This chapter shows how the interactions in and between channels may be calculated on the basis of some potential model for a few interacting bodies. That is, a Hamiltonian is defined whose matrix elements give rise to the channel couplings, also known as transition potentials. The parameters in this Hamiltonian may be found either from structure models (Chapter 5), or from fitting data (Chapter 15). It is also possible to directly fit the effects of these couplings on the asymptotic amplitudes of the wave functions, and this is the basis of the R matrix phenomenology discussed in Chapter 10.

4.1 Optical potentials

Before we can discuss more detailed reaction mechanisms, we need to see the typical kinds of potentials used for elastic scattering, and also the binding potentials needed to reproduce the usual single-particle structures of nucleons within a nucleus. We will start by describing the most commonly used optical optical potentials for elastic scattering, expanding on the introduction in Box 3.4.

4.1.1 Typical forms

The interaction potential between a nucleon and a spherical nucleus is usually described by an attractive nuclear well of depth $V_r$ with a radius $R_r$ slightly larger than the nuclear matter radius, and a diffuse nuclear surface. Most commonly we use the ‘Saxon-Woods’ shape†

$$V(R) = -\frac{V_r}{1 + \exp\left(\frac{R - R_r}{a_r}\right)}$$

(4.1.1)

† Also called a ‘Woods-Saxon’ (WS) or ‘Fermi’ shape.
As summarised in Box 3.4, the central depth $V_r$ is typically between 40 and 50 MeV, and the diffuseness $a_r$ about 0.6 fm. The radius $R_r$ is proportional to the size of the nucleus, and is commonly around $R_r = r_r A^{1/3}$ for a nucleus of $A$ nucleons, with $r_r \approx 1.2$ fm. Similar potentials can be used for the interaction between two nuclei with mass numbers $A_1$ and $A_2$, if the radii are scaled instead as $R_r = r_r (A_1^{1/3} + A_2^{1/3})$, since this is proportional to the sum of the individual radii.

This potential is usually combined with an imaginary and a spin-orbit part. The imaginary part, which is present at higher scattering energies as discussed in section 3.1.5, is also often given by a Woods-Saxon form

$$W(R) = -\frac{V_i}{1 + \exp \left( \frac{R-R_{ri}}{a_i} \right)}$$

for a similar geometry $R_i \gtrsim R_r$ and $a_i \approx a_r$, and a depth $V_i$ fitted to experiments giving $V_i \sim 10 - 20$ MeV depending on energy. Sometimes a surface-peaked imaginary contribution is also included, with a shape like the derivative of Eq. (4.1.2).

All the parameters (depth, radii and diffusenesses) should come from some model, or from fitting elastic scattering angular distributions. Usually the radii for the imaginary parts will be slightly larger than the real radii, reflecting the absorption that occurs from direct reactions just at and outside the nuclear surface.

The real parts of optical potential generally get weaker with increasing laboratory energy, with $\partial V_r / \partial E \approx -0.3$ for low energies up to about 20 MeV, but less rapidly at higher energies. At around 300 MeV the real part passes through zero, and becomes repulsive at higher energies, where the scattering tends to be dominated by the imaginary part.

The depth $V_r$ for neutrons is typically less than for protons, and this difference increases for targets with large neutron excess $N - Z$, as neutrons attract protons more than neutrons or protons attract each other. This effect may be parametrised in terms of the projectile isospin operator $t_z$ as

$$V(R) = V_0(R) + \frac{1}{2} t_z \frac{N - Z}{A} V_T(R)$$

for an attractive ‘isoscalar’ component $V_0(R)$ and a positive ‘isovector’ component $V_T(R)$. In this way, the central depths $V_p$ for protons and $V_n$ for neutrons differ by

$$V_p - V_n \approx 50 \left( \frac{N - Z}{A} \right) \text{ MeV}.$$
The compilation of Perey and Perey [1] is a useful listing of individual optical potentials for the many elastic scattering reactions fitted up to the year 1974.

The RIPL-2 Handbook [2] contains a list of the most important global potentials for both incident neutrons and protons:

- Koning and Delaroche [3] (Z=12-83, A=27-209, E=0.001-200 MeV), and

Other older global potentials that cover both incident neutrons and protons are those of

- Becchetti and Greenlees [5] (Z=20-92, A=40-238, E=10-50 MeV),
- Walter and Guss [6] (Z=26-82, A=54-208, E=10-80 MeV), and

Older global potentials developed exclusively for incident neutrons are those of

- Moldauer [8] (Z=20-83, A=40-209, E=0.001-5 MeV),
- Wilmore and Hodgson [9] (Z=20-92, A=40-238, E=0.01-25 MeV),
- Engelbrecht and Fiedeldey [10] (Z=20-83, A=40-210, E=0.001-155 MeV),

For incident protons alone, there are the potentials of

- Menet et al. [12] (Z=6-82, A=12-208, E=30-60 MeV),
- Perey [13] (Z=16-49, A=30-100, E=0.01-22 MeV), and

Box 4.1: Compilations and global optical potentials for nucleon-nucleus scattering.

There will also be spin-orbit forces that couple the nucleon spin to its orbital motion, of the kind to be discussed on page 129.

### 4.1.2 Global optical potentials

If experimental data for a projectile scattering on a range of nuclei $A$ at many incident energies are simultaneously fitted by an optical potential with coefficients that vary slowly with target mass number and energy, then a global optical potential is found that will be useful for interpolation (and sometimes extrapolation) to new reactions and energies.

The best known global potentials for the scattering of protons and neutrons on nuclei are listed in Box 4.1, and those for the scattering of deuterons, tritons, $^3$He and alpha particles are listed in Box 4.2.

Tests have shown [2] that in general the reproduction of elastic scattering data is worse with global potentials than using the specifically fitted potentials that appear in compilations. It may be argued, however, that op-
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For incident deuterons, the RIPL-2 compilation [2] lists the global potentials from

- Bojowald et al. [15] \((Z=6-82, A=12-208, E=20-100 \text{ MeV})\),
- Daehnick et al. [16] \((Z=20-82, A=40-208, E=11-90 \text{ MeV})\),
- Lohr and Haeberli [17] \((Z=20-83, A=40-209, E=8-13 \text{ MeV})\), and

For incident tritons and \(^3\text{He}\) particles, the global potentials of Becchetti and Greenlees [19] \((Z=20-82, A=40-208, E=1-40 \text{ MeV})\) are available.

In the case of incident alpha particles, there are global potentials from

- Avrigeanu et al. [21] \((Z=8-96, A=16-250, E=1-73 \text{ MeV})\),
- Huizenga and Igo [22] \((Z=10-92, A=20-235, E=1-46 \text{ MeV})\), and

Box 4.2: Global optical potentials for scattering of deuterons, tritons, \(^3\text{He}\) and alpha particles.

g.1.3 Folding potentials

Sometimes instead of specific or global potential, a density distribution of the target nucleus is known from electron scattering or from structure calculations. In this case, folding with for example the JLM potential [23] is one way of obtaining a nucleon-nucleus potential. The theory for folding will be given in section 5.2.

4.2 Single-nucleon binding potentials

Before we can discuss the transfer of nucleons from one interacting nucleus to another, we need to specify how they are bound (or unbound) in their initial and final states. Nucleons are bound inside nuclei, and as a first approximation we can consider them bound by the average attraction of all the other nucleons. This average potential will be very similar to the optical potentials we used above for scattering, but only the central and spin-orbit parts, without the imaginary component(s). (We discuss below a
4.2 Single-nucleon binding potentials

Fig. 4.1. Neutron eigenstates $n j^{\pi}$ in a Saxon-Woods binding potential, for spin $j$, parity $\pi$ and number of radial nodes $n$. The chosen potential has $V_r=56.6$ MeV, $r_v=1.17$ fm, $a_r=0.75$ fm, $V_{so}=6.2$ MeV, $r_{so}=1.01$ fm and $a_{so}=0.58$ fm, as a function of $A$ that scales both radii as $A^{1/3}$. The occupation numbers are $n_j = 2j + 1$.

possible significance for the imaginary parts.) The eigenstates in the average potential are called single-particle levels since they ignore higher-order interactions and correlations. We will use $r$ to describe the coordinates of the nucleons within a nucleus, along with previous $R$ to describe the relative coordinate of two scattering nuclei, and the respective angular momenta will be $\ell$ and $L$. Many kinds of forces are equally useful in both circumstances.

4.2.1 Neutron and proton single-particle states in nuclei

Let us take a typical combination of central and spin-orbit potentials. The central potentials are given as in Eq. (4.1.1), and details of the the spin-orbit forces will be given in section 4.3.2. The Fig. 4.1 then shows the eigenenergies of single-particle levels for the fixed potential parameters given in the caption, plotting the results as a function of nuclear mass $A$, by scaling both the radii as $A^{1/3}$. The multiple levels depend on the quantum numbers for motion in the $r$ coordinate: angular momentum $\ell$, intrinsic nucleon spin $s = 1/2$, and total spin $j = \ell \pm s$. The levels can be labeled by $n j^{\pi}$ for parity $\pi = (-1)^\ell$, or equivalently by $n \ell j \dagger$, and where $n = 1, 2, \ldots$ is the number of

† Following atomic spectroscopy, we use names $s, p, d, f, g$ etc., for $\ell = 0, 1, 2, 3, 4$ respectively. The lowest three levels in Fig. 4.1 are then equivalently named $1s_{1/2}$, $1p_{1/2}$ and $1p_{3/2}$. 
nodes in the radial wave function\(^\dagger\). The wave functions\(^\S\) \(\phi^m_{\ell sj;b}(r)\) for the bound state \(b\) have angular and radial factors as

\[
\phi^m_{\ell sj;b}(r) = [Y_{\ell}(\hat{r}) \otimes X_s]_{jm} u_{\ell sj;b}(r)/r ,
\]

with normalisation

\[
\int |\phi^m_{\ell sj;b}(r)|^2 dr = \int_0^\infty |u_{\ell sj;b}(r)|^2 dr = 1 \quad (4.2.2)
\]

for all magnetic substates \(m\), with \(X_s\) the state of the nucleon’s intrinsic spin.

We see that the lowest level is for \(\ell = 0\), labeled \(1/2^+\) (or \(s_{1/2}\)), followed by a closely-spaced pair of \(\ell = 1\) states \(3/2^-\) and \(1/2^-\). This pair is called the \(p\)-shell. Higher in energy is the \(sd\)-shell with three levels \(5/2^-\), \(1/2^+\) and \(3/2^+\), and in heavier nuclei there is a \(pf\)-shell higher again. The second \(1/2^+\) level, in the \(sd\)-shell, will have an additional radial node in its wave function, compared with the first such level in the \(s\)-shell. Single-particle levels for protons will be similar to these, but with a modified depth according to Eq. (4.1.3), and with their levels raised in energy by the average Coulomb repulsion.

Depending on the isotope \(A = N + Z\) are the number \(N\) of neutrons and \(Z\) of protons to occupy these single-particle levels. The maximum occupation numbers for each level are the values \(n_j = 2j + 1\) in Fig. 4.1. A \(^{16}\)O nucleus, for example, has \(N = Z = 8\), so each species of nucleon fills its \(s\) and \(p\) shells, if we adopt the simplest filling order from lowest to highest energies, and neglect changes caused by more complicated mixing and correlations.

We adopt for now this single-particle filling prescription.

Nucleons occupying states in a nucleus can be seen in pickup reactions that remove one of the low-energy nucleons. It will leave a hole, which, if the hole is below other occupied levels, amounts to an excitation energy of the residual nucleus.

The energy of the highest occupied level is called the Fermi energy \(E_F\), and levels above this are unoccupied. These unoccupied levels can be seen in transfer reactions that bring another nucleon into the nucleus\(^\dagger\). If a newly occupied level after a transfer reaction is above another unoccupied level, than it also represents an excited state in the residual nucleus.

In nature, nucleon bound states will never be exactly at the single-particle

\(^\dagger\) Here we include the node at the origin, so start counting with \(n = 1\). Another convention is to include only nodes at \(r > 0\), and therefore to start at \(n = 0\).

\(^\S\) These wave functions \(\phi(r)\) are of solely one nucleon, not the whole nucleus.

\(^\dagger\) Sometimes these are called ‘stripping’ reactions, but that word is often used for pickup reactions as well.
levels shown in the figure, because of higher-order effects which go beyond the spherical potential model. Levels may be shifted, and also fragmented among all the true eigenstates of the \( A \)-body system. The effect of this fragmentation is described by overlap functions, the square norm of which are the spectroscopic factors which will in general be different from unity.

To accommodate the energy shifts, it is normal to slightly adjust if necessary the precise strength \( V_r \) of the central potential to reproduce the experimental binding energies. This guarantees the correct exponential tail of the wave function (necessary, we will see, for sub-Coulomb transfer reactions), and also the post-prior equivalence that should be obtained for first-order DWBA transfer cross sections.

When the binding potentials are too weak to bind a given single-particle level, then it will become unbound, and while it is still not too far from threshold it will be visible as a resonance or a virtual state. As discussed on page 58, states with \( \ell = 0 \) and no Coulomb barrier will become virtual states, and the others will become resonances. We generally assume that the potentials for the continuum are the same as for bound states, apart from the energy dependence mentioned in page 120, though if necessary the central depth should be varied slightly to place a resonance at the experimental energy.

In between resonances there are still scattering states, as a Hamiltonian has eigenstates at every positive scattering energy \( E = \hbar^2 k^2 / 2\mu \), as described in the previous chapter. We may label the corresponding wave functions by \( \phi_{\ell sj}(r; k) \), or sometimes using \( E \) rather than \( k \). The wave functions, in the one-channel case, have been factorised as

\[
\phi_{\ell sj}^m(r; k) = [Y_{\ell}(\hat{r}) \otimes X_{sj}]_{jm} u_{\ell sj}(r; k)/r ,
\]

where the \( u_{\ell sj}(r; k) \) are normalised as usual for scattering wave functions†. This asymptotic normalisation gives the overlap integral of two radial functions as

\[
\int_0^\infty u_{\ell sj}(r; k)^* u_{\ell sj}(r; k')dr = \frac{\pi}{2} \delta(k - k') .
\]

It is therefore common to renormalise the continuum states using

\[
\hat{u}_{\ell sj}(r; k) = \sqrt{2/\pi} u_{\ell sj}(r; k)
\]

so that we have an orthonormal set

\[
\int_0^\infty \hat{u}_{\ell sj}(r; k)^* \hat{u}_{\ell sj}(r; k')dr = \delta(k - k') .
\]

† That is, in the same way as the \( \chi_L(R) \) of Chapter 3, by Eq. (3.1.37).
Other renormalisations are possible to give \( \delta \)-functions in energy \( E \). Using the \( \hat{u}(r) \) wave functions, we can give the completeness relation when all the bound and scattering states are combined:

\[
\sum_b \langle u_{\ell sj}^b(r) | u_{\ell sj}^b(r') \rangle + \int_0^\infty |\hat{u}_{\ell sj}(r; k)|^2 \langle \hat{u}_{\ell sj}(r'; k) | dk = \delta(r - r'),
\]

for each partial wave \( \ell sj \).

Resonances, in this basis, are just concentrations of strength at particular energies, as measured by the magnitudes of \( u_{\ell sj}(r; k) \) in the nuclear interior, so resonances should not be included as explicit terms in addition. A sum over resonances can only be given, Bergren [24] explains, if we deform the complex contour of the \( k \) integration in Eq. (4.2.7) so that it goes below some of the narrower resonance that are near the real \( k \) axis.

### 4.2.2 Optical potentials extended to bound states

In the previous section, we described the binding potentials giving all the single-particle levels for occupation in a simple uncorrelated structure model of the nucleus. Consider now a a full nucleus, where all the levels up to the Fermi energy are occupied. The occupied states above the Fermi energy are candidates for further nucleons that can be transferred in, and the occupied states below the Fermi energy are candidates for where nucleons can be picked out in a removal reaction. That is, in this ‘full nucleus’ case, we interpret single-particle states below the Fermi level to refer to nuclear states in the \( A - 1 \) system as specific holes in the \( A \)-nucleon state, and interpret single-particle states with \( E > E_F \) to refer to nuclear states in the \( A + 1 \) system as specific particles added to the \( A \)-nucleon state.

Holes or transfer states that are away from the Fermi level, and hence describe excited nuclei, will of course decay with some decay lifetime \( \tau \). This may occur by \( \gamma \) emission, or by excitation of other nucleons in the system, and the finite lifetime gives a spreading width to the excited state by \( \Gamma \approx \frac{\hbar}{\tau} \), as for a resonance. It is called a ‘spreading width’ because the excitation is spreading into other modes of nuclear excitation.

This decay lifetime may also be described by an imaginary potential \( W(R) < 0 \), according to the factor \( e^{-2|W|^\text{t}/\hbar} \) in Eq. (3.1.108). On this basis, therefore, we may extend the complex optical potential to bound states at negative energies, to describe the loss of flux from excited nucleonic states to more general nuclear excitations such as the statistical compound-nucleus modes. We choose the optical potential to give the correct decay times for excited states away from the Fermi level. Clearly this extended optical po-
4.3 Coupling potentials

Now that we have defined central potentials for scattering and bound states between nucleons and nuclei, we are able to give details of the off-diagonal potentials that couple together different partial waves within a multi-channel set. These coupling potentials describe processes that usually transfer angular momentum and energy from one interacting body to another, so we begin with their multipole analysis.

4.3.1 Multipole analysis of transition potentials

We are here concerned with an interaction part $H_{\text{int}}$ of the Hamiltonian that is responsible for the coupling between distinct states of a nucleus, and its orbital motion, as it interacts with another. The Hamiltonian itself is a scalar, which may be constructed as a sum of transition operators of multipoles $\lambda$ for both the nucleus and its spatial orbit. For each $\lambda$ value, let the orbital angular transition operator be the spherical harmonic $Y_{\lambda m}(\hat{R})$ and the nuclear transition operator be $T_{\lambda}(\xi)$ for the necessary internal coordinates $\xi$ of the nucleus (normalised so $T_0 = 1$, the identity operator). The $\lambda$-multipole part of $H_{\text{int}}$ is the scalar product of these:

$$H_{\text{int}}^{\lambda}(\xi, R) = \sqrt{4\pi} \mathcal{F}_\lambda(R) \sum_{m=-\lambda}^{+\lambda} T_{\lambda m}(\xi)^* Y_{\lambda m}^*(\hat{R})$$

$$\equiv \sqrt{4\pi} \mathcal{F}_\lambda(R) T_{\lambda}(\xi) \cdot Y_{\lambda}(\hat{R})$$

where $\sqrt{4\pi} \mathcal{F}_\lambda(R)$ expresses the as-yet-unknown radial dependence. The $\sqrt{4\pi}$ is a constant chosen to that a monopole equation gives $H_{\text{int}}^{0}(\xi, R) = \mathcal{F}_0(R)$ as $T_0 = 1$.

The quantity $\lambda$ is called the multipolarity of the reaction, or angular momentum transfer. The $\lambda = 0$ case describes monopole (scalar) transitions, which are either diagonal (like the optical potential), or off-diagonal couplings between states of the same spin. The $\lambda = 1$ describes dipole (vector) transitions and 2 the quadrupole (tensor) processes.

Most often, a given reaction mechanism changes the spin state of only one of the interacting pair of nuclei, the spin of the other nucleus not being dynamically coupled. In this simpler case, the J and S coupling schemes
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of section 3.2.1 both reduce to \( |(LJ)J_{\text{tot}}\rangle \). In this basis, the \( R \)-dependent matrix element of any coupling Hamiltonian (4.3.1) may be found by integrating over all variables except \( R \), obtaining

\[
\langle (L_fJ_f)J_{\text{tot}}|H_{\text{int}}^\lambda(\xi,R)|(L_iJ_i)J_{\text{tot}}\rangle = \sqrt{4\pi} \mathcal{F}_\lambda(R)
\]

\[
\times (-1)^{\lambda+J_{\text{tot}}+L_i+J_f} \left\{ \begin{array}{ccc} L_i & J_i & J_{\text{tot}} \\ J_f & L_f & \lambda \end{array} \right\} \langle L_f || Y_\lambda || L_i \rangle \langle J_f || T_\lambda || J_i \rangle
\]

(4.3.2)

by Eq. (1A-72) of Bohr & Mottelson [25]. The reduced matrix element of the spherical harmonic, with \( \hat{x} \equiv \sqrt{2}\hat{x}+1 \), is

\[
\langle L_f || Y_\lambda || L_i \rangle = (4\pi)^{-1/2} \hat{x}^\lambda \hat{L}_i^\dagger \langle L_i0,\lambda0|L_f0 \rangle
\]

(4.3.3)

using the definition of [25]:

\[
\langle j_m_f|\hat{O}_{jm_i}|j_m_i\rangle = \frac{\langle j_m_i, jm || j_m_f \rangle}{j_f} \langle j_f || \hat{O}_j || j_i \rangle
\]

(4.3.4)

(a different definition from that of [26]). We now define a transition potential to include the nuclear reduced matrix element for a specific pair of states:

\[
V_{\lambda}^{fi}(R) = \mathcal{F}_\lambda(R) \langle J_f || T_\lambda || J_i \rangle
\]

(4.3.5)

so we henceforth use the general form

\[
\langle (L_fJ_f)J_{\text{tot}}|H_{\text{int}}^\lambda(\xi,R)|(L_iJ_i)J_{\text{tot}}\rangle
\]

\[
= V_{\lambda}^{fi}(R)(-1)^{\lambda+J_{\text{tot}}+L_i+J_f} \hat{x}^\lambda \hat{L}_i^\dagger \langle L_i0,\lambda0|L_f0 \rangle \left\{ \begin{array}{ccc} L_i & J_i & J_{\text{tot}} \\ J_f & L_f & \lambda \end{array} \right\}
\]

\[
\equiv V_{\lambda}^{fi}(R)(-1)^{\lambda+J_{\text{tot}}-J_i-L_f} \hat{x}^\lambda \hat{L}_i^\dagger \langle L_i0,\lambda0|L_f0 \rangle W(L_iJ_iL_fJ_f;J_{\text{tot}}\lambda)
\]

(4.3.6)

in terms of the 6J or Racah coefficients \( W \) of angular momentum theory. The total coupling between two states is the sum over all multipoles of the terms in Eq. (4.3.6).

When there are spins in the set of coupled angular momenta that are spectators, in the sense of not participating in the dynamical transitions, rearrangements of the coupling order must be used in order to isolate those spins actually involved. Expressions are then obtained that are similar to Eqs. 4.3.6, but are more complicated, and contain sums over additional intermediate quantum numbers.

In later sections, we calculate the transition potentials \( V_{\lambda}^{fi}(R) \) using the \( \mathcal{F}_\lambda(R) \) form factors that may be common to several states \( i,f \). These functions of the internuclear separation \( R \) are determined by the physical reaction mechanism, and will be different for different target nuclei, and different
for single-particle and collective processes. For transfer reactions we will find that the form factors will be non-local.

4.3.2 Spin-dependent forces

The monopole diagonal potential \( \lambda = 0 \) has the simplest form \( F_0(R) \), such as a folded or optical potential that we have used to describe the interaction of two nuclei before any non-elastic reactions occur. Typical optical potentials have already been discussed, and folded potentials will be described in section 5.2.

When nuclei are interacting that have non-zero spins, non-zero multipole components may have an effect even if the nuclei stay in their ground state. Typically, \( \lambda = 1 \) vector potentials and \( \lambda = 2 \) tensor potentials may be present, up to \( \lambda \leq 2s \), which is twice the modulus of the sum \( s = \sum_i s_i \) of the spins \( s_i \) of the interacting bodies. Individual nucleons have spins 1/2, so \( \lambda = 1 \) multipoles may contribute to the scattering of nucleons on any other body. Pairs of nucleons may have total spin \( s = 0 \) or 1. The deuteron ground state has \( s = 1 \), for example, and in general \( \lambda = 2 \) tensor forces may operate during the scattering of any two nucleons in an \( s = 1 \) state. Neither of these vector or tensor forces changes the energy eigenstate of a particle or nucleus.

**Vector forces**

The most common vector potential is the spin-orbit force. This \( \lambda = 1 \) force can couple a single \( s = 1/2 \) nucleon with its orbital motion with \( T_1(\xi) = s \), the spin operator itself. This gives

\[
V_{so} = F_{1}^{so}(R) \, 2L \cdot s
\]

of which the matrix elements may be calculated in the J-basis \( J = L + s \) as

\[
\langle (Ls)J|2L \cdot s|(Ls)J \rangle = J(J+1) - L(L+1) - s(s+1) \\
= \begin{cases} 
+L & \text{for } J = L+1/2 \\
-L-1 & \text{for } J = L-1/2 
\end{cases}
\]

so only the diagonal \( \langle (Ls)J|V_{so}|(Ls)J \rangle = F_{1}^{so}(R)[J(J+1) - L(L+1) - s(s+1)] \) elements are non-zero. In the S-basis of Eq. (3.2.2) the spin-orbit force gives off-diagonal couplings\(^\dagger\).

In comparison with electrons in atoms, for which the spin-orbit potential

\(^\dagger\) This is one small but useful advantage of the J-basis formulation.
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is (Thomas [27])

\[ F_{so}(R) = -\frac{1}{2m_{\pi}c^2} \frac{1}{R} \frac{dV(R)}{dR} \]  

(4.3.9)

in terms of the central (electrostatic) potential \( V(R) \), the nuclear spin-orbit force is much stronger. If this formula is used directly for the nuclear spin-orbit forces, it is found to be too weak, requiring an amplification factor commonly taken as 25.

In practice, negative spin-orbit form factors \( F_{so}(R) \) are still taken as having this ‘Thomas form’ as derivatives of the form of a nuclear central potential, but with a scaling factor that is somewhat arbitrarily fixed in terms of the pion mass \( m_\pi \):

\[ F_{so}(R) = \left( \frac{\hbar}{m_\pi c} \right)^2 \frac{1}{R} \frac{d}{dR} \frac{V_{so}}{1 + \exp\left( \frac{R - R_{so}}{a_{so}} \right)} \]  

(4.3.10)

so \( \hbar^2/(m_\pi c)^2 = 2.00 \) \( \text{fm}^2 \), and \( F_{so}^2(R) \) has the same units as \( V_{so} \). This requires a potential factor of the order of \( V_{so} = 5 - 8 \text{ MeV} \) for nucleons†.

Another simple spin-dependent force possible with two nuclei both with spins \( J_p \) and \( J_t \) is the spin-spin force \( V_{ss} = F_{ss}^2(R)J_p \cdot J_t \). In the S (channel-spin) basis of Eq. (3.2.2) the spin-spin force is diagonal, with matrix elements

\[ \langle L(J_p, J_t; S; J_{tot})|V_{ss}|L(J_p, J_t; J_{tot}) \rangle = F_{ss}^2(R) \frac{1}{2} [S(S+1) - J_p(J_p+1) - J_t(J_t+1)] . \]  

(4.3.11)

In the J-basis (Eq. 3.2.1) there are off-diagonal couplings from the spin-spin force.

Tensor forces

An \( s = 1 \) cluster can be influenced in its orbital motion by a \( \lambda = 2 \) (rank two) tensor force, as can the motion between two \( s = 1/2 \) nucleons when they are together in an \( S = 1 \) (triplet) state. These tensor forces do couple together different \( L \neq L' \), in contrast to vector forces (rank one tensors).

For an object of spin \( s \), Satchler [28] showed that there are three generic kinds of rank-two operators that could be used for tensor forces. These couple the spin with either the radius, momentum or orbital angular momentum

† Note that sometimes spin-orbit forces are defined with \( L \cdot s \) in place of the \( 2L \cdot s \) in Eq. (4.3.7), and/or sometimes without the \( \hbar^2/(m_\pi c)^2 \), and these change the numerical strength factor \( V_{so} \).
vectors, yielding forms conventionally written as

\begin{align*}
T_r &= (s \cdot \hat{R})^2 - \frac{2}{3} \\
T_p &= (s \cdot p)^2 - \frac{2}{3}p^2 \\
T_L &= (s \cdot L)^2 - \frac{1}{2}s \cdot L - \frac{2}{3}L^2.
\end{align*}

(4.3.12)

(4.3.13)

(4.3.14)

where \( \hat{R} \) is the unit vector the direction of \( R \). All of these quadrupole forms
need to be combined with some radial form factor \( F_2(R) \) which may be
complex.

4.4 Inelastic couplings

An inelastic reaction is one where a nucleus changes its energy eigenstate,
from state \( i \) to state \( f \), in interacting with another nucleus. The inelastically
excited nucleus changes its energy from \( \epsilon_i \) to \( \epsilon_f \), determined according to
the eigenequations of Eq. (3.2.42). The energy \( Q \)-value for the reaction is
therefore \( Q = \epsilon_i - \epsilon_f \), as this is the energy released, and is only positive if
the nucleus is de-excited with \( \epsilon_f < \epsilon_i \).

An inelastic reaction will usually change the spin of the nucleus, by some
transfer of angular momentum that is the multipolarity \( \lambda \) of the reaction.
This angular momentum is usually transfer from the orbital angular momentum
of relative motion, but sometimes directly from the other interacting
nucleus. If the initial and final nuclear spins are \( I_i \) and \( I_f \) respectively†,
then there holds a triangular relationship \( \Delta(\lambda, I_i, I_f) \), or

\[ |I_i - I_f| \leq \lambda \leq I_i + I_f \]  

(4.4.1)

Thus monopole transitions keep \( I_i = I_f \), and dipole transitions (\( \lambda=1 \)) change
\( I_i \) by 1 at most.

Normal parity transitions change the nuclear parity only for odd \( \lambda \) multi-
poles, and all transitions caused by central forces are normal in this sense.
Other ‘non-normal’ transitions may occur when spin-dependent forces change
nuclear spin orientations during a reaction, but these are generally weaker
processes for relative velocities \( v \ll c \) (the speed of light).

Inelastic transitions in a nucleus are caused by the interaction with a
second nucleus, which may act by electromagnetic and/or nuclear forces.
The nuclear forces may be central, or may couple to the spins of individual
nucleons. The electromagnetic forces are almost entirely electric at low and
medium energies, and are therefore principally the Coulomb forces which
couple to the charges of the protons.

† It is conventional to use \( I \) in place of \( J \) or \( s \) for the collective processes we will be discussing.
Reaction mechanisms

If Coulomb forces contribute to the inelastic transition, then for \( R \) outside the nuclear radii \( R_n \), the inelastic form factors have characteristic forms as inverse powers of \( R^{\lambda+1} \). We will in Eq. (4.4.23) be defining a Coulomb reduced matrix element \( \langle I_f \| E\lambda \| I_i \rangle \), in terms of which

\[
V_{\lambda}^{fi}(R) \rightarrow R > R_n \frac{\langle I_f \| E\lambda \| I_i \rangle \sqrt{4\pi e^2 Z_i}}{R^{\lambda+1} (2\lambda+1)} \, , \tag{4.4.2}
\]

where we factorise out \( Z_i e \), the charge of the second nucleus.

The quantity

\[
B(E\lambda, I_i \rightarrow I_f) = \frac{1}{2I_i+1} |\langle I_f \| E\lambda \| I_i \rangle|^2 \tag{4.4.3}
\]

is called the reduced transition probability, and the rate for peripheral reactions depends only on \( B(E\lambda, I_i \rightarrow I_f) \). These may be low-energy photonuclear reactions, gamma decay processes, quadrupole moments, or very forward angle (or low energy) Coulomb excitation reactions. The value of \( B(E\lambda) \) is thus sometimes a good contact point between theory and experiment.

The reduced matrix element, as defined, will be symmetric: \( \langle I_f \| E\lambda \| I_i \rangle = \langle I_i \| E\lambda \| I_f \rangle \), so

\[
(2I_i+1) B(E\lambda, I_i \rightarrow I_f) = (2I_f+1) B(E\lambda, I_f \rightarrow I_i) \tag{4.4.4}
\]

and ‘up’ and ‘down’ \( B(E\lambda) \) values will be different.

To find \( V_{\lambda}^{fi}(R) \) at smaller radii, we need a potential model for the cause of the reaction. This model could be a rotational or vibrational collective model, a single-particle excitation model, or a more microscopic calculation of the many-body nuclear states and how they interact with the second nucleus. The collective and single-particle models are now described in turn, while the use of microscopic models is discussed in Chap. 5.

4.4.1 Collective inelastic processes

The most common collective models for nuclei consider the rotation a deformed nuclear shape, or the deformational vibration of a nucleus that initially spherical.

Rotational inelastic excitation

The simplest rotational model defines a deformed surface of a nucleus, and constructs the nuclear interaction potential as a function of the distance to the surface along a radial line. This gives natural parity transitions among members of a given rotational band of a nucleus. These bands are a set of
Fig. 4.2. A nucleus is rotating about the $z$ laboratory axis, and is itself a deformed object with axis of symmetry along the $z'$ axis that continuous varies with respect to $z$. The interaction potential of another body with a deformed object may depend on the radial distance to its surface.

states of spin $I$ that start from some bandhead $I = K$, and have excitation energies typically like

$$\epsilon_I = \frac{\hbar^2}{2\mathcal{I}} [I(I+1) - K(K+1)] ,$$  \hspace{1cm} (4.4.5)

where $\mathcal{I}$ is the moment of inertia of the nucleus. The moment $\mathcal{I}$ will be constant for the rotational band of a rigid rotor, with value

$$\mathcal{I}_{\text{rigid}} = \frac{2}{5} m_u A R_0^2$$  \hspace{1cm} (4.4.6)

for average surface radius $R_0$ and $m_u$ the unit atomic mass (amu). The value of $K$ is the projection of the intrinsic nuclear angular momentum on the rotational axis, and hence the smallest possible value of rotating spin $I$. Rotational bands with $K = 0$ will consist of the set of even levels $I = 0, 2, 4, \cdots$, whereas bands for $K > 0$ will have all levels $I = K, K+1, K+2, \cdots$.

A deformed nucleus is considered to have an intrinsic state in a body-fixed frame of reference $(r', \theta', \phi')$ that can be defined by placing the $z'$ axis along some axis of deformation symmetry. The varying radius $\tilde{R}$ of the surface of a deformed nucleus can always be expanded using spherical harmonics in terms of polar angles $\theta'$ and $\phi'$ in the body-fixed frame as

$$\tilde{R}(\theta', \phi') = R_0 + \sum_{q\mu} a_{q\mu} Y_q^\mu(\theta', \phi') .$$  \hspace{1cm} (4.4.7)
We are interested in $q \geq 2$ as $q = 0$ can be included in $R_0$ and $q = 1$ moves the centre of mass. Requiring $\tilde{R}(\theta', \phi')$ real means that $a_{q\mu} = (-1)^{\mu}a_{q-\mu}$, so not all the $a_{q\mu}$ are independent. The $R_0$ is the surface radius, which will in general be less than the potential radius.

The most common $a_{q\mu}$ is $q = 2$ for quadrupole deformations. Axially deformed nuclei only have $a_{20}$ non-zero, whereas a triaxially deformed nucleus has $a_{22} = a_{2-2}$ also non-zero. The $a_{q\mu}$ have units of length, and $a_{q0}$ is usually called the deformation length $\delta_q$. The fractional deformation is $\beta_q = \delta_q / R_r$ for the radius $R_r$ of the potential. A prolate deformed nucleus has $\beta_2 > 0$, and oblate deformation occurs with $\beta_2 < 0$.

In the simplest rotational model for a deformed surface, we describe the interaction $V$ with the second nucleus as depending on the distance to the surface as in Fig. 4.2:

$$V(R, \theta', \phi') = U(R) - U'(R) \sum_{q\mu} a_{q\mu} Y_q^{\mu}(\theta', \phi')$$

(4.4.8)

for some suitable nuclear potential function $U(R)$ such as an optical potential. The first step is to expand the potential of (4.4.8) into a sum of tensor components like Eq. (4.3.1). A simple approximate procedure is to expand $U(R-\tilde{R}(\theta', \phi')+R_0)$ to first order in the $a_{q\mu}$, which should be satisfactory for small deformations. This gives

$$V(R, \theta', \phi') = U(R) - U'(R) \sum_{q\mu} a_{q\mu} Y_q^{\mu}(\theta', \phi')$$

(4.4.9)

where the first term $U(R)$ is the diagonal optical potential.

The case of an axially-deformed rotator gives

$$V(R, \theta', \phi') = U(R) - \sum_{\lambda} \delta_\lambda U'(R) Y_0^{\lambda}(\theta', \phi') .$$

(4.4.10)

Now $(\theta', \phi')$ are the angles between the vector $\hat{R}$ and the axis $\hat{z}'$ of the body-fixed coordinate system, so (as $\mu = 0$ here)

$$Y_0^{\lambda}(\theta', \phi') = \frac{\hat{\lambda}}{\sqrt{4\pi}} P_\lambda(\cos \theta') = \frac{\hat{\lambda}}{\sqrt{4\pi}} \frac{4\pi}{2\lambda+1} \sum_{m} Y_{\lambda m}(\hat{\xi}) Y_{\lambda m}^{*}(\hat{R})$$

(4.4.11)

so the dependence of the potential $V$ is now

$$V(R, \hat{\xi}, \hat{R}) = U(R) - \sum_{\lambda} \delta_\lambda U'(R) \frac{\sqrt{4\pi}}{\lambda} \sum_{m} Y_{\lambda m}(\hat{\xi}) Y_{\lambda m}^{*}(\hat{R}) .$$

(4.4.12)

This interaction can change the state of the deformed nucleus by the non-zero matrix elements $\langle \phi_{I_f} | V(R, \hat{\xi}, \hat{R}) | \phi_{I_i} \rangle$. Here, $I_i$ is the initial spin and $I_f$...
the final spin of the rotating nucleus. In general, this nucleus will have some intrinsic spin $K$ even in its body-fixed frame, so all levels in its rotational band will have $I \geq K$. We construct a rotational excited state $IM$ as the intrinsic state $\phi_K$ operated on by a rotation matrix $D^{I}_{MK}(\omega)$ where $\omega$ is the set of Euler angles that transforms $(\theta', \phi')$ in the body-fixed frame to $(\theta, \phi)$ in the laboratory frame. From the normalisation property of the $D^{I}_{MK}(\omega)$, we have that a normalised state of a rotor is

$$\phi_{IM} = \frac{i}{\sqrt{8\pi^2}} D^{I}_{MK}(\omega)^* \phi_K \quad (4.4.13)$$

We are now able to calculate the matrix elements of the potential (4.4.12) between rotational states (4.4.13), using

$$\langle I_f M_f | Y^m_\lambda(\hat{\xi})^* | I_i M_i \rangle = \hat{\lambda} \sqrt{4\pi} \hat{I}_i \langle I_i, K, \lambda | I_f, K \rangle \langle I_i M_i, \lambda m | I_f M_f \rangle. \quad (4.4.14)$$

since $Y^m_\lambda = \hat{\lambda}/\sqrt{4\pi} D^{\lambda}_{m0}(\omega)^*$. Using the Bohr & Mottelson definition of a reduced matrix element we find that the integral over the core angle gives the factor

$$\langle I_f \parallel Y_\lambda \parallel I_i \rangle = \frac{\hat{\lambda}}{\sqrt{4\pi}} \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle. \quad (4.4.15)$$

These reduced matrix elements give a transition potential in the rotational model of

$$V^{fi}_{\lambda}(R) = -\frac{\delta_{\lambda}}{\sqrt{4\pi}} U^{i}(R) \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle. \quad (4.4.16)$$

which enters into Eq. (4.3.6) to complete the partial wave matrix element.

The Coulomb interaction between a deformed charge density of $Z_p$ protons and a second nucleus of $Z_t$ protons does not depend on just the distance to the surface as the nuclear force does, but on a more global integral. Let the combined proton probability density distribution $\rho(r_p)$ be normalised as $\int \rho(r_p) dr_p = Z_p$. The electrostatic potential from these protons is then

$$V_E(R) = e \int dr_p \frac{\rho(r_p)}{|R - r_p|}. \quad (4.4.17)$$
where these coordinates are in the laboratory frame. Using

\[ \frac{1}{|R - r_p|} = \sum_{\lambda \mu} \frac{4\pi}{2\lambda + 1} Y_\lambda^\mu(r_p)^* Y^\mu_\lambda(R)f(R, r_p) \]  

(4.4.18)

with the ‘near field’ and ‘far field’ forms

\[ f(R, r_p) \equiv \begin{cases} R^\lambda/r^\lambda_{p+1} & \text{for } R \leq r_p \\ r^\lambda_{p}/R^\lambda+1 & \text{for } R \geq r_p \end{cases} \]  

(4.4.19)

respectively, then

\[ H_{\text{int}} = eZ_t V_E(R) \]

\[ = \sum_{\lambda \mu} \frac{4\pi Z_t e^2}{2\lambda + 1} Y_\lambda^\mu(R) \int Y^\mu_\lambda(r_p)f(R, r_p)\rho(r_p)dr_p . \]  

(4.4.20)

Comparing this with Eq. (4.3.1), we find that the rotational form factor is

\[ F^\lambda_\mu(R) T^\mu_\lambda(\xi) = \sqrt{\frac{4\pi}{2\lambda + 1}} \int Y^\mu_\lambda(r_p)f(R, r_p)\rho(r_p)dr_p , \]

(4.4.21)

so the matrix elements for transitions from nuclear spin \( I_i \) to \( I_f \) are

\[ V^f_i(\lambda) = \langle I_f \| F_\lambda \| I_i \rangle = \frac{\sqrt{4\pi Z_t e^2}}{2\lambda + 1} \langle I_f \| Y^\mu_\lambda(r_p)f(R, r_p)\rho(r_p)\| I_i \rangle \]

\[ \rightarrow \frac{\sqrt{4\pi e^2 Z_t}}{2\lambda + 1} \frac{\langle I_f \| Y^\mu_\lambda(r_p)\rho(r_p)\| I_i \rangle}{R^{\lambda + 1}} . \]  

(4.4.22)

We define the numerator of the second term as the *Coulomb reduced matrix element*

\[ \langle I_f \| E\lambda \| I_i \rangle = \langle I_f \| Y^\mu_\lambda(r_p)r^\lambda_{p}\rho(r_p)\| I_i \rangle . \]  

(4.4.23)

For a deformed sphere of constant internal density with deformation length \( \delta_\lambda \) and mean radius \( R_c \), the rotational picture gives an approximate value for this matrix element of

\[ \langle I_f \| E\lambda \| I_i \rangle = \frac{3Z_0 \delta_\lambda R_c^{\lambda - 1}}{4\pi} \hat{I}_i(I_i, K, \lambda 0 | I_f K) . \]  

(4.4.24)

The accuracy may be improved by taking the Taylor series of Eq. (4.4.9) to second order in the \( a_{\mu\mu} \). This gives rise to tensor products \( Y_q \otimes Y_{q'} \), which couple up to \( \lambda \) satisfying \( |q - q'| \leq \lambda \leq q + q' \). This means that second-order terms contribute also to the monopole parts, changing the diagonal distorting potential. There are also now \( \lambda = 2q \) multipoles, allowing direct excitation of, for example, \( 0^+ \rightarrow 4^+ \) transitions for \( q = 2 \) quadrupole deformations.
For best accuracy, a result accurate to all orders in the $a_{q\mu}$ may be found by carrying out a numerical angular integration in the body-fixed frame for the overlap $\langle Y_\lambda(\hat{R}')|U(R-\hat{R}(\theta',\phi')+R_0)\rangle$. For axial deformations, this is an integral over $\theta'$ only:

$$F_\lambda(R) = \frac{\hat{\lambda}}{2} \int_{-1}^{+1} U(R-\hat{R}(\theta')-R_0) P_\lambda(\cos \theta') d(\cos \theta') \, ,$$  \number{4.4.25} for deformation lengths $\delta_q$.

Sometimes, particular information about specific transitions is available from structure calculations, or from other experiments such as lifetime measurements. Including this knowledge will require changing by hand the values of $V_{f'i}^{fi}(R)$ for specific $I_i \rightarrow I_f$ transitions. This is a way in which previous measurements of $B(E\lambda, I_i \rightarrow I_f)$ may be taken into account.

**Vibrational models** of nuclei give rise to inelastic transition potentials with forms similar to those from the rotational model. The excitation of quadrupole one-phonon states in a nucleus will give a $2^+_1$ state, and then two-phonon states consist of two phonons of energy which can be coupled together to give a triplet of $0^+$, $2^+_2$ and $4^+_3$ states at approximately twice the first $2^+_1$ energy. Octupole phonons also may occur, generating a $3^-$ excited state with one phonon of energy.

The theory of vibrational nuclei may be developed along the same lines as above for rotational couplings, but now the $a_{q\mu}$ are operators which change the state of the nucleus by exciting phonon states, with one or more phonons. Vibrational models are described in detail in for example Eisenberg and Greiner [29].

For a given $B(E\lambda)$, the vibrational model yields the same off-diagonal couplings as does the rotational model, but the diagonal re-orientation couplings are set to zero. This reflects the fact that a purely vibrational $2^+$ state, for example, has zero static quadrupole moment. Real nuclei are usually between the pure limits, and have some small static quadrupole moments even for vibrational excited states.

**4.4.2 Single-particle inelastic processes**

Sometimes the state of a nucleon (or cluster of nucleons) in a nucleus changes because of differential tidal forces when interacting with a second nucleus.
Consider the change of state of a valence nucleon or cluster \( v \) from initial state \( \phi_{i,s_j}(r) \) to \( \phi_{f,s_j}(r) \), so that, when bound to a core \( c \) of spin \( I \), the inelastic transition is from \( (I_{j_i})_I \) to \( (I_{j_f})_f \).

The potential model to generate this transition describes a three-body system with core \( c \), valence \( v \) and target \( t \). The model is defined by the three masses \( m_c, m_v \) and \( m_t \) respectively, and the three pair-wise potentials \( V_{ct}(r), V_{vt}(r_{vt}) \) and \( V_{ct}(r_{ct}) \), where the coordinates \( r_{ct} \) and \( r_{ct} \), as shown in the diagram, are related to \( R \) and \( r \) by

\[
\begin{align*}
    r_{vt} &= R + \frac{m_c}{m_p} r \\
    r_{ct} &= R - \frac{m_v}{m_p} r
\end{align*}
\]

where \( m_p = m_c + m_v \), the total mass of the nucleus being excited. The transition arises as a tidal effect because neither \( V_{vt} \) nor \( V_{ct} \) act just on the centre of mass at \( R \). If we consider the principal central parts of these two potentials, then the core spin \( I \), nucleon spin \( s \) and target spin \( J_t \) are all spectators. The inelastic transition potential is therefore

\[
V^{fi}(R) = \langle (L_f J_f)\Lambda | V_{vt} + V_{ct} | (L_i J_i)\Lambda \rangle
\]

where \( L_i, L_f \) are the initial and final orbital angular momenta of relative motion, and \( \Lambda = L + J \) is their sum which is a conserved angular momentum. The sum \( J \) is to be combined with \( s, I \) and \( J_t \) to give the total spin \( J_{tot} \) for the whole system.

With \( V_{vt} \) and \( V_{ct} \) depending only on the length square roots of

\[
\begin{align*}
    r_{vt}^2 &= R^2 + \frac{m_c^2}{m_p} r^2 + 2m_c R r z \\
    r_{ct}^2 &= R^2 + \frac{m_v^2}{m_p} r^2 - 2m_v R r z
\end{align*}
\]

where \( z = \hat{R} \cdot \hat{r} \), the cosine of the angle between \( R \) and \( r \). All the effects of \( V_{vt} \) and \( V_{ct} \) are in the two-variable multipole function

\[
F_\lambda(R,r) = \frac{1}{2} \int_{-1}^{+1} [V_{vt}(r_{vt}) + V_{ct}(r_{ct})] P_\lambda(z) dz .
\]

In terms of this \( F_\lambda(R,r) \) the potentials in the matrix element (4.4.28) are

\[
V_{vt} + V_{ct} = \sum_\lambda (2\lambda+1) F_\lambda(R,r) P_\lambda(z)
\]

\[
= \sum_\lambda (2\lambda+1) F_\lambda(R,r) \sum_m Y^m_\lambda(\hat{r})^* Y^m_\lambda(\hat{R}) ,
\]

where \( m \) is the magnetic quantum number.
If the single particle states have angular and radial parts given by Eq. (4.2.3), then the transition potential for multipole component $\lambda$ is

$$V_{\lambda f i}(R) = \int_0^{\infty} u_{\ell f s j f}(r) F_{\lambda}(R,r) u_{\ell i s j i}(r) dr \hat{\lambda} \hat{\ell} \langle \ell i 0, \lambda 0 | \ell f 0 \rangle$$  \hspace{1cm} (4.4.32)

### 4.5 Mass rearrangements

#### 4.5.1 Transfer reactions

In a transfer reaction, a valence nucleon (or cluster) is transferred from the projectile to the target, or vice versa. When this is transferred from the projectile it is often called *stripping*, and when it is added to the projectile from the target this is *pickup*. (Note that sometimes the word ‘stripping’ is used to refer to any removal process from the projectile, including say breakup and more complicated reactions, but in this section it refers only to the transfer to a specific bound state around the target).

For transfer processes, as for single-particle inelastic excitations, we use $R$ to refer to the difference of the projectile and target positions, and $r$ to the relative coordinates of the valence cluster to its core. For mass transfer reactions, there will be distinct pairs of vectors in the initial state $R, r$, and in the final state $R', r'$ because, by recoil effects, the vectors $R \neq R'$ as shown in Fig. 4.3.

Let us consider the specific case of stripping, that is removing a nucleon from the projectile. Here, the initial bound state of the projectile $\phi_p(r)$ and the final bound state of the residual nucleus $\phi_t(r')$ satisfy the eigenvalue...
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equations

\[ h_p - \varepsilon_p \phi_p(r) = 0 \quad \text{where} \quad h_p = T_p + V_p(r) \]
\[ h_t - \varepsilon_t \phi_t(r') = 0 \quad \text{where} \quad h_t = T_t' + V_t(r') \]  \hspace{1cm} (4.5.1)

The binding potentials \( V_p(r) \) are usually fitted so the eigenenergies agree with experimental separation energies, using the methods described in section 6.4.

The Q-value for the reaction, the amount of energy released during the transfer, is \( Q = \varepsilon_p - \varepsilon_t \). In terms of the binding energies \( B \equiv -\varepsilon \) which are positive for bound states, the Q-value is \( Q = B_t - B_p \), so \( B_t = B_p + Q \).

The dynamical details of the transfer coupling arise from the matrix elements of the Hamiltonian for the three bodies involved: the initial and final cores, and the valence particle. This Hamiltonian is

\[ H = T_r + T_R + V_p(r) + V_t(r') + U_{cc}(R_c) \]  \hspace{1cm} (4.5.2)

where \( U_{cc}(R_c) \) is the core-core optical potential. The pair of kinetic energy terms can equivalently be written \( T_r + T_R = T_{r'} + T_{R'} \), so there will henceforth be two ways of expanding the Hamiltonian as we have already seen in section 3.2.2. These are called prior and post:

\[ H = H_{\text{prior}} = T_R + U_i(R) + h_p(r) + V_i(R, r) \]
\[ = H_{\text{post}} = T_{R'} + U_f(R') + h_t(r') + V_f(R', r') \]  \hspace{1cm} (4.5.3)

where the \( U_{i,f} \) are the entrance and exit diagonal potentials respectively, so the interaction terms \( V_i \) and \( V_f \) to be used must be

\[ V_i(R, r) = V_i(r') + U_{cc}(R_c) - U_i(R) \]
\[ \text{or} \quad V_f(R', r') = V_p(r) + U_{cc}(R_c) - U_f(R') \]  \hspace{1cm} (4.5.4)

It is these interaction terms which cause the transition from one bound state to another, since the other terms in the post and prior Hamiltonians are diagonal with respect to transfer channels. The first part of the \( V_{i,f} \) is the binding potential, and the remaining two terms are called the remnant terms, which, as they are often similar in magnitude and contain complex potentials, are sometimes neglected for convenience. The post and prior forms give exactly the same results in first order DWBA, as was shown on page 95. With sufficient numerical accuracy this equality should hold also in practice, and this equality requires that the remnant terms be always included.

The matrix elements of these interaction terms are now found for the case
of two transfer channels
\[ \Psi = \left[ \phi_p(r) \otimes Y_L(\hat{R}) \right]_\Lambda \psi_i(R)/R + \left[ \phi_t(r') \otimes Y_L(\hat{R}') \right]_\Lambda \psi_f(R')/R' \] (4.5.5)
in which model space† we need to find the transition terms such as
\[ V^{fi}_w(R', R) = R' \langle [\phi_t(r') \otimes Y_L(\hat{R}')] | V_w | [\phi_p(r) \otimes Y_L(\hat{R})] \rangle R^{-1} \] (4.5.6)
with \( w = i, f \) for prior and post interactions respectively in Eq. (4.5.4).

**Finite-range transfers**

This matrix element is a non-local integral operator, as it operates on the function \( \psi_i(R) \) to produce a function of \( R' \). We therefore derive the non-local kernel \( V^{fi}_w \) so that the matrix element operation on a wave function, which initially involves a five-dimensional integral over \( r' \) and \( \hat{R}' \), may be calculated by means of a one-dimensional integral over \( R \):
\[ \Omega_f(R') = \int V^{fi}_w(R', R) \psi_i(R) dR. \] (4.5.7)
Such a source term may be used, for example, in the Green function integral methods of Chapter 3.

Note that when the initial and final single-particle states are real, then the kernel function is symmetric
\[ V^{fi}_w(R', R) = V^{if}_w(R, R') \] (4.5.8)
whether \( w \) is post or prior.

Since the \( r \) and \( r' \) are linear combinations of the channel vectors \( R \) and \( R' \), we have
\[ r = aR' + bR \quad \text{and} \quad r' = a'R' + b'R \] (4.5.9)
where \( a = -\omega, \ b = \nu_t \omega, \ a' = -\nu_p \omega, \ \text{and} \ b' = \omega \), with \( \nu_p = m_c/m_p, \ \nu_t = m_c'/m_t' \), and \( \omega = (1 - \nu_p \nu_t)^{-1} \). The 'core-core' vector is \( R_c = r' - r = (a' - a)R' + (b' - b)R \).

To calculate the matrix element of Eq. (4.5.6), we first convert the bound states \( \phi_t(r') \) and \( \phi_p(r) \) into functions of \( R \) and \( R' \) using Eq. (4.5.9). To do this, we must transform the spherical harmonics \( Y_{\ell}(\hat{r}) \) and \( Y'_{\ell}(\hat{r}') \) into linear combinations of the spherical harmonics \( Y_n(R) \) and \( Y'_m(\hat{R}') \). This is done by means of the Moshinsky solid-harmonic expansion [30]:
\[ Y_{\ell}^m(\hat{r}) = \sqrt{\frac{4\pi}{2\ell+1}} \sum_{n, \lambda} c(\ell, n) \left( \frac{aR'}{r} \right)^{\ell-n} (bR')^n \sum_{\lambda} Y_{\ell-n}^{m-\lambda}(\hat{R}') \langle \ell-n, m-\lambda, n|n\rangle \] (4.5.10)
† We omit the two core states \( \phi_c \) and \( \phi_c \) since here they are spectators with no dynamical role.
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where \( c(\ell, n) = \left( \frac{(2\ell + 1)!}{(2n + 1)!(2(\ell-n)+1)!} \right)^{1/2}. \) (4.5.11)

Considering only the case where the remnant part of the interaction \( V_w \) of Eq. (4.5.6) contains just scalar potentials, we perform the Legendre expansion

\[
V_w \frac{u_{\ell'}(r')}{r^\ell'+1} \frac{u_{\ell}(r)}{r^\ell+1} = \sum_{T=0}^{T_{\text{max}}} (2T+1) q_{\ell',\ell}^{T,w}(R', R) P_T(z) \] (4.5.12)

where the \( u_{\ell}(r) \) are the radial parts of the bound wave functions according to Eq. (4.2.3). The limit \( T_{\text{max}} \) is chosen large enough to generate all the couplings for partial waves to be used. Here, the Legendre polynomials \( P_T(z) \) are functions of \( z \), the cosine of the angle between \( R \) and \( R' \). According to Eq. (4.5.4) with scalar potentials, the \( V_w \) depends only on the lengths of the vectors \( R_c \) and \{\( r', R \)\} or \{\( r, R' \)\}, all of which may be calculated in terms of \{\( R, R' \)\} and \( z \) according to formulae such as 

\[
r = \left( a^2 r'^2 + b^2 R^2 + 2abRR'z \right)^{1/2}
\]

in the numerical quadrature for the integral

\[
q_{\ell',\ell}^{T,w}(R', R) = \frac{1}{2} \int_{-1}^{+1} V_w \frac{u_{\ell'}(r')}{r^{\ell'+1}} \frac{u_{\ell}(r)}{r^{\ell+1}} P_T(z) \, dz \] (4.5.13)

Using the Legendre expansion, the radial kernel function

\[
V_{\ell L':\ell L}^{\Lambda w}(R', R) = (-1)^{L+L'} \hat{L}\hat{L}' \sum_{T KK'} \left( \begin{array}{ccc} K & L & T \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} K' & L' & T \\ 0 & 0 & 0 \end{array} \right) \times \sum_{\lambda} W(\ell L L'; \Lambda \lambda) W(K L K'; T \lambda) \mathcal{F}_{\lambda, K; KT}^{\ell L}(R', R) \] (4.5.14)

where we use the non-local form factor

\[
\mathcal{F}_{\lambda, K; KT}^{\ell L}(R', R) = \frac{|b|^3}{2} \sum_{nn'} RR'(aR')^{\ell-n}(bR)^n(a'R')^{\ell'-n'}(b'R)^{n'} \times (2T+1)(-1)^{\Lambda+T} \hat{\ell}\hat{\ell}' (\ell-n) (\ell'-n')(2K+1)(2K'+1) \times c(\ell, n)c(\ell', n') \left( \begin{array}{ccc} \ell-n & n' & K \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} \ell'-n' & K' & n' \\ 0 & 0 & 0 \end{array} \right) \times (2\lambda+1) \left( \begin{array}{ccc} \ell' & \lambda & \ell \\ n' & K & \ell-n \end{array} \right) q_{\ell',\ell}^{T,w}(R', R) \] (4.5.15)

Combining these factors, we are able to calculate the non-local kernels \( V_{\ell L':\ell L}^{\Lambda w}(R', R) \) to calculate transfer reactions using an exact treatment of
the finite range of the potentials and all the recoil terms from the finite masses of the cores $c$ and $c'$.

**Zero-range transfers**

When the wave functions $\phi_\ell(r)$ are all $s$-states ($\ell = 0$), when the remnant terms can be neglected, and when the interaction potential is of zero-range, $V_\phi(r) \sim D_0 \delta(r)$, then the form factor $V_{LL'}^{L':LL}(R', R)$ of equation (4.5.14) can be simplified to

$$V_{LL':LL}^{L':LL}(R', R) = D_0 \left( \frac{-1}{L} \right) L' \hat{L} \hat{L}' \left( \begin{array}{ccc} \ell' & L & L' \\ 0 & 0 & 0 \end{array} \right)$$

$$\times \frac{1}{R} u_\ell'(R') \frac{b^2}{a} \delta(aR' + bR). \quad (4.5.16)$$

This can be made local by defining a new step size $h' = -ah/b \equiv \nu h$ in the stripping channel $f$, and this considerably simplifies the problem of solving the coupled equations.

**Local energy approximation**

If the interaction potential is of small range, though not zero, and the projectile still contains only $s$-states, then a first-order correction may be made to the above form factor. This correction will depend on the rate of oscillation of the source wave function $\psi_i(R)$ within a 'finite-range effective radius' parameterised as $\rho$. The rate of oscillation is estimated from the local energy in the entrance and exit channels, which is proportional to

$$K_i(R)^2 = \left[ \frac{2\mu_p}{h^2} (E_i - U_i) \right]. \quad (4.5.17)$$

and the result [31] is to multiply $u_\ell'(R')$ in Eq. (4.5.16) by a factor

$$\left[ 1 + \rho^2 \frac{2\mu_p}{h^2} \left( U_{cc}(R') + V_{\ell'}(R') - U_f(R') + \varepsilon_p \right) \right] \quad (4.5.18)$$

where $\mu_p$ is the reduced mass of the valence particle in the projectile, and $V_{\ell'}(R')$ is the binding potential for $u_\ell'(R')$.

At sub-Coulomb incident energies [32], the details of the nuclear potentials in Eq. (4.5.18) become invisible, and as the longer-ranged Coulomb potentials cancel in Eq. (4.5.18), the form factor can be simplified to

$$u_\ell'(R') D_0 \left[ 1 + \rho^2 \frac{2\mu_p}{h^2} \varepsilon_p \right] = Du_\ell'(R') \quad (4.5.19)$$

where

$$D = (1 + k_p^2 \rho^2) D_0 \quad (4.5.20)$$
is the effective zero-range coupling constant for sub-Coulomb transfers, using the bound state wave number $k_p^2 = 2\mu_p\epsilon_p/\hbar^2$.

The parameters $D_0$ and $D$ can be derived from the details of the projectile bound state $\phi_0(r)$. The zero-range constant $D_0$ is the integral

$$D_0 = \sqrt{4\pi} \int_0^\infty rV_0(r)u_0(r)dr.$$  \hspace{1cm} (4.5.21)

The parameter $D$, on the other hand, reflects the asymptotic strength of the wave function $u_0(r)$ as $r \to \infty$, as it is the magnitude of this tail which is important in sub-Coulomb reactions:

$$u_0(r) = r \to \infty \frac{2\mu_p}{\hbar^2} \frac{1}{\sqrt{4\pi}} De^{-k_i^{(p)}r}.$$  \hspace{1cm} (4.5.22)

It may be also found, using Schrödinger’s equation, from the integral

$$D = \sqrt{4\pi} \int_0^\infty \frac{\sinh(k_i^{(p)}r)}{k_i^{(p)}} V_0(r)u_0(r)dr.$$  \hspace{1cm} (4.5.23)

From this equation we can see that as the range of the potential becomes smaller, $D$ approaches $D_0$. The ‘finite-range effective radius’ $\rho$ of equation (4.5.20) is thus some measure of the mean radius of the potential $V_0(r)$.

The Asymptotic Normalisation Coefficient (ANC) of the bound state wave function of any $\ell$ is defined as the asymptotic coefficient $C_\ell$ of the Whittaker function, or the decaying exponential for uncharged particles:

$$u_\ell(r) = r \to \infty W_{\ell+1/2,0\eta}(k_p r)_{\eta=0} = C_\ell e^{-k_p r},$$  \hspace{1cm} (4.5.24)

so is related to the $D$ of $s$-wave states by

$$C_\ell = \frac{2\mu_p}{\hbar^2} \frac{1}{\sqrt{4\pi}} D.$$  \hspace{1cm} (4.5.25)

### 4.5.2 Knockout reactions

A variety of different reactions are commonly called ‘knockout reactions’:

- any removal of a nucleon (cluster) from a nucleus, whether by transfer, breakup, or more complicated nonelastic reaction leaving one or more residual nuclei in excited states,
- quasi-elastic knockout such as the breakup reaction $^{12}\text{C}(p,pp)^{11}\text{B}$ with high energy projectiles so the incident and bound protons scatter as if elastically to $\sim 90^\circ$ relative final angles in a three-body final state, and
• knockout of a cluster which is replaced in the target by the projectile, such as $^{14}\text{C}(p,\alpha)^{11}\text{B}$ to a two-body final state with the nuclei in specific energy levels.

This last kind of knockout (transfer) reaction is considered here, as it is the more important for low incident proton energies, and is usually exothermic ($Q > 0$).

A $(p,\alpha)$ reaction can be regarded as a superposition of amplitudes for two different mechanisms. The first is a triton transfer:

$$ (^{14}\text{C} = ^{11}\text{B} + ^{3}\text{H}) + p \rightarrow ^{11}\text{B} + (\alpha = p + ^{3}\text{H}) , $$ (4.5.26)

and the second is again a transfer, this time of a $^{10}\text{Be}$ cluster from $^{12}\text{C}$ to the proton:

$$ (^{14}\text{C} = ^{10}\text{Be} + \alpha) + p \rightarrow \alpha + (^{11}\text{B} = p + ^{10}\text{Be}) . $$ (4.5.27)

This second mechanism, called heavy particle transfer, produces the final two nuclei with the new projectile identical to the initial target, and vice versa, so its amplitude needs to be permuted by an operator $P_{\alpha, ^{11}\text{B}}$ before combining with the first amplitude, as explained in section 3.4.2.

The heavy particle stripping interaction terms are

\[
\begin{align*}
\text{prior } V_i &= V_{p-\text{Be}} + U_{\alpha \text{Be}} - U_{\text{pC}} \\
\text{post } V_f &= V_{\alpha-\text{Be}} + U_{\alpha \text{Be}} - U_{\alpha B}
\end{align*}
\] (4.5.28)

of which the most important terms are not the binding potentials $V_{p-\text{Be}}$ or $V_{\alpha \text{Be}}$ as usual for normal transfers, but the second term $U_{\alpha \text{Be}}$ describing the interaction between the incident proton and the knocked out $\alpha$ particle. This is the potential we would expect to be dominant if we picture the proton interacting with and knocking out the $\alpha$ particle from its bound state.

### 4.5.3 Breakup reactions

When a nucleus in a reaction breaks up into two or pieces which are both detected separately, we have breakup reactions as defined in Chapter 2. Breakup reactions necessarily involve the continuum final states in some nucleus.

One way of populating continuum states is to use the single-particle excitation mechanism of section 4.4.2 to moving a nucleon from a bound state $u_{\ell s j b}(r)$ to a continuum state $u_{\ell' s j' f}(r; k')$ for some scattering momentum $k'$. This scattering state now extends to infinity in $r$, and hence allows the detection of one or both pieces moving with relative coordinate $r$ and relative
momentum \( k' \). Ideally we will want to calculate the cross sections to all possible final states \( \ell' s j' \); \( k' \), so, to avoid omission or over-counting they should constitute a complete set. For that reason the renormalised wave functions \( \hat{u}_{\ell' s j'}(r; k') \) of Eq. (4.2.5) are generally used, up to some maximum \( k' \) value judged suitable for each particular reaction.

Strictly speaking, there should also be single-particle excitation mechanisms, not only from the ground state to the continuum, but also between all continuum states. The \( \hat{u}_{\ell' s j'}(r; k') \) wave functions are suitable for the first purpose, but, because they extend to infinity, not for the second task of calculating continuum-to-continuum couplings. The Coupled Discretized Continuum Channels (CDCC) method has been devised to solve this problem and allow all physically important couplings to be included, and will be discussed in Chapter 8.

Transfer mechanisms can also populate the continuum, using the methods of section 4.5.1 respectively. Convergence is more difficult in this case, however, because of continuing small contributions from very large \( R \) values. Ideally, as discussed in section 8.3, the radial integrals should be deformed in the complex plane by the method of Vincent and Fortune [33] in order to obtain numerically stable results.

4.5.4 Capture reactions

Two nuclei may approach each other and fuse together, and the rate of fusion depends on the potential in the entrance channel as well as the capture mechanism. The important factor in the entrance channel is clearly the penetration of any Coulomb or centrifugal barrier present at middle distances. If the scattering wave comes in from large relative distances, then it will be attenuated by the time it has tunneled through any Coulomb or centrifugal barrier, and we will calculate a penetrability factor to describe this reduction.

There are different mechanisms for trapping the particles permanently, once they have come inside the Coulomb barrier. One way is for a \( \gamma \)-ray to be emitted, and the particles lose that energy and fall down into a bound state of relative motion. This is a direct capture process, to be discussed further in section 4.7. Another direct mechanism is for one of the nuclei to be pushed up to an excited state, and energy absorbed this way that makes escape less likely. This mechanism can produce doorway resonances.

A third way is for the particles to be captured by some of the long-living resonances of the compound nucleus that is formed of both the nuclei together. This will happen especially with heavier nuclei, where there is a high level density of these resonances: many per MeV. Compound nucleus
4.6 Isospin transitions

4.6.1 Charge exchange reactions

These are reactions in which the participating nuclei keep their masses constant, but change their charge, such as the reaction $^{12}\text{C}(p,n)^{12}\text{N}$. Various mechanisms may contribute to such a transition:

- heavy particle transfer of $^{11}\text{C}$ from $^{12}\text{C} = ^{11}\text{C} + n$ to $^{12}\text{N} = ^{11}\text{C} + p$, as discussed in section 4.5.2,
- two-step transfers via a $^{11}\text{C} + d$ intermediate state, the first transfer adding a neutron to the projectile and the second step removing a proton, and
- direct conversion of a proton to a neutron, for example by a meson $\pi^+$.
being emitted from the proton, and absorbed by the $^{12}$C where it changes a neutron there into a proton.

The first two mechanisms can be calculated by the transfer interactions described previously, but we need to combine these with the last direct charge exchange mechanism, to be described in this section. It can be modelled by an isospin operator which raises the projectile isospin from $t_z = -1/2$ (p) to $+1/2$ (n), alongside lowering the target isospin from $T_z = 0$ for $^{12}$C to $-1$ for $^{12}$N. Such an effect would be caused by a component of the Hamiltonian that couples together the two isospins $t$ and $T$, the isospin operators for the projectile and target respectively.

Lane [35] showed that the optical potential for the scattering of protons and neutrons on nuclei appears to have a $t \cdot T$ contribution. In heavy nuclei, for example, the additional attraction between protons and neutrons means that the optical potential for the scattering of protons on neutron-rich nuclei is more attractive than for neutron scattering. As discussed in section 4.1.1, this is commonly parametrised as

$$V(R) = V_0(R) + \frac{1}{2} t_z N - Z A V_T(R)$$

(4.6.1)

where $t_z$ is the projectile operator, and the target has proton and neutron numbers $Z, N$. Such a force implies a corresponding neutron-proton charge-exchange interaction, if we generalise the expression to an invariant form

$$V(R) = V_0(R) + \frac{t \cdot T}{A} V_T(R) .$$

(4.6.2)

The $V_T(R)$ potential can therefore cause transitions to isobaric analogue states of the target, as well as producing scalar shifts in the neutron and proton optical potentials.

**Fermi transitions**

The Fermi transitions are the simplest form of isospin coupling, whereby the interaction term is

$$H_F = V_F(R)t \cdot T ,$$

(4.6.3)

for $t$ the isospin operator for the projectile, and $T$ for the target. The operator

$$t \cdot T = t_x T_x + t_y T_y + t_z T_z = \frac{1}{2} [t_+ T_- + t_- T_+] + t_z T_z$$

(4.6.4)
where \( t_\pm = t_x \pm it_y \) is the raising (+) and lowering (–) operator for the projectile, and \( T_\pm \) similarly for the target. The three terms in Eq. (4.6.4) have different effects: the first \( t_+T_- \) converts a projectile proton to a neutron and a target neutron to a proton, the second does the opposite, and the third makes no conversions of protons or neutrons. In general, the effect of the \( T_\pm \) operators on is

\[
T_\pm |T, T_z\rangle = \sqrt{T(T+1) - T_z(T_z \pm 1)}|T, T_z \pm 1\rangle.
\]

(4.6.5)

For arbitrary initial and final state of the target nucleus \( \phi_i \) and \( \phi_f \), the magnitude of the charge exchange transition depends on the matrix element \( \langle \phi_f | T | \phi_i \rangle \). The isospin-raising and lowering parts of this matrix element will be used in Chap. 5 in connection with \( \beta \)-decay processes.

**Gamow-Teller transitions**

The Gamow-Teller operators are the next-simplest form of charge exchange. These involve a spin as well as an isospin transition:

\[
H_{GT} = V_{GT}(R) (s \cdot S) (t \cdot T)
\]

(4.6.6)

where \( s \) and \( S \) are spin operators for the projectile and target respectively. These operators do not change the spatial configurations within a nucleus (they keep the same partial wave \( \ell \) and preserve the radial wave functions). The \( H_{GT} \) operator is of vector form, so cannot, for example, couple \( 0^+ \) states together, in contrast to the Fermi operator \( H_F \).

The single-particle reduced matrix elements of \( H_{GT} \) for one nucleus (projectile or target) are, for \( n \rightarrow p \) transitions for example:

\[
\langle u_{\ell'sj}(r) | s \cdot (t \cdot T) | u_{\ell'sj}(r) \rangle
\]

\[
= \frac{1}{2} \langle u_{\ell'sj}(r) | s \cdot (t \cdot T) | u_{\ell'sj}(r) \rangle
\]

\[
= \frac{1}{2} \int_0^\infty u_{\ell'sj}(r)^* u_{\ell'sj}(r) dr
\]

\[
\times \hat{s}^\dagger \hat{t}^\dagger \hat{j}^\dagger (-1)^{\ell + s + j + 1} \left\{ \begin{array}{ccc} \ell & s & j \\ 1 & j' & s \end{array} \right\} \langle s | s \rangle \delta_{ll'} \]

(4.6.7)

with \( \langle s | s \rangle = \sqrt{s(s+1)(2s+1)} \). The radial integral is a measure of the spatial similarity between the initial and final states.

The Gamow-Teller transition is not limited to single isobaric analogue states like the Fermi transition, but will populate many spin and charge-exchange states over a range of energies, with varying transition rates. The GT measurements therefore depend on more details of nuclear structure, and measurements of GT transitions can be used to probe that structure.
The strength of $V_F(R)$ and $V_{GT}$ can be found by fitting to experiment, or derived theoretically by folding nuclear wave functions with effective NN forces $V_{NN}$. The $V_F(R)$ and $V_{GT}$ strengths therefore depend directly on the isovector and spin-flip-isovector components of $V_{NN}$ respectively.

4.6.2 Generalised multipole transitions

Nucleon-nucleon interactions $V_{NN}$ are known to have in general spatial-, spin- and isospin-transition components. When all of these are included for the transition potential that couples a given pair of initial and final nuclear states, we have a large set of generalised multipole transitions. These can be classified according to spatial angular momentum transfer ($\lambda$, here also called $L$), spin transfer (here $S$), and isospin transfer $T$. As well, there may be dependence on the vector sum $L + S = J$. The general form may therefore be written schematically as

$$H_{LSJT} = V_{LSJT}(R) \left[ Y_L(\hat{R}) \otimes T_S(s, S) \right] \mathcal{T}_{tot}(t, T)$$

(4.6.8)

where $L \geq 0$, and $T_S$, $\mathcal{T}_{tot}$ are tensors of rank $S$ and $T$ composed of their vector arguments.

There are long descriptive names for each of these combinations: $S=1$: 'spin flip'; $T=0$: 'isoscalar' and $T=1$: 'isovector'; and $L=0$: 'monopole'; $L=1$: 'dipole'; $L=2$: 'quadrupole'. Thus the $LST = 210$ transition will be called the 'isoscalar spin-flip quadrupole' reaction.

4.7 Photo-nuclear couplings

4.7.1 Single-photon reactions

Section 3.5.1 derived the equations that couple a single-photon of energy $E_\gamma = \hbar \omega$ with a particle of charge $q$ moving to or from a bound state $\Phi_b$. From Eqs. (3.5.33, 3.5.34) these are:

$$\left[ -\frac{\hbar c}{k} \nabla^2 - E_\gamma \right] Z(r) = 2\sqrt{\hbar/\omega} \langle \Phi_b | \hat{j}_q(r) | \Psi(R) \rangle ,$$

$$[\hat{T} + V - E_i] \Psi(R) = \sqrt{\hbar/\omega} (\hat{j}_q \Phi_b) \cdot Z(r) .$$

(4.7.1)

Here, the coordinate $r$ is the distance from the centre of mass of the whole system, and the current operator $\hat{j}(r)$ gives at each position $r$ the current after integrating over all the charged-particle positions $r_i$, which are again their distances from that centre of mass. For the bound state $\Phi_b$ at energy $E_b$, and continuum state $\Psi$ at $E_i$, the photon energy is $E_\gamma = E_i - E_b$. The $\hat{j}_q$
is the electric current operator defined in section 3.1.4, and \( Z(r) = \sqrt{k/\hbar c} A \) is the normalised vector potential of Eq. (3.5.31) for the electromagnetic field.

The 3-dimensional wave functions of Eq. (4.7.1) have been expanded in partial waves in Eq. (3.5.38), from which the partial wave T matrix is given by Eq. (3.5.43), and a vector-form T matrix is also given by Eq. (3.5.45). In order to satisfy the Coulomb gauge condition, section 3.5.5 explains how this vector-form T matrix is replaced by that of Eq. (3.5.61):

\[
T^{(E,M)}_{JM} = \langle \sqrt{4\pi} \sqrt{2J+1} i^J A_{JM}(r; (E, M))|V_{\gamma p}|\Psi(R, k_i) \rangle (4.7.2)
\]

which we must now evaluate. These matrix elements of the derivative current operator can be evaluated exactly (see for example [36] or [37]), but the complications of derivatives mean that simplified alternatives are often used. One minor detail is that the centre-of-mass coordinates \( r \) for the position of the photon and \( r_i \) for the charged particles are different from \( R \), the relative separation of the two nuclei, so the couplings are spatially nonlocal. The most common simplifying approximations used are (a) the long wavelength approximation, and (b) using Siegert’s theorem to transform current into charge density operators.

The long wavelength approximation means that the lowest \( \Lambda \) partial wave in the allowed range \( |J-1| \leq \Lambda \leq J+1 \) is taken to be dominant. This is reasonable if the photon wave lengths \( 2\pi \hbar c/E_\gamma \) are much larger than the nuclear bound states: even large nuclear halo states of \( \sim 20 \text{ fm} \) in extent allow photon energies up to \( E_\gamma \sim 60 \text{ MeV} \) before the long-wavelength approximation is unusable.

Siegert’s theorem [38, 39] uses the continuity equation satisfied by the particle density to allow integration by parts to transform the current operators into second derivatives, where they may be replaced by using the Schrödinger equation satisfied by the particle wave function \( \Psi(R) \).

### 4.7.2 Electric transitions using the Siegert theorem

The Siegert theorem uses the long wavelength approximation of \( kr \ll 1 \) to replace the electric plane wave component by a derivative of the longitudinal plane wave. Since \( F_\Lambda(0, kr) \propto (kr)^{\Lambda+1} \), this limit allows us to neglect the \( \Lambda = J+1 \) terms of both \( A_{JM}(r; E) \) and \( A_{JM}(r; \text{long}) \). The two remaining \( \Lambda = J - 1 \) terms are thus proportional to each other, as explained on page 116, allowing the replacement

\[
A_{JM}(r; E) \simeq \sqrt{\frac{J+1}{J}} A_{JM}(r; \text{long}) = \sqrt{\frac{J+1}{J}} \frac{1}{k} \nabla \frac{1}{kr} F_j(0, kr) Y^M_j(\hat{r}) (4.7.3)
\]
to a good approximation. If we use the abbreviations

\[ s_{kJ} = \sqrt{4\pi\sqrt{2J+1}} \frac{i^{-J}}{k^2} \left( \frac{J+1}{J} \right)^{1/2} \]  
(4.7.4)

\[ t_{kJ} = s_{kJ} 2\sqrt{\frac{\hbar}{\omega}} \]  
(4.7.5)

\[ f_{JM}(r) = r^{-1} F_j(0, kr) Y^M_j(r), \]  
(4.7.6)

the electric T matrix from Eq. (4.7.2) can be calculated as

\[ T_{JM}^{(E)} = s_{kJ} \left( \nabla f_{JM}(r) \right) | \nabla \cdot \hat{j}_q(r) | \Phi \rangle \]  
(4.7.7)

\[ = t_{kJ} \left( \nabla f_{JM}(r) \right) \Phi \rangle | \Phi \rangle \]  
(4.7.8)

Remember that the \( \langle \ldots | \rangle \) notation implies integration over all the variables on the left: integration over both photon position \( r \) and particle positions \( r_i \) is implied.

We now use integration by parts to move the \( \nabla \) from acting on the \( f_{J} \) to act on the current operator. The limits of the integrand give zero since the bound state \( \Phi_b(r) \rightarrow 0 \) as \( R \rightarrow \infty \), so

\[ T_{JM}^{(E)} = -t_{kJ} \langle \Phi_b f_{JM}(r) | \nabla \cdot \hat{j}_q(r) | \Phi \rangle \]  
(4.7.9)

The divergence of the current \( \nabla \cdot \hat{j}_q(r) \) is then replaced using the continuity equation (3.1.104), yielding

\[ T_{JM}^{(E)} = -t_{kJ} \left( \frac{iq}{\hbar} \left[ \delta(r-r_i) H - H^\dagger \delta(r-r_i) \right] \right) | \Phi \rangle \]  
(4.7.10)

In the last step we used the Schrödinger equations satisfied by \( \Phi_b \) and \( \Psi \),

\[ [H - E_b] \Phi_b = [H - E_i] \Psi = 0, \]  
and rewritten the integral as a matrix element in the \( r \) coordinate. Since \( E_i - E_b = E_\gamma = \hbar \omega = \hbar kc \) for the photon, and \( E_b \)

is real for a bound state, the constant factors become

\[ -t_{kJ} \frac{iq}{\hbar} E_\gamma = -i\sqrt{4\pi\sqrt{2J+1}} \frac{i^{-J}}{k^2} \left( \frac{J+1}{J} \right)^{1/2} \sqrt{\frac{\hbar}{\omega}} \frac{iq}{\hbar} E_\gamma \]  
\[ = -iq\sqrt{4\pi\sqrt{2J+1}} \frac{i^{-J}}{k^2} \left( \frac{8\hbar c(J+1)}{k^3 J} \right). \]  
(4.7.11)

The electric photon vector-form T matrix is thus

\[ T_{JM}^{(E)} = -i\sqrt{4\pi\sqrt{2J+1}} \left( \frac{8\hbar c(J+1)}{k^3 J} \right) \langle \Phi_b q i^J f_{JM} | \Psi \rangle. \]  
(4.7.12)
In the long wavelength approximation, we use Eq. (3.1.17) to give
\[ F_J(0, kr) \simeq \frac{1}{(2J+1)!!} (kr)^{J+1} \] thus \[ f_{JM}(r) \simeq \frac{k^{J+1}}{(2J+1)!!} r^J Y^M_J, \] (4.7.13)

The Seigert theorem therefore gives us a matrix element that is proportional to the standard multipole operator of order \( J \) for the charge density of Eq. (4.4.23):
\[ \langle \Phi_b | q f_{JM} | \Psi \rangle \simeq \frac{k^J}{(2J+1)!!} \langle \Phi_b | q r^J Y^M_J | \Psi \rangle. \] (4.7.14)

The conjugation of \( f_{JM} \) and the sign of \( M \) in this equation depend on whether the photon is in the initial or the final state.

**Reconstructing a local form factor for coupled equations**

Having made the reduction to charge-density form, it is sometimes convenient for the uniform treatment of all reaction channels to reconstitute a coupled photon equation which gives the same partial-wave \( T_{\gamma \alpha} \) matrix element as Eq. (4.7.12). Now, for photon partial wave \( \gamma \) as in Eq. (3.5.40),
\[ \sqrt{4\pi} \sqrt{2J+1} = 4\pi Y^0_J(+) = kA(\gamma; +z), \] (4.7.15)
and \( T^{(\xi)}_{J\alpha} \) is the amplitude for the emission of photons in the \( +z \) direction, so using Eq. (3.3.44) gives
\[ T_{\gamma \alpha} = -\frac{1}{k} \frac{k}{\hbar c} \frac{q}{\sqrt{8\pi \hbar c (J+1)}} \sum_{\alpha} \langle q^J F_J | \phi_b | \psi_{\alpha \alpha} \rangle. \] (4.7.16)

The same \( T \) matrix numerical value can be obtained from the asymptotic solution of a reformulated photon channel equation
\[ \left[ -\frac{\hbar c}{k} \frac{d^2}{dr^2} - \frac{J(J+1)}{r^2} - E_\gamma \right] \tilde{\zeta}_\gamma(r) = \frac{q}{4} \sqrt{8\pi \hbar c (J+1)/(Jk)} \phi_b \psi_\alpha(R). \] (4.7.17)

The new photon wave functions \( \tilde{\zeta}_\gamma(r) \) are not equal to the previous functions \( \zeta_\gamma(r) \), to start with being in partial wave \( \Lambda' = J \) rather than \( \Lambda = J-1 \), but they have been constructed to have the same asymptotic \( T \) matrix amplitude.

Since the wave functions \( \Psi(R) \) are written as functions of the two-body separation \( R \) and not on the distance \( r \) from their centre of mass, Eq. (4.7.17)
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is a non-local equation. It may be approximated by a local equation if we can use the power series of Eq. (4.7.14) to include the scaling factor in \( r = \nu R \):

\[
\left[ -\frac{\hbar c}{k} \left( \frac{d^2}{dR^2} - \frac{J(J + 1)}{R^2} \right) - E_\gamma \right] \tilde{\zeta}_\gamma(R) = \nu^{J+2} \frac{q}{1} \sqrt{8\pi \hbar c(J+1)/(Jk)} \phi_\nu \psi_\alpha(R). \tag{4.7.18}
\]

This equation, valid in the long-wavelength approximation or when \( \nu \simeq 1 \), implies that the transition potential from particle to gamma channels is

\[
V_{\lambda p}^\gamma(R) = \nu^{J+2} \frac{q}{1} \sqrt{8\pi \hbar c(J+1)/(Jk)} \phi_\nu(R) \tag{4.7.19}
\]

for multipolarity \( \lambda \) equal to the bound state orbital angular momentum.

From the \( T \) matrix element calculated either by Eq. (4.7.12) or by Eq. (4.7.18), the photon cross section is (as stated in Chapter 3)

\[
\sigma_{\gamma p}^J = \frac{4\pi}{k^2} \frac{1}{(2J_p+1)(2J_t+1)} v_i \sum_{J_{\text{tot}}} (2J_{\text{tot}}+1) |T_{\gamma t_{\text{tot}}}^J|^2 \tag{4.7.20}
\]

4.7.3 Combining multiple particle and \( \gamma \) channels

The coupled channels formalism that we have used as a general framework for reaction theory is designed to have only two bodies in relative motion in each partition. This means that if a two-body composite system or a continuum state decays by \( \gamma \) emission, the theory describes the relative motion of the emitted photon, and the remaining nucleus has to remain as one body, as effectively bound. Any further decays of that residual system will have to be described statistically the Hauser-Feshbach methods given in Chapter 11.

What the coupled channels framework can describe well is the production of a composite system, \( A+B \rightarrow (A+B)^* \), and then the couplings between all the decays of this system to two-body channels. Such channels include particle channels \( C+D, C'+D' \), etc, as well as \( \gamma \) decay channels \( \gamma+E, \gamma'+E' \), etc, where all the \( C, D \) and \( E \) nuclei are effectively bound. The coupled channels set for total spin and parity \( J_{\text{tot}}^\pi \) will have partial waves for each of the particle channels \( C+D \), as well as a set \( \gamma \) channels, one or more for each residual nucleus \( E \) after \( \gamma \) emission. Medium and heavy nuclei will typically have very many states \( E \) to which the composite system could decay by photon emission, and each state will contribute channels to the coupled channels set.

If the composite \( (A+B)^* \) is a resonance, then the coupled channels framework as present here will correctly generate the branching ratios to all the
particle and decay channels. In the language to be developed in Chapter 10, it will generate all the partial widths for both kinds of exit channels. The partial widths to all the possible residual states E will add together, and combine with the particle partial widths, to form the total width of the resonance that is observed. In section 11.2 we will discuss some approximations which may help to simplify the problem when there are very many $\gamma$ emission channels in heavier nuclei.

### 4.7.4 Connecting photon cross sections and $B(EJ)$

Let us define the electric multipole operator

$$
\mathcal{M}^{\text{elec}}_{JM} = qr^{J} Y_{J}^{M}(\hat{r})
$$

(4.7.21)
as that which appears in the earlier matrix element (4.4.23) for Coulomb inelastic scattering. Then the function defined above in Eq. (4.7.6) becomes in the long-wavelength approximation

$$
q_{f}(r) \simeq \frac{k^{J+1}}{(2J+1)!!} M^{\text{elec}}_{JM}.
$$

(4.7.22)

From Eq. (4.7.20), the photon cross section is

$$
\sigma_{\gamma p}^{J} = \frac{16\pi^{3}}{k_{i}^{2}} \frac{1}{(2J_{p}+1)(2J_{t}+1)} J+1 \frac{k^{2J+1}}{(2J+1)!!} \frac{1}{\hbar v_{i}}
\times \sum_{m_{b}M_{i}} \left| \langle \Phi_{b}^{m_{b}} | \mathcal{M}_{JM}^{\text{elec}} | \sqrt{\frac{2}{\pi}} \Psi_{i}^{M_{i}} \rangle \right|^{2}.
\times \int dE B(EJ, i \rightarrow \gamma).
$$

(4.7.23)

The incoming scattering waves $\sqrt{2/\pi}\Psi_{i}$ are normalised as delta functions in $k$, and $\frac{d}{dE} = \hbar v_{i} \frac{d}{dE}$, so

$$
\frac{1}{\hbar v_{i}} \sum_{m_{b}M_{i}} \left| \langle \Phi_{b}^{m_{b}} | \mathcal{M}_{JM}^{\text{elec}} | \sqrt{\frac{2}{\pi}} \Psi_{i}^{M_{i}} \rangle \right|^{2} = (2J_{i}+1) \frac{d}{dE} B(EJ, i \rightarrow \gamma).
$$

(4.7.24)

This gives the $\gamma \rightarrow p$ cross section

$$
\sigma_{\gamma p}^{J} = \frac{16\pi^{3}}{k_{i}^{2}} \frac{(2J_{i}+1)}{(2J_{p}+1)(2J_{t}+1)} J+1 \frac{k^{2J+1}}{(2J+1)!!} \frac{1}{\hbar v_{i}} \frac{d}{dE} B(EJ, i \rightarrow \gamma).
$$

(4.7.25)
Using the detailed balance for photo-nuclear reactions as given by Eq. (3.2.4), the reverse \((p\rightarrow\gamma)\) photo-disintegration cross section is

\[
\sigma_{\gamma p}^J = \frac{k^2(2J_p+1)(2J_t+1)}{k^22(2J_B+1)} \sigma_{\gamma p}^J
\]

\[
= \frac{(2\pi)^3(J+1) 2J_t + 1}{J[(2J+1)!!]^2} \frac{k^2 2J_B + 1}{k^2 2J_t - 1} \frac{dB(EJ, p \rightarrow \gamma)}{dE}
\]

\[
= \frac{(2\pi)^3(J+1) k^2 2J_B + 1}{J[(2J+1)!!]^2} \frac{dB(EJ, \gamma \rightarrow p)}{dE}.
\]  

(4.7.26)

For transitions between \textit{discrete} states, see section 5.4.

**Transitions for two-body nuclei**

Consider a nucleus consisting of two bodies as in section 4.4.2, with a charge \(Z_c\) at \(r = -\frac{m_v}{m_p} R\) and another \(Z_v\) at \(r = \frac{m_c}{m_p} R\), where \(R\) is the vector separation between the clusters. If its initial scattering state is \(\Psi_i(R)\) and final bound state is \(\Phi_b(R)\), then the electric matrix element \(EJ\) is

\[
\langle \Phi_b | M^{EM}_{J}\rangle \Psi_i = \int dR \Phi_b^* R^{J} \Psi_i (R) \delta \left( \frac{m_v}{m_p} R \right) + Z_v e \left( \frac{m_c}{m_p} R \right) \]

\[
= e_J \int \Phi_b^* R^{J} \Psi_i (R) \ dR
\]  

(4.7.27)

where the \textit{effective multipole charge} is

\[
e_J = Z_v e \left( -\frac{m_v}{m_p} R \right) + Z_v e \left( \frac{m_c}{m_p} R \right).
\]  

(4.7.28)

The spatial integral is

\[
\int \Phi_b^* R^{J} \Psi_i \ dR = \frac{1}{\sqrt{4\pi}} \frac{J_i}{L_i} \langle L_i M_i, J M | L_f M_f \rangle \langle L_i 0, J 0 | L_f 0 \rangle \langle \Phi_b | R^J | \psi_i \rangle
\]  

(4.7.29)

and the reduced transition probabilities are

\[
B(EJ, i \rightarrow b) = e_J^2 \frac{2J+1}{4\pi} \left| \langle L_i 0, J 0 | L_f 0 \rangle \langle \Phi_b(R) | R^J | \psi_i(R) \rangle \right|^2.
\]  

(4.7.30)

More general Racah algebra expressions may be obtained when the core and/or valence particles have spins, when these are spectators to the electric transition. When the core or valence particles themselves undergo electric transitions, further terms must be added that involve internal matrix elements for one or both of the bodies.
4.7 Photo-nuclear couplings

Limiting cases for neutrons

Consider $L_f = 0$ s-wave bound states of a neutron with a delta-function binding potential $V(R) = \delta(R)$ at the origin. The wave function is $\Phi_b(R) = \sqrt{2\gamma} R^{-1} \exp(-\gamma R) Y_0^0(R)$ for wave number $\gamma = \sqrt{2\mu E_b/\hbar^2}$, and the scattering states are p-wave plane waves. The dipole reduced transition probability for photo-disintegration is

$$\frac{d\mathcal{B}(E1, 0^+ \rightarrow 1^-)}{dE} = \frac{2\hbar^2}{\pi^2\mu} \left( \frac{Z_e e}{A_{c+1}} \right)^2 \sqrt{E_b E^3/2} \frac{(E + E_b)^4}{(E + E_b)^4}. \quad (4.7.31)$$

If instead $\Phi_b(R) = C_0 R^{-1} \exp(-\gamma R) Y_0^0(R)$ for asymptotic normalisation coefficient (ANC) $C_0$ defined by Eq. (4.5.24), then this result will be scaled by $C_0^2/2\gamma$.

Limiting cases for protons

For low energy capture of charged particles, the dominating feature is the Coulomb barrier in the entrance channel. At very low energies, if there are no resonances then the entrance phase shifts will be zero, and the incoming wave function will be essentially $\psi_{\alpha\alpha}(R) = \delta_{\alpha\alpha} F_L(\eta_i, k_i R)$. According to Box 3.2, in the $\eta \gg L_i$ and $k_i R \ll 1$ limits this becomes

$$F_{L_i}(\eta_i, k_i R) \approx \frac{(\eta_i k_i)^{L_i}}{L_i!(2L_i+1)!} \frac{2\pi \eta_i}{e^{2\pi \eta_i} k_i R^{L_i+1}}, \quad (4.7.32)$$

noting that $\eta_i k_i$ is independent of energy. Since now this elastic wave function is the only non-zero channel in Eq. (4.7.16), we get from Eq. (4.7.20) the cross section for a specific $J_{tot}$, entrance channel $L_i$ and multipole $J$ of

$$\sigma_J^{\gamma p} = \frac{4\pi}{k_i^2} g_{J_{tot}} \left[ \frac{q}{\hbar c^2} \sqrt{8\pi \hbar c (J+1)} \frac{(F_J(0, kr)|\phi_b|F_{L_i}(\eta_i, k_i R))}{J} \right]^2$$

$$= \frac{e^{-2\pi \eta_i}}{E_i} \left[ \frac{32\pi^3 \hbar c}{k} \right]^{2} \left[ \frac{g_{J_{tot}}}{L_i!(2L_i+1)!} \right]^2 \frac{\eta_i k_i^{2L_i+1}}{(J+1)} \frac{(J+1)}{\hbar c} \left[ \frac{|\phi_b|}{h c} \right]^{2}, \quad (4.7.33)$$

omitting the angular momentum coupling factors. At low proton energies $E_i$, the photon energy $E_\gamma = E_i - E_b$ will tend to a constant, so the factor governing the low-energy behaviour here is just the $e^{-2\pi \eta_i}/E_i$, which is that built into the definition of the astrophysical S-factor in Eq. (1.2.3). The same factor thus describes the energy dependence for all partial waves without resonances.
The integrand of the final matrix element will be dominated by the long-range behaviour of the bound state wave function $\phi_b(R)$, which can be characterised by its asymptotic normalisation coefficient $C_b$ defined according to Eq. (4.5.24). At low incident energies, therefore, the astrophysical S factor will tend to by constant with energy, and the constant will be proportional to $C_b^2$. For more details, see references [40, 41, 42, 43].

4.7.5 Magnetic transitions

The coupled equations Eq. (4.7.1) describe the combined electric and magnetic contributions to the photon cross section. The separated electric part was calculated from Eq. (4.7.7), and now we look at $M_J$ transitions which give overall parity changes of $(-1)^{J-1}$ as explained in Box 3.5.4. The magnetic part of the $M_J$ transition integral depends only on the $\Lambda = J$ components, and is

$$T_{JM}^{(M)} = \langle \sqrt{4\pi} \sqrt{2J+1} \ i^J \ A_{JM}(r; M) | V_{\gamma p} | \Psi(R; k) \rangle$$  (4.7.34)

where, from Eq. (3.5.51), the magnetic plane wave part is

$$A_{JM}(r; M) = (kr)^{-1} F_J(0, kr) \ Y_J^M(\hat{r})$$  (4.7.35)

and, from Eq. (3.5.35), the coupling operator is

$$V_{\gamma p} = 2\sqrt{\hbar/\omega} \langle \Phi_b | \hat{j}_q(r) \rangle = 2 \sqrt{2\pi \hbar | q | \omega / 2\mu} \langle \Phi_b | \{ \delta(r-r_i) \ \nabla - \nabla \delta(r-r_i) \} \rangle.$$  (4.7.36)

Now the vector spherical harmonic in Eq. (4.7.35) may be obtained also by acting on a spherical harmonic with the angular momentum vector operator,

$$L Y_J^M(\hat{r}) = [J(J+1)]^{1/2} Y_J^M(\hat{r}),$$  (4.7.37)

so

$$A_{JM}(r; M) = [J(J+1)]^{-1/2} F_J(0, kr) \ L Y_J^M(\hat{r})$$
$$= [J(J+1)]^{-1/2} L (kr)^{-1} F_J(0, kr) \ Y_J^M(\hat{r})$$
$$\equiv [J(J+1)]^{-1/2} k^{-1} L f_{JM}(r)$$  (4.7.38)

using the fact that $L$ commutes with any purely radial function, and then the same definition of $f_{JM}(r)$ as in Eq. (4.7.4).
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The T matrix is therefore

\[ T_{JM}^{(M)} = \sqrt{\frac{4\pi(2J+1)}{k^2 J(J+1)}} \langle i^J l f_{JM}(r) | V_{\gamma p} | \Psi(R, k_i) \rangle \] (4.7.40)

\[ = 2 \sqrt{\frac{2\pi \hbar q}{kc}} \frac{4\pi(2J+1)}{k^2 J(J+1)} i^{-J} I \] (4.7.41)

where \( I \) is the integral

\[ I = \int dr \left[ \Phi_b^* \mathbf{L} f_{JM}(r)^* \cdot \nabla \Psi + \Psi \mathbf{L} f_{JM}(r)^* \cdot \nabla \Phi_b^* \right]. \] (4.7.42)

Now \( \mathbf{L} \cdot \nabla = \mathbf{r} \times \mathbf{p} \cdot \nabla = 0 \), so \( \left[ \mathbf{L} f(r) \right] \cdot \nabla = -\left[ \nabla f(r) \right] \cdot \mathbf{L} \) for any spatial function \( f(r) \). Thus

\[ I = -\int dr \left[ \Phi_b^* \mathbf{L} f_{JM}(r)^* \cdot \nabla \Psi + \Psi \mathbf{L} f_{JM}(r)^* \cdot \nabla \Phi_b^* \right], \] (4.7.43)

where from Eq. (3.5.63) we find that

\[ \nabla f_{JM}(r) = k^2 A_{JM}(r; \text{long}), \] (4.7.44)

\[ \approx \sqrt{\frac{J}{2J+1}} \frac{k}{r} F_{J-1}(0, kr) Y_{J-1, J}^M(\hat{r}) \] (4.7.45)

in the long-wavelength approximation which allows us to neglect the second term in Eq. (3.5.63). These steps remove all the derivative operators from the matrix element of \( I \), giving

\[ I = -\sqrt{\frac{J}{2J+1}} \frac{k}{r} \int dr F_{\Lambda'}(0, kr) \left[ \Phi_b^* Y_{\Lambda', J}^M \cdot \mathbf{L} \Psi + \Psi Y_{\Lambda', J}^M \cdot \mathbf{L} \Phi_b^* \right], \] (4.7.46)

where we define \( \Lambda' = J-1 \). The photon spatial wave functions are now in partial wave \( \Lambda' \) rather than the original \( \Lambda = J \). The parity change between \( \Psi \) and \( \Phi_b \) is now simply \((-1)^{\Lambda'}\), which agrees with Eq. (4.7.46) as there are no longer has derivative operators which reverse the parity.

Note that this transformation of photon partial waves is the opposite of that accomplished by the Siegert Theorem, which transforms an original \( \Lambda = J-1 \) to \( \Lambda' = J \) in order to remove the derivative operators in the electric matrix element. In both cases, the electromagnetic parity change is simply now \((-1)^{\Lambda'}\).

The angular parts of the matrix elements have reduced matrix elements

\[ \langle L_b | Y_{\Lambda', J}^M \cdot \mathbf{L} | L_i \rangle = \frac{(2L_i+1)\hat{\Lambda'} \hat{J} \sqrt{L_i(L_i+1)}}{\sqrt{4\pi}} \langle L_b 0, \Lambda' 0 | L_i 0 \rangle W(L_i 1 L_b \Lambda'; L_i J). \] (4.7.47)
We may now follow the same procedures as above for electric transitions, and derive transition potentials for use in coupled particle-photon equations, and also define a magnetic multipole operator $M^\text{mag}_{JM}$ to construct a reduced transition probability $dB(MJ, \gamma \to p)/dE$.

References

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