Approximate methods
Adiabatic approximation: examples

- elastic scattering with the Johnson special three-body model

\[ T_{el}^{ad} = F(Q) T_{el}^{\text{point}} \]
Adiabatic approximation: examples

- transfer reactions

**FIG. 4.** Cross sections of the $^{10}\text{Be}(d,p)^{11}\text{Be}$ reaction at $E_d = 25$ MeV calculated with adiabatic deuteron wave function. (a) Calculations have been done with proton optical potential P1 and deuteron adiabatic potential D1; spectroscopic factor $S = 1$ was used. (b) Different sets of proton and deuteron optical potentials were used and theoretical curves are normalized to the experimental data.

**FIG. 1.** Cross sections of the $^{16}\text{O}(d,p)^{17}\text{O}$ reaction at $E_d = 36$ MeV calculated with optical (a) and adiabatic (b) deuteron wave functions.
Eikonal approximation

Reaction timescales – in surface grazing collisions

For 100 and 250 MeV/u incident energy:

\[ \gamma = 1.1, \quad v/c = 0.42, \quad \gamma = 1.25, \quad v/c = 0.6, \]
\[ \Delta t = 7.9 \times d \times 10^{-24} \text{s}, \quad \Delta t = 5.6 \times d \times 10^{-24} \text{s} \]
Eikonal approximation:

\[ u_{KL}(r) \to (i/2) \left[ \frac{1}{2} H_L^{(-)} - S_L H_L^{(+)} \right] \]
Eikonal approximation:

Semi-classical approaches – many L-values

large $k$ – e.g. nucleus-nucleus

$R_P$, $k$, $L$

$|S_L|$

absorption

transmission

semi-classical: $S(b)$, $L + \frac{1}{2} = kb$

$$\sum_{L=0}^{\infty} (2L + 1) \ldots \rightarrow 2k^2 \int_{0}^{\infty} db \ b \ldots \frac{L(L + 1)}{r^2} \rightarrow \frac{k^2 b^2}{r^2} - \frac{1}{4}$$
Eikonal approximation:

Point particle scattering – cross sections

All cross sections, etc. can be computed from the S-matrix, in either the partial wave or the semi-classical (impact parameter) representations, for example (spinless case):

\[
\sigma_{el} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1)|1 - S_{\ell}|^2 \approx \int d^2\vec{b} \ |1 - S(b)|^2
\]

\[
\sigma_R = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1)(1 - |S_{\ell}|^2) \approx \int d^2\vec{b} \ (1 - |S(b)|^2)
\]

\[
\sigma_{tot} = \sigma_{el} + \sigma_R = 2 \int d^2\vec{b} \ [1 - \text{Re}.S(b)] \quad \text{etc.}
\]

and where (cylindrical coordinates)

\[
\int d^2\vec{b} \equiv \int_0^\infty bdb \int_0^{2\pi} d\phi = 2\pi \int_0^\infty bdb
\]
Eikonal approximation: for point particles (1)

Approximate (semi-classical) scattering solution of

\[
\left( -\frac{\hbar^2}{2\mu} \nabla_r^2 + U(r) - E_{cm} \right) \chi^+_k(\vec{r}) = 0, \quad \mu = \frac{m_cm_v}{m_c + m_v}
\]

\[
\left( \nabla_r^2 - \frac{2\mu}{\hbar^2} U(r) + k^2 \right) \chi^+_k(\vec{r}) = 0
\]

small wavelength

valid when \(|U|/E \ll 1, \quad ka \gg 1\) \rightarrow high energy

Key steps are: (1) the distorted wave function is written

\[
\chi^+_k(\vec{r}) = \exp(i\vec{k} \cdot \vec{r}) \omega(\vec{r})
\]

all effects due to \(U(r)\), modulation function

(2) Substituting this product form in the Schrodinger Eq.

\[
\left[ 2i\vec{k} \cdot \nabla \omega(\vec{r}) - \frac{2\mu}{\hbar^2} U(r) \omega(\vec{r}) + \nabla^2 \omega(\vec{r}) \right] \exp(i\vec{k} \cdot \vec{r}) = 0
\]
Eikonal approximation: point neutral particles (2)

\[
\left[ 2i\vec{k} \cdot \nabla \omega(\vec{r}) - \frac{2\mu}{\hbar^2} U(r)\omega(\vec{r}) + \nabla^2 \omega(\vec{r}) \right] \exp(i\vec{k} \cdot \vec{r}) = 0
\]

The conditions \( |U|/E \ll 1, \ k a \gg 1 \) \( \rightarrow \) imply that

\[ 2\vec{k} \cdot \nabla \omega(\vec{r}) \gg \nabla^2 \omega(\vec{r}) \]

Slow spatial variation cf. \( k \)

and choosing the z-axis in the beam direction \( \vec{k} \)

\[
\frac{d\omega}{dz} \approx -\frac{i\mu}{\hbar^2 k} U(r)\omega(\vec{r})
\]

with solution

\[
\omega(\vec{r}) = \exp \left[ -\frac{i\mu}{\hbar^2 k} \int_{-\infty}^{z} U(r')dz' \right]
\]

1D integral over a straight line path through \( U \) at the impact parameter \( b \)
Eikonal approximation: point neutral particles (3)

\[ \chi_{k}^{+}(\vec{r}) = \exp(i\vec{k} \cdot \vec{r}) \omega(\vec{r}) \approx \exp(i\vec{k} \cdot \vec{r}) \exp \left[ -\frac{i\mu}{\hbar^2 k} \int_{-\infty}^{z} U(\vec{r}') d\vec{z}' \right] \]

So, after the interaction and as \( z \to \infty \)

\[ \chi_{k}^{+}(\vec{r}) \to \exp(i\vec{k} \cdot \vec{r}) \exp \left[ -\frac{i\mu}{\hbar^2 k} \int_{-\infty}^{\infty} U(\vec{r}') d\vec{z}' \right] = S(b) \exp(i\vec{k} \cdot \vec{r}) \]

\[ \chi_{k}^{+}(\vec{r}) \to S(b) \exp(i\vec{k} \cdot \vec{r}) \]

S(b) is amplitude of the forward going outgoing waves from the scattering at impact parameter b

Eikonal approximation to the S-matrix \( S(b) \)

\[ S(b) = \exp \left[ -\frac{i}{\hbar v} \int_{-\infty}^{\infty} U(\vec{r}') d\vec{z}' \right] \]

Moreover, the structure of the theory generalises simply to few-body projectiles
Eikonal approximation: point particles - summary

\[ \chi^\pm_k(\vec{r}) = \exp(i\vec{k} \cdot \vec{r}) \exp \left[ -\frac{i\mu}{\hbar^2 k} \int_{-\infty}^{\tilde{z}} U(r')dz' \right] \]

\[ \nu = \frac{\hbar k}{m} \]

\[ \chi(b) = -\frac{1}{\hbar \nu} \int_{-\infty}^{\infty} U(r')dz \]

Limit of range of finite ranged potential

\[ \chi^\pm_k(\vec{r}) \rightarrow S(b) \exp(i\vec{k} \cdot \vec{r}) \]

\[ S(b) = \exp [i\chi(b)] = \exp \left[ -\frac{i}{\hbar \nu} \int_{-\infty}^{\infty} U(r')dz' \right] \]
Using the standard result from scattering theory, the elastic scattering amplitude is

\[ f(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d\vec{r} \exp(-i\vec{k}' \cdot \vec{r}) U(r) \chi_{\vec{k}}^+ (\vec{r}) \]

\[ = -\frac{\mu}{2\pi\hbar^2} \int d\vec{r} \exp(-i\vec{k}' \cdot \vec{r}) U(r) \exp(i\vec{k} \cdot \vec{r}) \omega(\vec{r}) \]

\[ = -\frac{\mu}{2\pi\hbar^2} \int d\vec{r} \exp(i\vec{q} \cdot \vec{r}) U(r) \omega(\vec{r}) \]

with \( \vec{q} = \vec{k} - \vec{k}' \), \( q = 2k \sin(\theta/2) \) is the momentum transfer. Consistent with the earlier high energy (forward scattering) approximation

\[ \vec{q} \cdot \vec{r} \approx \vec{q} \cdot \vec{b} \]

\[ \vec{q} \cdot \vec{k} \approx 0 \]
Point particles – the differential cross section

So, the elastic scattering amplitude

\[ f(\theta) = -\frac{\mu}{2\pi \hbar^2} \int d\vec{r} \exp(i\vec{q} \cdot \vec{r}) U(r) \omega(\vec{r}) \]

is approximated by

\[ f_{eik}(\theta) = -\frac{ik}{2\pi} \int d^2\vec{b} \exp(i\vec{q} \cdot \vec{b}) \int_{-\infty}^{\infty} \frac{d\omega}{dz} d\omega \]

Performing the z- and azimuthal \( \phi \) integrals

\[ f_{eik}(\theta) = -ik \int_{0}^{\infty} b \, db \, J_0(qb) [S(b) - 1] \]

\[ S(b) = \exp[i\chi(b)] = \exp \left[ -\frac{i}{\hbar \nu} \int_{-\infty}^{\infty} U(r') dz' \right] \]

\[ \frac{d\omega}{dz} = -\frac{i\mu}{\hbar^2 k} U(r) \omega(\vec{r}) \]

\[ U(r) \omega(\vec{r}) = \frac{i\hbar^2 k}{\mu} \frac{d\omega}{dz} \]
Point particle – the Coulomb interaction

Treatment of the Coulomb interaction (as in partial wave analysis) requires a little care. Problem is, eikonal phase integral due to Coulomb potential diverges logarithmically.

$$\chi_C(b) = -\frac{1}{\hbar v} \int_{-a}^{+a} V_C(r') dz$$

Must 'screen' the potential at some large screening radius

$$f_{eik}(\theta) = e^{i\chi} a \left[ f_{pt}(\theta) - ik \int_{0}^{\infty} b \, db \, J_0(qb) \, e^{i\chi_{pt}} [\bar{S}(b) - 1] \right]$$

overall unobservable screening phase

usual Coulomb (Rutherford) point charge amplitude

nuclear scattering in the presence of Coulomb

$$\bar{\chi}(b) = \chi_N(b) + \chi_{\rho}(b) - \chi_{pt}(b)$$

nuclear phase

Due to finite charge distribution

$$\chi_{pt}(b) = 2\eta \ln(kb)$$

See e.g. J.M. Brooke, J.S. Al-Khalili, and J.A. Tostevin PRC 59 1560
Eikonal approximation: several particles (preview)

\[ \chi_i(b) = -\frac{1}{\hbar v} \int_{-\infty}^{\infty} U_i(r') dz \]

Total interaction energy

\[ U(r_1, \ldots) = \sum_i U_i(r_i) \]

\[ S_i(b_i) = \exp \left[ i \chi_i(b_i) \right] = \exp \left[ -\frac{i}{\hbar v} \int_{-\infty}^{\infty} U_i(r'_i) dz' \right] \]

\[ \chi(b_1, \ldots) = -\frac{1}{\hbar v} \int_{-\infty}^{\infty} \sum_i U_i(r'_i) dz \]

with composite objects we will get products of the S-matrices

\[ \exp[i \chi(b_1, \ldots)] = \prod_i S_i(b_i) \]
Eikonal approximation for composite systems

- generalized Eikonal scattering amplitude

\[ f_{fi}(\theta) = -\frac{iK_0}{2\pi} \int d^2 b \ e^{i\mathbf{q} \cdot \mathbf{b}} \langle \Phi_f(\mathbf{r}) | e^{i\chi(\mathbf{b} - \mathbf{b}_r)} - 1 | \Phi_i(\mathbf{r}) \rangle \]

Wfn of the projectile \( \Phi_i(\mathbf{r}) \equiv \Phi_i(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-1}) \)

Eikonal phase \( \chi(\mathbf{b} - \mathbf{b}_r) \equiv \sum_{j=1}^{n} \chi(\mathbf{b} - \mathbf{b}_{r_j}) \)

- introduce S-matrices for each body:

\[ S_i(b_i) = e^{i\chi_i(b_i)} \quad b_i = b - b_{r_i} \]
Eikonal approximation for composite systems

- The resulting scattering amplitude becomes:

\[
f_{fi}(\theta) = -\frac{iK_0}{2\pi} \int d^2b \, e^{i\mathbf{q} \cdot \mathbf{b}} \langle \Phi_f(\mathbf{r}) | e^{i\chi(\mathbf{b} - \mathbf{b}_r)} - 1 | \Phi_i(\mathbf{r}) \rangle
\]

\[
f_{fi}(\theta) = -\frac{iK_0}{2\pi} \int d^2b \, e^{i\mathbf{q} \cdot \mathbf{b}} \int \cdots \int d\mathbf{r}_1 d\mathbf{r}_2 \cdots d\mathbf{r}_{n-1}

\Phi_f(\mathbf{r}_1, \ldots, \mathbf{r}_{n-1})^* [S_1(\mathbf{b}_1) \ldots S_n(\mathbf{b}_n) - 1] \Phi_i(\mathbf{r}_1, \ldots, \mathbf{r}_{n-1})
\]

- n-body scattering S-matrix

\[
S^n_{fi}(\mathbf{b}) = \langle \Phi_f | \prod_{j=1}^n S_j(\mathbf{b}_j) | \Phi_i \rangle
\]

\[
S_i(\mathbf{b}_i) = e^{i\chi_i(\mathbf{b}_i)}
\]

- Contains all correlations between particles
- Contains all orders of V (exponential!)
Eikonal approximation: optical limit

- neglect all correlations

\[ S^n_{fi}(\mathbf{b}) = \langle \Phi_f | \prod_{j=1}^n S_j(\mathbf{b}_j) | \Phi_i \rangle \]

\[ S_i(\mathbf{b}_i) = e^{i\chi_i(\mathbf{b}_i)} \]

- for elastic often expressed in terms of g.s. density

\[ S^{n\text{OL}}_{fi}(\mathbf{b}) = \exp \left( i \sum_{j=1}^n \chi_j(\mathbf{b} - \mathbf{b}_j) | \Phi_i \rangle \right) \]
Eikonal approximation: integrated cross sections

- elastic scattering

$$\sigma_{el} = \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta \sin \theta \sigma(\theta)$$

$$= 2\pi \int_{0}^{\pi} d\theta \sin \theta |f(\theta)|^2$$

$$= \frac{\pi}{k^2} \sum_{L=0}^{\infty} (2L+1) |1 - S_{L}|^2$$

- reaction cross section

$$\sigma_R = \frac{\pi}{k^2} \sum_{L} (2L+1)(1 - |S_{L}|^2)$$

\[
\sigma_{el} = \int d^2b \left| 1 - S^{n}_{fi}(b) \right|^2
\]

\[
= \int d^2b \left| 1 - \langle \Phi_0 | \prod_{j=1}^{n} S_j(b - b_j) | \Phi_0 \rangle \right|^2
\]

\[
\sigma_R = \int d^2b \left( 1 - |\langle \Phi_0 | \prod_{j=1}^{n} S_j(b - b_j) | \Phi_0 \rangle|^2 \right)
\]
Accuracy of the eikonal $S(b)$ and cross sections

J.M. Brooke, J.S. Al-Khalili, and J.A. Tostevin PRC 59 1560
Accuracy of the eikonal $S(b)$ and cross sections

J.M. Brooke, J.S. Al-Khalili, and J.A. Tostevin PRC 59 1560
Example: $^9\text{Be} (^{17}\text{C},^{16}\text{C} \gamma) \text{X}$ at 70 MeV/A
Eikonal approximation: integrated cross sections

- stripping $B(A,c)X$ can be constructed
  - probability of core surviving $|S_c|^2$
  - probability of valence being absorbed $1 - |S_v|^2$

$$\sigma_{str} = \frac{1}{2I_p+1} \int dB \sum_m \langle \Phi_{0m} | \left[ |S_c|^2 (1 - |S_v|^2) \right] | \Phi_{0m} \rangle$$
Eikonal approximation: integrated cross sections

- breakup scattering amplitude

\[ f_{bu}(k, \theta) = -\frac{iK_0}{2\pi} \int d^2b \ e^{i\mathbf{q} \cdot \mathbf{b}} \langle \Phi_{k,m'} | S_{2b} | \Phi_{0m} \rangle \]

- use completeness relation

\[ \sum_{m'} \int dk \ | \Phi_{k,m'} \rangle \langle \Phi_{k,m'} | = 1 - \sum_m | \Phi_{0m} \rangle \langle \Phi_{0m} | \]

\[ S_{2b} = S_c(b_c)S_v(b_v) \]

- obtain integrated breakup cross section

\[ \sigma_{bu} = \frac{1}{2I_p+1} \int d^2b \ \sum_{mm'} \left( \langle \Phi_{0m'} | |S_{2b}|^2 | \Phi_{0m} \rangle \delta_{mm'} - |\langle \Phi_{0m'} | S_{2b} | \Phi_{0m} \rangle|^2 \right) \]
Contributions are from nuclear surface and beyond

$^{12}\text{Be} + ^{9}\text{Be} \rightarrow ^{11}\text{Be(gs)} + X, \text{80A MeV}$

$ b \geq R_C + R_T $
Non-point particles: such as in knockout reactions

Elastic scattering of composite nuclei or description of one or two-nucleon removal – at ~100 MeV/nucleon

How to describe? - what can we learn from these?

Absorptive cross sections - target excitation

Since our effective interactions are complex all our $S(b)$ include the effects of absorption due to inelastic channels

$$\sigma_{\text{abs}} = \sigma_R - \sigma_{\text{diff}} = \int \mathrm{d}b \, \left\langle \phi_0 \left| 1 - |S_c S_v|^2 \right| \phi_0 \right\rangle$$

\[
\begin{align*}
|S_v|^2 (1 - |S_c|^2) + \\
|S_c|^2 (1 - |S_v|^2) + \\
(1 - |S_c|^2)(1 - |S_v|^2)
\end{align*}
\]

- $v$ survives, $c$ absorbed
- $v$ absorbed, $c$ survives
- $v$ absorbed, $c$ absorbed

$$\sigma_{\text{strip}} = \int \mathrm{d}b \, \left\langle \phi_0 \left| S_c^2 (1 - |S_v|^2) \right| \phi_0 \right\rangle$$

Related equations exist for the differential cross sections, etc.
Stripping of a nucleon – nucleon ‘absorbed’

\[ \sigma_{\text{strip}} = \int \, db \, \langle \phi_0 \parallel S_c \parallel^2 (1 - |S_1|^2) \parallel \phi_0 \rangle \]
Diffractive dissociation of composite systems

The total cross section for removal of the valence particle from the projectile due to the break-up (also called diffractive dissociation) mechanism is the break-up amplitude, summed over all final continuum states, i.e.

$$\sigma_{\text{diff}} = \int dk \int db \left| \langle \phi_k | S_c(b_c) S_v(b_v) | \phi_0 \rangle \right|^2$$

but, using completeness of the break-up states

$$\int dk \left| \phi_k \right> \left< \phi_k \right| = 1 \left| \phi_0 \right> \left< \phi_0 \right| + \left| \phi_1 \right> \left< \phi_1 \right| \ldots$$

If > 1 bound state

can (for a weakly bound system with a single bound state) be expressed in terms of only the projectile ground state wave function as:

$$\sigma_{\text{diff}} = \int db \left\{ \left| \langle \phi_0 | S_c S_v \right| \left| \phi_0 \rangle \right|^2 - \left| \langle \phi_0 | S_c S_v | \phi_0 \rangle \right|^2 \right\}$$
Diffractive (breakup) removal of a nucleon

\[ \sigma_{\text{diff}} = \int db \left\{ |\langle \phi_0 | S_c S_v | \phi_0 \rangle|^2 - |\langle \phi_0 | S_c S_v \phi_0 \rangle|^2 \right\} \]