Scattering theory: identical particles

Identical particles: symmetrization

- if two groups of nucleons are far apart, their wavefunctions do not overlap – exchange of nucleons cannot affect observables
- antisymmetrization tends to be less important in scattering than in structure
- exception if projectile and target are the same – exchange amplitudes!
- mass of proton almost the same as mass of neutron: treat protons and neutrons as almost identical particles

- isospin – distinguishes protons and neutrons

\[ \hat{\tau}_z |n\rangle = +1 |n\rangle \quad \text{and} \quad \hat{\tau}_z |p\rangle = -1 |p\rangle. \]
Nucleon’s isospin

- similarity with spin ½ algebra
  \[ |n\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |p\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
  \[ \hat{t}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{t}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]
  \[ \hat{t}_z = |p\rangle \langle n| + |n\rangle \langle p| \]
  \[ \hat{t}_y = i(|p\rangle \langle n| - |n\rangle \langle p|) \]
  \[ \hat{t}_z = |n\rangle \langle n| - |p\rangle \langle p| \]
  \[ [\tau_x, \tau_y] = 2i\tau_y \]

- isotopic spin operators
  \[ \hat{t} = \frac{1}{2} \hat{t} = \frac{1}{2}(\hat{t}_x, \hat{t}_y, \hat{t}_z) \]
  \[ t_z = +1/2 \text{ for the neutron and } -1/2 \text{ for the proton} \]

- raising and lowering operators
  \[ \hat{t}_+ = t_x + it_y \]
  \[ \hat{t}_- = t_x - it_y \]
  \[ \hat{t}_+ = \frac{1}{2} |n\rangle \langle p| \text{ and } \hat{t}_- = \frac{1}{2} |p\rangle \langle n| \]

Isospin for composite systems

- total isospin: \[ \hat{T} = \sum_k \hat{t}_k \]

Consider the 2-nucleon system => T=0 or T=1
- T=0 \( T_z=0 \) \( n+p \)
- T=1 \( T_z=-1 \) \( p+p \); \( T_z=0 \) \( n+p \); \( T_z=+1 \) \( n+n \)

Generalized antisymmetrization principle
- gives symmetry of wfns with respect to interchange of all coordinates (space, spin and isospin).
- For two nucleons \[ \Psi(1,2) = -\Psi(2,1) \]
Generalization antisymmm principle

- Considering a general case 1+2
  \[ \Psi_J(1, 2) = [Y_L(R) \otimes [s_1 \otimes s_2]_J]_L [T_1 \otimes T_2]_T \]

- Interchanging 1 and 2
  \[ \Psi_J(2, 1) = [Y_L(-R) \otimes [s_2 \otimes s_1]_J]_L [T_2 \otimes T_1]_T \]
  \[ = (-1)^L (-1)^{L+S+T}(-1)^{T-T_1-T_2}\Psi_J(1, 2) \]
  \[ = (-1)^{L+S+T}\Psi_J(1, 2) \]

- More complex nuclei – T still good quantum number although Coulomb effects are larger. Still find many similar states when replacing a neutron by a proton (isobaric analogue states)

Isobaric analogue states
Direct and exchange amplitudes

- so far we have considered $p + t \rightarrow p' + t'$
- now we need to consider:
  - (a) Scattering of identical fermions: $p = t$ of odd baryon number;
  - (b) Scattering of identical bosons: $p = t$ of even baryon number; and
  - (c) Exchange scattering: $p' = t$ and $t' = p$, and $p$ is distinguishable from $t$,

- consider an exchange index
  - $\varepsilon = +1$ for boson-boson
  - $\varepsilon = -1$ for fermion-fermion

\[
\tilde{P}_{pt} \tilde{\Psi}_{x, M_{pt}}^{\text{out}}(R_x, \xi_p, \xi_t) - \varepsilon \tilde{\Psi}_{x, M_{pt}}^{\text{out}}(R_x, \xi_t, \xi_p)
\]

\[\varepsilon = (-1)^{2I_p}\]

Direct and exchange amplitudes

- first identical spinless particles

\[
\psi^{\text{asym}}(R) = A \left[ e^{ikz} + f(\theta) \frac{e^{ikR}}{R} \right]
\]

- for two identical particles the wfn should be

\[
\Psi_{\varepsilon}^{\text{asym}}(R) = \psi^{\text{asym}}(R) + \varepsilon \psi^{\text{asym}}(-R)
\]

- scattered outgoing wave properly symmetrized should be:

\[
\Psi_{\varepsilon}^{\text{out}}(R) = A[f(\theta) + \varepsilon f(\pi - \theta)] \frac{e^{ikR}}{R} \equiv A f_{\varepsilon}(\theta) \frac{e^{ikR}}{R}
\]
Direct and exchange amplitudes

- cross section for identical particle scattering
  \[ \sigma(\theta) = |f_E(\theta)|^2 = |f(\theta) + \epsilon f(\pi - \theta)|^2 = |f(\theta)|^2 + |f(\pi - \theta)|^2 + 2\epsilon \text{Re} f(\theta)^* f(\pi - \theta). \]

- the partial wave expansion for the scattering amplitude is:
  \[ f_E(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1)P_L(\cos \theta)T_L[1 + \epsilon (-1)^L]. \]
  Odd partial waves do not contribute!
  Even partial waves are doubled!

\[ P_L(\cos(\pi - \theta)) = (-1)^L P_L(\cos \theta) \]

Direct and exchange with spin

- permutation best done in LS coupling:
  \[ \hat{P}_{ll'}[L(l_p,l_s);J_{tot}(\lambda)] = (-1)^L (-1)^{S-l_p-l_h} [L(l_p,l_s);J_{tot}(\lambda)] \]

- the partial wave expansion for the scattering amplitude is:
  \[ f_S(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1)P_L(\cos \theta)T_L[1 + \epsilon (-1)^{L+S-l_p-l_h}] = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1)P_L(\cos \theta)T_L[1 + (-1)^{L+S}] \]
  For S=0 odd partial waves do not contribute!
  For S=1 even partial waves do not contribute!
Direct and exchange with spin

- characteristic interference patterns for different spin states!

![Graph showing classical and quantum cross sections for different spin states](image)

Fig. 3.9. Fermion singlet (a) and triplet (b) nucleon-nucleon scattering cross sections, assuming pure Coulomb scattering with $\eta = 5$. Case (a) also applies for boson scattering. The cross section is in units of $\eta^2/4\pi^2$.

Integrated cross sections (2nd week!)

- use properties of Legendre polynomials

$$
\sigma_{cl} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \sigma(\theta)
= 2\pi \int_0^\pi d\theta \sin \theta |f(\theta)|^2
= \frac{\pi}{k^2} \sum_{L=0}^\infty (2L+1)(1 - S_L^2)
= \frac{4\pi}{k^2} \sum_{L=0}^\infty (2L+1) \sin^2 \delta_L.
$$
Integrated cross section and symmetrization...

- Elastic cross section is removal from the beam

\[ \sigma_{el} = \frac{1}{2} \int_0^{2\pi} d\phi \int_0^\pi d\theta |\sin \theta| \sigma(\theta) \]
\[ = \frac{2\pi}{k^2} \sum_{L+5 \text{ even}} (2L+1)[1 - |S_L|^2] \]

- Reaction cross section

\[ \sigma_R = 2 \frac{\pi}{k^2} \sum_{L+5 \text{ even}} (2L+1)[1 - |S_L|^2] \]

- Channel cross section

\[ \sigma_{\text{channel}} = \frac{\pi}{k^2} \sum_{J_{\text{ex}}=0,1} \frac{1 + \delta_{p_1t_i}}{(2J_{\text{ex}}+1)(2J_{\text{ex}}+1)} \sum_{J_{\text{ex}}=0,1} \left| S_{J_{\text{ex}}} \right|^2 \]

\[ 1 + \delta_{p_1t_i} \text{ Doubles cross section if projectile and target are identical and in the same state.} \]

Zero otherwise.

Exchange transfer

- How would you picture this reaction?

\[ ^6\text{He} + ^4\text{He} \rightarrow ^4\text{He} + ^6\text{He} \]

- Elastic or transfer? Both add coherently to the amplitude

\[ \Psi_\epsilon = \psi_{p_1} + \epsilon \hat{P}_{p'_1 \tau'_1} \psi_{p'_1} \]