \phi + \hat{G}_0 U \hat{G}_0 U \phi + \hat{G}_0 U \hat{G}_0 U \hat{G}_0 U \phi + \cdots
\]
(3.162)
from which the outgoing \( T \) matrix amplitude is
\[
T = -\frac{2\mu}{\hbar^2 k} \left[ \langle \phi | U \phi \rangle + \langle \phi | \hat{G}_0 U \phi \rangle + \cdots \right].
\]
(3.163)
The equation (3.162) may be illustrated by the figure 3.4, where each node of the graph is an action of the potential \( U \) and each line a propagation by \( \hat{G}_0 \).

If the potential \( U(R) \) is weak, in the sense that we could treat it as a perturbation, then we might truncate these series and still achieve sufficient precision. The first term is called the plane wave Born approximation (PWBA):
\[
T_{PWBA} = -\frac{2\mu}{\hbar^2 k} \langle \phi | U | \phi \rangle.
\]
(3.164)
This partial wave PWBA, when written explicitly with the radial wave functions, is

$$T_{L}^{PWBA} = -\frac{2\mu}{\hbar^2 k} \int_0^\infty F_L(0,kR) \ U(R) \ F_L(0,kR) \ dR . \quad (3.165)$$

Substituting these T matrix elements into Eq. (3.36), the three-dimensional form for the PWBA scattering amplitude is

$$f_{PWBA}(\theta) = -\frac{\mu}{2\pi\hbar^2} \int dR \ e^{-i \mathbf{q} \cdot \mathbf{R}} \ U(\mathbf{R}) , \quad (3.166)$$

where the momentum transfer $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ so $q = 2k \sin \theta / 2$. The PWBA amplitude is thus simply proportional to the Fourier transform of the potential. The PWBA is expected to be more accurate at very high energies when potentials are weak, such as in electron-nucleus scattering.
3.3 Integral forms

Two-potential scattering

From Eq. (3.159), the exact T matrix expression is again

\[ T_{12} = T_1 - \frac{2\mu}{\hbar^2 k} \langle \chi^{-} | U_2 | \psi \rangle \]  

(3.167)

where, from Eq. (3.161), the exact wave function is the solution of the implicit equation \( \psi = \chi + \hat{G}_1 U_2 \psi \). We may therefore again form a Born series

\[ T_{12} = T_1 - \frac{2\mu}{\hbar^2 k} \left[ \langle \chi^{-} | U_2 | \chi \rangle + \langle \chi^{-} | U_2 \hat{G}_1 U_2 | \chi \rangle + \cdots \right]. \]  

(3.168)

**Distorted wave Born approximation (DWBA)**

If this series is truncated after the first term, linear in \( U_2 \), then

\[ T_{12}^{\text{DWBA}} = T_1 - \frac{2\mu}{\hbar^2 k} \langle \chi^{-} | U_2 | \chi \rangle \]  

(3.169)

is called the *Distorted Wave Born approximation* (DWBA), because it is a matrix element using wave functions \( \chi(R) \) which include \( U_1 \) as a distorting potential. It is a *first order* DWBA because \( U_2 \) appears only linearly. It is particularly useful for exit channels where \( U_1 \) might be say a central optical potential that cannot by itself cause the transition. In this case \( T_1 = 0 \), and we have the convenient DWBA expression for the T matrix from incoming channel \( \alpha_i \) to exit channel \( \alpha \):

\[ T_{\alpha\alpha_i}^{\text{DWBA}} = -\frac{2\mu_\alpha}{\hbar^2 k_\alpha} \langle \chi^{-}_\alpha | U_2 | \chi_{\alpha_i} \rangle. \]  

(3.170)

This DWBA may be a good approximation depending on the size of \( U_2 \), as we see in Fig. 3.5.

**Approximate coupled channels solutions**

As discussed at the beginning of §3.3.1, the value of these integral expressions may be equivalently derived by solving inhomogeneous differential equations. This means that the above approximations may be reached by modifications to the coupled channels equations. In particular, the first-order DWBA result may be identically obtained by removing all couplings except the diagonal potentials in the elastic \( (\alpha_i) \) and final \( (\alpha) \) channel \( (U_1 \text{ say}) \), and except the couplings from the elastic channel to the final channel \( (U_2 \text{ say}) \).

Allowing for a distinct potential \( U_{\alpha_i} \) in the entrance channel, this modified coupling set is

\[
\begin{align*}
[T_{\alpha_i} + U_{\alpha_i} - E_{\alpha_i}]\psi_{\alpha_i} &= 0 \\
[T_{\alpha} + U_1 - E_{\alpha}]\psi_{\alpha} + U_2 \psi_{\alpha_i} &= 0.
\end{align*}
\]  

(3.171)
These equations may be solved explicitly. The elastic channel wave function
is just the uncoupled solution \( \psi_{\alpha i} = \chi_{\alpha i} \). The T-matrix amplitude for the
final channel may then be equivalently found by the DWBA integral (3.170),
or by solving the inhomogeneous differential equation (3.171) by the method
of \S 6.3.3.

3.4 Identical particles
3.4.1 Isospin

The proton and neutron are almost the same in mass \((\Delta m/m = 1.4 \times 10^{-3})\), and
their different charges and magnetic moments make only a small
difference compared with the strong nuclear forces. This suggests treating
them as identical in some sense, and including the electromagnetic effects
(etc.) by perturbation theory.

The only sense that the neutron and proton need to be distinguished,
then, is for the Pauli Principle, and to make antisymmetric wave functions.
For this purpose (alone) we follow the standard derivation of two ‘states’ for
a nucleon, to be regarded as in one of two states depending on an isobaric
variable \( \tau_z \). This distinguishes a neutron \((\tau_z = 1)\) from a proton \((\tau_z =
-1)\). Thus a full specification of a nucleon depends on position \( r \), spin
z-component \( \sigma_z \), and now also isospin z-component \( \tau_z \) as \( \psi(r, \sigma_z, \tau_z) \).

If we define the neutron state vector as \(|n\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \(|p\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), then
the \( \hat{\tau}_z \) operator \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) has as eigenstates the above vectors:
\[ \hat{\tau}_z |n\rangle = +1 |n\rangle \text{ and } \hat{\tau}_z |p\rangle = -1 |p\rangle . \] (3.172)

The \( \hat{\tau}_z \) may be supplemented by \( \hat{\tau}_x \) and \( \hat{\tau}_y \) which follow the familiar algebra
for components of a spin-1/2 particle, namely
\[ \hat{\tau}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \hat{\tau}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \] (3.173)
satisfying the commutators \[ [\tau_p, \tau_q] = 2i \tau_r \] for \((pqr)\) cyclic permutations of
\((xyz)\), by analogy with the Pauli spin operators. This justifies the name of
isospin. In terms of the original \(|n\rangle\) and \(|p\rangle\) states, the isospin operators
may be written
\[ \hat{\tau}_x = |p\rangle\langle n| + |n\rangle\langle p| \]
\[ \hat{\tau}_y = i(|p\rangle\langle n| - |n\rangle\langle p|) \]
\[ \hat{\tau}_z = |n\rangle\langle n| - |p\rangle\langle p| . \] (3.174)
3.4 Identical particles

From the $\hat{\tau}_i$ matrix operators, we obtain the isotopic spin operators

$$\hat{t} = \frac{1}{2} \hat{\tau} = \frac{1}{2} (\hat{\tau}_x, \hat{\tau}_y, \hat{\tau}_z). \quad (3.175)$$

Since $\hat{t}^2 = 3/4$, the nucleon has total isospin $t = 1/2$, and $z$-components $m_t = t_z = +1/2$ for the neutron and $-1/2$ for the proton. The operators $\hat{t}_+ = t_x + it_y$ and $\hat{t}_- = t_x - it_y$ are the raising and lowering operators respectively, as from Eq. (3.174) we see that $\hat{t}_+ = \frac{1}{2} |n\rangle \langle p|$ and $\hat{t}_- = \frac{1}{2} |p\rangle \langle n|$.

Composite systems

For systems of two or more nucleons ($k = 1, 2, ...$), the isospins may be coupled to a total isospin

$$\hat{T} = \sum_k \hat{t}_k \quad (3.176)$$

with $z$-component $M_T = T_z = \frac{1}{2} (N - Z)$ for $N$ neutrons and $Z$ protons. For even numbers of nucleons, $T = 0, 1, ...$, and for odd numbers $T = \frac{1}{2}, \frac{3}{2}, ...$. Two protons or two neutrons have $T_z = \pm 1$, hence $T = 1$, whereas a neutron and a proton together have $T_z = 0$, and hence either $T = 0$ or 1.

The spin-statistics theorem is a fundamental result in quantum field theory. This states that spin-half objects obey Fermi-Dirac statistics and have antisymmetric wave functions, whereas integer-spin objects obey Bose-Einstein statistics and have symmetric wave functions. The (anti-)symmetry of the collective wave functions means that, on interchanging the coordinates of any two particles, the wave function changes sign (antisymmetric for fermions), or remains the same (symmetric, for bosons).

Antisymmetrisation can be applied to neutrons and to protons separately, but we can make for nucleons what Bohr and Mottelson [8] call a generalised antisymmetrisation principle, that the wave function for a set of fermions is antisymmetric with respect to the interchange of all coordinates (space, spin, and isospin) of any pair of nucleons.

Consider the case of two nucleons with coordinates $(r_i, \sigma_{zi}, \tau_{zi})$, $i = 1, 2$. Antisymmetry requires $\Psi(1,2) = -\Psi(2,1)$. Let the spins $s_1, s_2$ be coupled to $S$, the isospins $t_1, t_2$ to $T$, the relative angular momentum be $L$, and $L + S = J$ the total spin. Then

$$\Psi_J(1,2) = [Y_L(R) \otimes [s_1 \otimes s_2]_S] J \ [t_1 \otimes t_2]_T \quad (3.177)$$
which gives, on interchanging the particles 1 and 2,

\[
\Psi_J(2, 1) = [Y_L(-R) \otimes [s_2 \otimes s_1]_J [t_2 \otimes t_1]_T =
\begin{align*}
&= (-1)^s (-1)^{S-s_1-s_2} (-1)^{T-t_1-t_2} \Psi_J(1, 2) \\
&= (-1)^{L+S+T} \Psi_J(1, 2)
\end{align*}
\] (3.178)

as all \( s_i = t_i = 1/2 \). Antisymmetrisation for two nucleons then requires that they be in a state where \( L + S + T \) is odd.

For more complex nuclei composed of antisymmetric pairs of nucleons, \( T \) is still a good quantum number even though there are greater effects of Coulomb forces, which depend on the projections \( t_z \). We find that isobaric sets of nuclei (those with constant \( A \)) have many sets of energy levels that are almost constant when \( T \) is constant (varying slightly because of the Coulomb energies). These sets of levels, illustrated in Fig. 3.6, are called isobaric analogue states, and have very similar internal nuclear structure. There is little mixing between \( T \) values, only slight energy shifts that hardly change the structure.

### 3.4.2 Direct and exchange amplitudes

The scattering analysis so far has dealt with reactions of the form \( p + t \rightarrow p' + t' \) for projectile \( p \), target \( t \), ejectile \( p' \) and residue \( t' \) that are considered distinguishable nuclei. In general, however, we have to consider the cases of

a) Scattering of identical fermions: \( p = t \) are of odd baryon number,

b) Scattering of identical bosons: \( p = t \) are of even baryon number, and
c) Exchange scattering: \( p' = t \) & \( t' = p \), and \( p \) is distinguishable from \( t \),

where \( p \) and \( t \) may be clusters, not just individual nucleons.

Let us define an ‘exchange index’ \( \epsilon = +1 \) for boson-boson and \( \epsilon = -1 \) for fermion-fermion collisions, so the a) and b) cases may be considered with one formalism. Define the operator \( P_{pt} \) which exchanges projectile and target coordinates in a many-body wave function. The spin-statistics
Identical particles

A complete wave function $\Psi_\epsilon$ should therefore satisfy

$$\hat{P}_p \Psi_\epsilon = \epsilon \Psi_\epsilon.$$  \hfill (3.179)

**Identical spinless scattering**

Consider first the case of spinless scattering, but keep explicit reference to $\epsilon$. If the asymptotic form of Eq. (3.1) for spherical potentials is

$$\psi_{\text{asym}}(R) = A \left[ e^{ikz} + f(\theta) \right]$$

then a suitable wave function for identical particles is

$$\Psi_{\text{asym}}(R) = \psi_{\text{asym}}(R) + \epsilon \psi_{\text{asym}}(-R).$$  \hfill (3.180)

We now calculate the initial flux $j_i$ and final flux $j_f$, in order to calculate the cross section

$$\sigma = j_f/j_i.$$  \hfill (3.181)

and the scattered outgoing radial wave is

$$\Psi_{\text{inc}}^\epsilon(R) = A [e^{ikz} + \epsilon e^{-ikz}]$$

The total flux in $\Psi_{\text{inc}}^\epsilon(R)$ is numerically zero. A more detailed analysis uses wave packets [9], and shows that that the incident flux is a combination of a projectile flux $\hbar k |A|^2$ in the $+z$ direction, and an (identical) target flux in the $-z$ direction. The projectile flux in the beam is therefore $j_i = \frac{\hbar k}{\mu} |A|^2$. The scattered flux is $j_f = \frac{\hbar k}{\mu} |A|^2 |f_\epsilon(\theta)|^2$, so the needed cross section for identical particle scattering is

$$\sigma(\theta) = |f_\epsilon(\theta)|^2 = |f(\theta) + \epsilon f(\pi - \theta)|^2 = |f(\theta)|^2 + |f(\pi - \theta)|^2 + 2 |f(\theta)| |f(\pi - \theta)| \cos(\epsilon (\pi - \theta)) .$$  \hfill (3.183)

Fig. 3.7 shows interfering classical paths.

Since $P_L(\cos(\pi - \theta)) = (-1)^L P_L(\cos \theta)$, the new amplitude has the partial wave expansion

$$f_\epsilon(\theta) = \frac{1}{k} \sum_{L=0}^\infty (2L + 1) P_L(\cos \theta) T_L[1 + \epsilon (-1)^L] .$$  \hfill (3.185)

This implies for the scattering of identical bosons, where $\epsilon = 1$, that odd partial waves should be removed, and that the amplitude for the remaining even partial waves should be doubled.
Fig. 3.7. Direct and exchange paths which interfere with each other, in the scattering of identical particles.

Fig. 3.8. Singlet and triplet nucleon-nucleon scattering cross sections.

Identical particles with spin

The above analysis for $J_p = J_t = 0$ is necessarily restricted to $\epsilon = 1$ for identical bosons, and for identical fermions, it is essential to include spin. This is easiest in the $S$ (channel spin) partial wave basis, where the effect of the $\hat{P}_{pt}$ exchange operator is

$$\hat{P}_{pt}|L(J_p, J_t)S; J_T\kappa\rangle = (-1)^L(-1)^{S-J_p-J_t}\epsilon|L(J_t, J_p)S; J_T\kappa\rangle$$

(3.186)

with the radial wave function itself, $\psi_{\kappa}(R_\kappa)$, not being affected. The $(-1)^L$ factor comes from reversing the direction of the radius vector in the spherical harmonic, and $(-1)^{S-J_p-J_t}$ comes from reordering the coupling of particle spins to make the channel spin. The expression of Eq. (3.96) for the scattering amplitude therefore has an extra factor

$$[1 + \epsilon(-1)^{L+S-J_p-J_t}]$$

$$= [1 + (-1)^{L+S}] \text{ since } J_p = J_t \text{ and } \epsilon = (-1)^{2J_p}.$$  

(3.187)

Thus nucleon-nucleon scattering therefore is different in singlet ($S = 0$) and triplet ($S = 1$) states, and gives rise to the characteristic interference patterns shown in Fig. 3.8.

Total cross sections

The required quantity has to be specified carefully in this case, since for every nucleus removed from the beam, two identical nuclei are produced in
3.4 Identical particles

Fig. 3.9. Elastic transfer effects in $^6\text{He} + ^4\text{He}$ scattering.

scattering states. If we define the elastic $\sigma_T$ as the cross section for removal from the beam, then for spinless bosons, following Eq. (3.40),

$$\sigma_T = \frac{1}{2} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta \sigma(\theta)$$

$$\sigma_T = \frac{2}{k^2} \sum_{L \text{ even}} (2L + 1)|1 - |S_L|^2$$  \hspace{1cm} (3.188)

The boson reaction cross section is given by a similar expression

$$\sigma_R = \frac{2}{k^2} \sum_{L \text{ even}} (2L + 1)|1 - |S_L|^2$$  \hspace{1cm} (3.189)

### 3.4.3 Exchange transfer

This occurs in transfer reactions such as $^6\text{He} + ^4\text{He} \rightarrow ^4\text{He} + ^6\text{He}$, where the transfer process adds coherently to the elastic scattering amplitude, but with revised relative coordinate. In general, consider $p + t \rightarrow t' + p'$ reactions where the primes indicate the transfer channels. We construct a total wave function

$$\Psi_\epsilon = \psi_{pt} + \epsilon \hat{P}_{p't'} \psi_{t'p'}$$  \hspace{1cm} (3.190)

where $\epsilon = 1$ for $p \neq t$ and $\epsilon = (-1)^{2J_p}$ for $p = t$, and where again $\hat{P}_{p't'}$ is operator which exchanges the $p'$ and $t'$ nuclear coordinates. In the $S$ basis, $\hat{P}_{pt} \psi_{tp}$ generates a diagonal factor $(-1)^{L+S-J_p-J_t}$. In the $J$ basis, it generates a linear combination of $|\alpha\rangle$ basis states. The effect of elastic transfer is generally to give a backward-angle peak in the elastic scattering distribution, as illustrated in Fig. 3.9.
3.5 Electromagnetic channels

3.5.1 Photon channels

The Maxwell equations for the magnetic $H$ and electric $E$ field vectors are, in Gaussian units,

\[
\nabla \times H = \frac{4\pi}{c} j + \frac{1}{c} \frac{dE}{dt} \\
\nabla \times E = -\frac{1}{c} \frac{dH}{dt} \\
\n\nabla \cdot H = 0 \\
\n\nabla \cdot E = 4\pi \rho ,
\]

(3.191)

where the source terms $j$ and $\rho$ are the charge current and charge density respectively. These describe the classical electromagnetic field surrounding a nucleus. At low velocities, this is predominantly the electrostatic field arising from $\rho$, the charge density of the protons in the nucleus.

In addition to this electrostatic field, there can be radiative photons of higher energies produced by capture reactions, and similar photons that may lead to the breakup of nuclei. To a first approximation, these need only be considered one at a time, since (unlike a coherent laser field), radiative photons will generally react individually. In this section, therefore, we consider the coupling of one radiative photon with the charges in a nucleus that may lead to its excitation, and also the reverse: the production of a photon by a decaying nuclear state. The higher energy states in a nucleus may be in the continuum, in which case these reactions are photodisintegration and photoproduction reactions, respectively. The photon by itself will have a specific energy $E = \hbar \omega$ and momentum $p_\gamma = \hbar k_\gamma = E_\gamma / c$.

Vector potential

We define, in the standard manner, the vector potential $A(t)$ and scalar potential $\phi(t)$ such that

\[
H = \nabla \times A(t) , \quad (3.192a)
\]

\[
E = -\nabla \phi(t) - \frac{1}{c} \frac{dA}{dt} . \quad (3.192b)
\]

Substituting these into the Maxwell equations (3.191) we find

\[
\nabla^2 A(t) + \frac{1}{c^2} \frac{d^2 A}{dt^2} = -\frac{4\pi}{c} j + \nabla (\nabla \cdot A(t)) + \frac{1}{c} \frac{d\phi}{dt} \quad (3.193)
\]

\[
\nabla^2 \phi(t) = -4\pi \rho(t) - \frac{1}{c} \frac{d}{dt} (\nabla \cdot A(t)) , \quad (3.194)
\]
where we have used the identity $\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$.

The potentials $A(t)$ and $\phi(t)$ are not fixed, but may be changed according to some arbitrary spatial gauge function $\chi(t)$ by

$$A'(t) = A(t) + \nabla \chi(t) \quad (3.195)$$

$$\phi'(t) = \phi(t) + \partial \chi(t)/\partial t \quad . \quad (3.196)$$

This means that we are free to choose the gauge $\chi(t)$, for example so that

$$\nabla \cdot A(t) = 0 \quad , \quad (3.197)$$

which is called the Coulomb or transverse gauge. In this gauge, the scalar potential satisfies Poisson’s equation

$$\nabla^2 \phi(t) = -4\pi \rho(t) \quad (3.198)$$

and the vector potential satisfies an inhomogeneous wave equation

$$\nabla^2 A(t) + \frac{1}{c^2} \frac{d^2 A}{dt^2} = -\frac{4\pi}{c} j(t) \quad . \quad (3.199)$$

We now make our physical approximation, separating the electrostatic field caused by the charges $\rho$ from the radiative photon that couples to the current $j$. In this case, $\phi$ becomes time-independent like $\rho$, and the electromagnetic wave equation for a radiative single photon becomes

$$\nabla^2 A(t) + \frac{1}{c^2} \frac{d^2 A}{dt^2} = -\frac{4\pi}{c} j \quad . \quad (3.200)$$

For a fixed monochromatic photon energy $E = h\omega$, we can write the time-dependent vector potential $A(t)$ in terms of a time-independent $A$ and current $j$ as

$$A(t) = Ae^{-i\omega t} + A^* e^{i\omega t} \quad (3.201)$$

$$j(t) = je^{-i\omega t} + j^* e^{i\omega t} \quad . \quad (3.202)$$

The wave equation for the stationary state $A$ vector potential is thus

$$\nabla^2 A + k^2 A = -\frac{4\pi}{c} j \quad . \quad (3.203)$$

### 3.5.2 Coupling photons and particles

Classically, the charge current $j = vq\rho$, the product of the particle velocity, charge, and number density. In quantum mechanics, the number density is...
\( \rho = \psi^* \psi \), and the velocity operator is \( \hat{v} = \frac{\hat{p}}{m} = \frac{\hbar}{im} \nabla \), so the quantum mechanical charge current operator is

\[
\hat{j} = \Re \frac{\hbar q}{im} \nabla .
\] (3.204)

The analytic form of this operator is

\[
\vec{j} = \frac{\hbar q}{2im} (\vec{\nabla} - \vec{\nabla}) ,
\] (3.205)

which has local matrix elements at each spatial position

\[
(\phi | \vec{j} | \psi) = \frac{\hbar q}{2im} [\phi^*(\nabla \psi) - (\nabla \phi)^* \psi] .
\] (3.206)

The parentheses in the notation \( (\phi | \vec{j} | \psi) \) mean that no spatial integration is implied. This local current expression satisfies the continuity equation

\[
\nabla \cdot (\psi | \vec{j} | \psi) + \partial (\psi^* \psi)/\partial t = 0 .
\] (3.207)

**Photo-production**

Photonuclear couplings may work in two directions. A current of charged particles may produce photons, for example in the photo-production reaction \(^7\text{Be}(p, \gamma)^8\text{B}\). At other times, the photons may cause the movement of charged particles and the breakup of bound states as in \(^8\text{B}(\gamma, p)^7\text{Be}\), and this is called photo-disintegration and described in the following subsection.

The coupling from particle current to photon production, for particle initial scattering state \( \psi \) and final bound state \( \phi_b \), is therefore described by

\[
\nabla^2 A + k_r^2 A = -\frac{4\pi}{c} (\phi_b | \vec{j} | \psi) .
\] (3.208)

It is convenient to multiply this equation by \(-\hbar c/k_\gamma\), so that the coefficient of \( A \) becomes \( h c/k_\gamma \times k_r^2 = h c k_\gamma = E_\gamma \), and has the same units of energy as in a Schrödinger equation:

\[
\left[ -\frac{\hbar c}{k_\gamma} \nabla^2 - E_\gamma \right] A = \frac{4\pi\hbar}{k_\gamma} (\phi_b | \vec{j} | \psi) .
\] (3.209)

This equation has the form of a particle-to-photon coupled equation, linking the initial particle state \( \psi \) to an outgoing photon field \( A \) with an ‘interaction Hamiltonian’ \( H_{\gamma p} = \frac{4\pi\hbar}{k_\gamma} (\phi_b | \vec{j} | \psi) . \)
3.5 Electromagnetic channels

Photo-disintegration

The standard minimal gauge coupling describing the influence of an electromagnetic field on particle motion is found by transforming the free-field Schrödinger equation \[ \frac{\hat{p}^2}{2m} + V - E \psi = 0 \] by replacing
\[ \hat{p} \rightarrow \hat{p} - \frac{q}{c} \mathbf{A}, \] (3.210)
where \( \mathbf{A} \) is the vector potential for the electromagnetic field. Expanding the square,
\[ \left[ \frac{1}{2m} \left( \hat{p}^2 - \frac{2q}{c} \mathbf{A} \cdot \hat{p} + \frac{q^2}{c^2} |\mathbf{A}|^2 \right) + V - E \right] \psi = 0 \] (3.211)
since \( \hat{p} \) and \( \mathbf{A} \) commute in the Coulomb gauge Eq. (3.197) with \( \nabla \cdot \mathbf{A} = 0 \).

Neglecting the \( |\mathbf{A}|^2 \) term according to our one-photon approximation, this particle equation becomes
\[ \left[ \hat{T} + V - E \right] \psi - \frac{1}{c} \mathbf{A} \cdot \mathbf{\leftrightarrow j} \phi_b = 0 \] (3.213)
where we use the analytic form of the current operator \( \mathbf{\leftrightarrow j} \) defined by Eq. (3.205), both terms of which contribute equal real parts to any matrix element.

For a photo-disintegration reaction with particle initial state \( \phi_b \) and final scattering state \( \psi \), this equation is often used in the form
\[ \left[ \hat{T} + V - E \right] \psi - \frac{1}{c} \mathbf{A} \cdot \mathbf{\leftrightarrow j} \phi_b = 0 \] (3.213)
where we have an effective ‘interaction Hamiltonian’ \( H_{\text{int}} = -\frac{1}{c} \mathbf{A} \cdot \mathbf{\leftrightarrow j} \) that couples between two particle states.

Equally, if Eq. (3.212) is written as
\[ \left[ \hat{T} + V - E \right] \psi - \frac{1}{c} \mathbf{\leftrightarrow j} \cdot \mathbf{\leftrightarrow j} \phi_b = 0 \] (3.214)
for fixed initial particle state \( \phi_b \), then this again appears as an equation coupling the incoming photon field \( \mathbf{A} \) to the particle state \( \psi \) in the continuum.

The coupling operator is \( H_{\gamma\gamma} = -\frac{1}{c} \mathbf{\leftrightarrow j} \cdot \mathbf{\leftrightarrow j} \).

3.5.3 Photon cross sections

When the photon outgoing wave is \( \mathbf{A} \) in direction \( \mathbf{k} \), so \( \mathbf{A}(r) = a e^{i \mathbf{k} \cdot r} \), the cross section depends on the number flux of photons. In order to determine
this from the magnitude of $A$, we calculate the Poynting vector, which is the energy flux, and then divide by the energy of each photon.

Given $A$, the physical vector potential of Eq. (3.201) is

$$A(t) = 2\Re(e^{i\omega t}) = 2\Re(e^{i(k \cdot r - \omega t)}) = 2a \cos (k \cdot r - \omega t). \quad (3.215)$$

The time independence of $\phi$, as used to derive Eq. (3.200), thus uses

$$E(t) = -\nabla \phi(t) - \frac{1}{c} \frac{dA}{dt},$$

to give

$$E(t) = 2k_a \sin (k \cdot r - \omega t) \quad (3.216)$$

and

$$H(t) = 2(k \times a) \sin (k \cdot r - \omega t). \quad (3.217)$$

The Poynting vector is

$$S = \frac{c}{4\pi} E \times H. \quad (3.218)$$

Far from the source, where the radiative field has $E$ and $H$ mutually perpendicular, the magnitudes $|H|$ and $|E|$ become equal, so

$$|S| \simeq \frac{c}{4\pi} |E|^2 = \frac{c}{4\pi} |H|^2. \quad (3.219)$$

As $\langle |\sin(t)|^2 \rangle = 1/2$, the time-averaged $\langle |E(t)|^2 \rangle = 1/2 |2k_a|^2 = 2k_a^2 |a|^2$, and the time-averaged energy flux is

$$\langle |S(t)|^2 \rangle = \frac{k_a^2 c}{2\pi} |a|^2. \quad (3.220)$$

Dividing by the photon energy $E_\gamma = \hbar k_\gamma c$, we find the outgoing photon number flux

$$j_\gamma = \frac{k_a}{2\pi \hbar} |a|^2. \quad (3.221)$$

Because the flux has factors additional to the usual expression of Eq. (3.2) in quantum mechanics, it is now convenient to define a vector photon \textit{wave function}

$$\zeta(r) = \sqrt{\frac{k_a}{2\pi \hbar c}} A \quad (3.222)$$

so that photon number density is simply $|\zeta(r)|^2$, and the photon number flux now appears as

$$j_\gamma = c|\zeta(r)|^2. \quad (3.223)$$
By Eq. (3.2) this is appropriate for a quantum mechanical object moving at the speed of light \( c \).

In terms of this new photon wave function, the photo-production ‘coupled equation’ (3.209) becomes

\[
\left[-\frac{\hbar c}{k_\gamma} \nabla^2 - E_\gamma\right] \zeta = \sqrt{\frac{k_\gamma}{2\hbar c}} \frac{4\pi \hbar}{k_\gamma} (\phi_b | \vec{j} | \psi) \\
= 2 \sqrt{\frac{2\pi \hbar}{k_\gamma c}} (\phi_b | \vec{j} | \psi) \\
= 2 \sqrt{\frac{\hbar}{\omega}} (\phi_b | \vec{j} | \psi), \quad (3.224)
\]

and the photo-disintegration equation (3.214) becomes

\[
[\hat{T} + V - E] \psi = \sqrt{\frac{2\pi \hbar}{k_\gamma c}} (\vec{j} \phi_b) \cdot \zeta \\
= \sqrt{\frac{\hbar}{\omega}} (\vec{j} \phi_b) \cdot \zeta \quad (3.225)
\]

These two equations (3.224, 3.225) have almost the same kind of coupling interaction, namely

\[
V_{\gamma p} = 2 \sqrt{\frac{\hbar}{\omega}} (\phi_b | \vec{j} |), \quad (3.226)
\]
\[
V_{p\gamma} = \sqrt{\frac{\hbar}{\omega}} (\vec{j} \phi_b). \quad (3.227)
\]

respectively, using the bound particle state \( \phi_b \) in each case. The factor of 2 difference in magnitude between these two couplings arises from the relativistic kinematics for photons, as explained below.

The electromagnetic coupling operators of Eq. (3.226) are written in their current form, as they use the derivative current operator of Eq. (3.205). This is commonly judged as too complicated for everyday use, so in section 4.5 an approximate form for the couplings will be derived. This form will be valid for long photon wavelengths (low energy photons).

The photon T matrix element is defined as the asymptotic amplitude \( T_{J_J}^{\gamma} \) of the outgoing solution of Eq. (3.224), following the pattern of Eq. (3.35) established for particle channels, and gives the photo-production cross section

\[
\sigma_{\gamma p}^{J_J} = \frac{4\pi}{k_t^2} \frac{1}{(2J_p + 1)(2J_t + 1)} \frac{c}{v_t} \sum_{J_T} (2J_T + 1) |T_{J_J}^{\gamma}|^2 \quad (3.228)
\]

since the outgoing photon velocity is the speed of light \( c \).
For low-energy nuclear reactions, Newtonian non-relativistic kinematics is sufficiently accurate. Later, however, we discuss breakup reactions at intermediate energies, and photons themselves, with zero rest mass, are necessarily relativistic, so we need to know the relativistic extensions of coupled-channels scattering theory.

In a coupled channels scheme \[ [T_\alpha - E_\alpha] \psi_\alpha + \sum_\beta V_{\alpha\beta} \psi_\beta = 0, \]
the channel kinetic energy is \( E_\alpha = E_T - E_0 \) for total relativistic energy \( E_T^2 = (\hbar c)^2 + E_0^2 \) and rest energy \( E_0 = m_0 c^2 \).

The kinetic energy term \( T_\alpha \) can be written in general as
\[
t_\alpha = (E_T - E_0)/k^2 = E_0 (\gamma - 1)/k^2
\]
for \( k = m_0 \gamma v/\hbar \) with \( \gamma = (1 - v^2/c^2)^{-1/2} \). In the non-relativistic limit, \( t_\alpha \to \hbar^2/2m_0 \).

The \( T \) matrix integrals of Eq. (3.149) can be also written more generally, for elastic channel \( \alpha_i \), as
\[
T_{\alpha\alpha_i} = -2i \frac{v_\alpha}{v_{\alpha_i}} T_{\alpha\alpha_i} \int \tilde{F}_\alpha \sum_\beta V_{\alpha\beta} \psi_\beta dR ,
\]
from which the symmetric \( S \) matrix form of Eq. (3.92) is
\[
\tilde{S}_{\alpha\alpha_i} = 2i \frac{v_\alpha}{v_{\alpha_i}} T_{\alpha\alpha_i}
= -\frac{2i}{\hbar \sqrt{v_\alpha v_{\alpha_i}}} \frac{h v_\alpha}{t_\alpha k_\alpha} \int \tilde{F}_\alpha \sum_\beta V_{\alpha\beta} \psi_\beta dR .
\]
Non-relativistically, the quantity \( w_\alpha = h v_\alpha/t_\alpha k_\alpha = 2 \), the same for all channels. The general relativistic expression, however, is \( w_\alpha = 1 + (1 - v_\alpha^2/c^2)^{1/2} \), namely \( w_\alpha = 2 \) for \( v_\alpha \ll c \), but \( w_\alpha = 1 \) for \( v_\alpha = c \) with photons. This implies that for a symmetric \( S \) matrix we must have \( w_\alpha V_{\alpha\beta} = w_\beta V_{\beta\alpha} \). The photon and particle coupling interactions of Eq. (3.226) indeed satisfy
\[
V_{\gamma p} = 2 V_{p\gamma} ,
\]
in contrast to nonrelativistic particle-particle couplings which must all be symmetric.

### 3.5.4 Vector spherical harmonics

The photon vector potential \( \mathbf{A}(\mathbf{r}) \) and its normalised form \( \zeta(\mathbf{r}) \) are described by a three-dimensional vector at every spatial position \( \mathbf{r} \). This three coordinates may be equivalently mapped onto the three \( m \)-state amplitudes
\( \mu = -1, 0, +1 \) for the photon as a spin-1 object. Let \( \xi_\mu \) be three (complex) unit vectors in 3D space, such that the complex coefficients \( A_\mu \) in \( \mathbf{A} = \sum_\mu A_\mu \xi_\mu \) in this basis transform as the components of a spin-1 vector. One choice for the \( \xi_\mu \) is \( \xi_0 = \mathbf{\hat{z}} \) and \( \xi_{\pm 1} = \mp (\mathbf{\hat{x}} \pm i\mathbf{\hat{y}}) / \sqrt{2} \). If this coordinate system is specifically chosen so that the \( \mathbf{\hat{z}} \) axis coincides with the local direction of photon propagation, the Coulomb gauge condition \( \nabla \cdot \mathbf{A} = 0 \) is simply \( A_0 = 0 \), as the polarisation vectors \( \xi_{\pm 1} \) map the allowed transverse directions.

The vector field \( \mathbf{A}(r) \) can therefore be expanded as the coupling of a spatial angular momentum \( \lambda \) with photon spin 1 to form a total spin \( J_\gamma \). For each value of \( J_\gamma \) and its \( z \)-axis projection \( M_\gamma \) there is a multipole field \( \mathbf{A}_{J_\gamma M_\gamma}(r) \) of the form

\[
\mathbf{A}_{J_\gamma M_\gamma}(r) = \sum_\lambda \xi_\mu \left( \lambda M_\gamma - \mu, 1\mu | J_\lambda M_\gamma \right) Y^M_{\lambda \mu}(\mathbf{\hat{r}}) \mathcal{X}_{J_\lambda}(r) / r \tag{3.234}
\]

for some radial functions \( \mathcal{X}_{J_\lambda}(r) \) for each \( \lambda = J_\gamma - 1, J_\gamma, J_\gamma + 1 \).

We can define vector spherical harmonics as the basis functions

\[
Y^M_{\lambda \mu}(\mathbf{\hat{r}}) = \sum_\mu \xi_\mu \left( \lambda M - \mu 1\mu | JM \right) Y^M_{\lambda \mu}(\mathbf{\hat{r}}) \tag{3.235}
\]

with parity \( (-1)^\lambda \), so the multipole fields become a linear combination of vector spherical harmonics

\[
\mathbf{A}_{J_\gamma M_\gamma}(r) = \sum_\lambda Y^M_{\lambda \mu}(\mathbf{\hat{r}}) \mathcal{X}_{J_\lambda}(r) / r . \tag{3.236}
\]

### 3.5.5 Electric and magnetic plane wave components

For plane wave vector field in the \( z \)-direction, \( \mathbf{A}_\mu = \xi_\mu e^{ikz} \), the polarisation states allowed by the gauge condition are \( \mu = \pm 1 \). Following [10], this wave may be decomposed into parts of definite parity according to

\[
\mathbf{A}_\mu = \mu(2\pi)^{1/2} \sum_{J_\gamma} (2J + 1)^{1/2} i^{J_\gamma} \left[ \mathbf{A}_{J_\gamma \mu}(r; \mathcal{M}) + i\mu \mathbf{A}_{J_\gamma \mu}(r; \mathcal{E}) \right] \tag{3.237}
\]

where the electric part of \( \mathbf{A} \) is defined as that with parity \( (-1)^{J_\gamma + 1} \), and the magnetic part as that with parity \( (-1)^{J_\gamma} \). The expansion for a wave travelling in an arbitrary direction \( \mathbf{k} \) can be obtained by rotating this expansion:

\[
\xi_\mu e^{i\mathbf{k} \cdot \mathbf{r}} = \mu(2\pi)^{1/2} \sum_{J_\gamma M} (2J + 1)^{1/2} i^{J_\gamma} \left[ \mathbf{A}_{J_\gamma M}(r; \mathcal{M}) + i\mu \mathbf{A}_{J_\gamma M}(r; \mathcal{E}) \right] D^{J_\gamma}_{M \mu}(\mathbf{R}) , \tag{3.238}
\]
where $R$ is the rotation taking the $z$-axis to the direction $\hat{k}$.

Henceforth with simplified notation $J \equiv J_{\gamma}$, the magnetic part is the $\lambda = J$ term

$$A_{JM}(r; M) = r^{-1}F_J(0, kr) Y_{JM}^M(\hat{r})$$

and the electric part is the combination of $\lambda = J \pm 1$ terms

$$A_{JM}(r; E) = \frac{\sqrt{J+1}}{2J+1} r^{-1} F_{J-1}(0, kr) Y_{J-1,J}^{M}(\hat{r}) - \frac{J}{2J+1} r^{-1} F_{J+1}(0, kr) Y_{J+1,J}^{M}(\hat{r}),$$

where the coefficients have been determined from the transverse gauge condition $\nabla \cdot A_{J_{\gamma}M,\gamma}(r; e) = \nabla \cdot A_{J_{\gamma}M}(r; M) = 0$.

Both components are normalised like plane waves,

$$\int A_{JM}(r; e) \cdot A_{JM}(r; e') dr = \delta(k - k')\delta_{JJ'}\delta_{MM'}\delta_{ee'}$$

where $e, e' = M$ or $E$, and they are related by

$$A_{JM}(r; M) = \frac{1}{ik} \nabla \times A_{JM}(r; E)$$

$$A_{JM}(r; E) = \frac{1}{ik} \nabla \times A_{JM}(r; M).$$

We may also define a longitudinal part of the field, another combination of $\lambda = J \pm 1$ terms, as

$$A_{JM}(r; \text{long}) = \frac{1}{k} \nabla \left( r^{-1} F_J(0, kr) Y_{J}^{M}(\hat{r}) \right) = \sqrt{\frac{J}{2J+1}} r^{-1} F_{J-1}(0, kr) Y_{J-1,J}^{M}(\hat{r}) + \sqrt{\frac{J+1}{2J+1}} r^{-1} F_{J+1}(0, kr) Y_{J+1,J}^{M}(\hat{r}),$$

which, though $\nabla \cdot A_{J_{\gamma}M,\gamma}(r; \text{long}) \neq 0$, and hence not satisfying the transverse gauge condition, the longitudinal component will be useful later since it differs in form to Eq. (3.240) only by the coefficients of the two terms.

References

[3] Coulomb wfs as 1F1
[4] COULFG
[5] COULCC
Table 3.6. Electromagnetic multipoles

Nuclear transitions, we will see in general in Ch. 4, are classified as dipole, quadrupole, etc. Electric and magnetic transitions are classified not according to their spatial part $\lambda$ but the total angular momentum $J_\gamma$ that is transferred to or from the photon. E1 and M1 photons have $J_\gamma = 1$, whereas the E2 and M2 have $J_\gamma = 2$, etc.

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[9] Wavepacket scattering

Exercises

3.1 Derive Eq. (3.89).

Still to come in this chapter (or elsewhere e.g. ch. 4)?

(i) Polarisation: definitions, etc.
(ii) How the partial wave expansion of the e/m field satisfies the Coulomb gauge (7)
(iii) Separation Eq. (3.237) of the e/m field into electric and magnetic parts: why?
(iv) Figures showing complex k and E planes
(v) Relativistic kinematics in general? – ch. 1