PHY982 12th week

Electromagnetic field and coupling with photons

Maxwell’s equations

- magnetic H and electric E fields are related
  
  Charge current and charge density \( \mathbf{j}_q \) and \( \rho_q \)

\[
\begin{align*}
\nabla \times \mathbf{H} &= \frac{4\pi}{c} \mathbf{j}_q + \frac{1}{c} \frac{d\mathbf{E}}{dt} \\
\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{d\mathbf{H}}{dt} \\
\n\nabla \cdot \mathbf{H} &= 0 \\
\n\nabla \cdot \mathbf{E} &= 4\pi \rho_q.
\end{align*}
\]

- radiative capture reactions: coupling of photons and the nucleus
  one radiative photon at a time

\[
E_\gamma = \hbar \omega
\]

\[
p_\gamma = \hbar k_\gamma = E_\gamma / c
\]
Maxwell’s equations

- vector potential and scalar potential
  \[ \mathbf{H} = \nabla \times \mathbf{A}(t), \]
  \[ \mathbf{E} = -\nabla \phi(t) - \frac{1}{c} \frac{d\mathbf{A}}{dt} \]

- gauge invariance
  \[ \mathbf{A}'(t) = \mathbf{A}(t) + \nabla \chi(t) \]
  \[ \phi'(t) = \phi(t) + \partial \chi(t)/\partial t \]

- transverse gauge (Coulomb gauge)
  \[ \nabla \cdot \mathbf{A}(t) = 0 \]
  \[ \nabla^2 \phi(t) = -4\pi \rho_q(t) \]
  \[ \nabla^2 \mathbf{A}(t) + \frac{1}{c^2} \frac{d^2 \mathbf{A}}{dt^2} = -\frac{4\pi}{c} \mathbf{j}_q + \frac{1}{c} \nabla \frac{d\phi}{dt} \]

Maxwell’s equations

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- gauge invariance
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  \[ \phi'(t) = \phi(t) + \partial \chi(t)/\partial t \]

- transverse gauge (Coulomb gauge)
  - one photon approximation (density and scalar field at time independent)
  \[ \nabla^2 \mathbf{A}(t) + \frac{1}{c^2} \frac{d^2 \mathbf{A}}{dt^2} = -\frac{4\pi}{c} \mathbf{j}_q \]
  \[ \mathbf{A}(t) = \mathbf{A} e^{-i\omega t} + \mathbf{A}^* e^{i\omega t} \]
  \[ \mathbf{j}_q(t) = \mathbf{j} e^{-i\omega t} + \mathbf{j}^* e^{i\omega t} \]
  \[ \nabla^2 \mathbf{A} + k_e^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{j} \]
Coupling photons and particles

Connecting the classical EM description with quantum particle dynamics

- free field source flux
  \[ (\Phi | \hat{j}_{\text{free}} | \Psi) = \frac{\hbar q}{2i\mu} [\Phi^*(\nabla \Psi) - (\nabla \Phi)^* \Psi] \]

- particle current to photon production (capture)
  \[ \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\frac{4\pi}{c} \langle \Phi_b | \hat{j}_{\text{free}} | \Psi \rangle \]
  \[ \left[ -\frac{\hbar c}{k\gamma} \nabla^2 - E_\gamma \right] \mathbf{A}(\mathbf{r}) = \frac{4\pi \hbar}{k\gamma} \langle \Phi_b | \hat{j}_{\text{free}}(\mathbf{r}) | \Psi \rangle \]
  \[ H_{\gamma p} = \frac{4\pi \hbar}{k\gamma} \langle \Phi_b | \hat{j}_{\text{free}} | \Psi \rangle \]

Coupling photons and particles

- photon current to particle flux
  minimal gauge coupling
  \[ \hat{p} \to \hat{p} - \frac{q}{c} \mathbf{A} \]
  \[ [\hat{p}^2/2m + V - E] \Psi = 0 \]
  \[ \left[ \frac{1}{2\mu} \left( \frac{\hat{p}^2}{c^2} [\mathbf{A} \cdot \hat{p} + \hat{p} \cdot \mathbf{A}] + \frac{q^2}{c^2} \mathbf{A}^2 \right) + V - E \right] \Psi = 0 \]

- photo-disintegration equation
  \[ [\hat{T} + V - E] \Psi - \frac{1}{c} \int d\mathbf{r} \mathbf{A}(\mathbf{r}) \cdot \hat{j}(\mathbf{r}) \Psi = 0 \]

- initial state is bound state and we need equation coupling the incoming photon field to the particle in the continuum
  \[ [\hat{T} + V - E] \Psi - \frac{1}{c} \int d\mathbf{r} (\hat{j}(\mathbf{r}) \Phi_b) \cdot \mathbf{A}(\mathbf{r}) = 0 \]
Photon flux and normalization

- photon outgoing wave
  \[ A(r) = a e^{ik \cdot r} \]

- The Poynting vector provides energy flux
  \[ S_P = \frac{c}{4\pi} E \times H \]
  \[ |S_P| \approx \frac{c^{2}}{4\pi} |E|^2 = \frac{c}{4\pi} |H|^2 \]
  \[ \langle |S_P(t)| \rangle = \frac{k_{\gamma}^2 c}{2\pi} |a|^2 \]

- outgoing photon flux is energy flux per photon energy
  \[ j_\gamma = \frac{k_{\gamma}}{2\pi \hbar} |a|^2 = c |Z(r)|^2 \]

- normalized photon wave
  \[ Z(r) = \sqrt{\frac{k_{\gamma}}{2\pi \hbar c}} A(r) \]

Coupled photon-particle equations

- photo-production
  \[ \left[-\frac{\hbar c}{k_{\gamma}} \nabla^2 - E_\gamma\right] Z(r) = 2\sqrt{\hbar/\omega} \langle \Phi_b | \hat{j}(r) \rangle \Psi \]

- photo-disintegration
  \[ \left[ \hat{T} + V - E \right] \Psi = \sqrt{\hbar/\omega} \int dr \hat{j}(r) \Phi_b \cdot Z. \]

- generalization of wavefunction to incorporate photon channel
  \[ \Psi_{j_{tot}}^{M_{tot}} = \Psi_{\gamma \eta_{tot}}^{M_{tot}} + \Psi_{\mu j_{tot}}^{M_{tot}} = \sum_{\gamma} |\gamma\rangle \zeta_{\gamma}(r)/r + \sum_{\alpha} |\alpha\rangle \psi_{\alpha}(R)/R, \]
Partial wave decomposition for photons

- Expanding photon into vector spherical harmonics
  \[
  Y^M_{\Lambda J}(\mathbf{r}) = \langle \Lambda 1 | J \rangle \sum_{m\mu} \langle \Lambda m, 1 \mu | JM \rangle \xi_{\mu}(\mathbf{r}) Y^m_{\Lambda}(\mathbf{r})
  \]

- Radial coupled channel equations
  \[
  |T_\Lambda - E_{\gamma'}| \xi_{\gamma'}(r) = \sum_{\alpha} \langle \gamma' | V_{\gamma'\alpha} | \alpha \rangle \psi_{\alpha}(R)
  \]
  \[
  [T_{\gamma} \quad V_{\gamma}] \psi_{\alpha}(R) = \sum_{\gamma'} \langle \alpha | V_{\gamma'\gamma} | \gamma' \rangle \xi_{\gamma'}(r),
  \]
  \[
  |\gamma'\rangle = \langle \Lambda 1 | J_{\gamma'}, J_{\beta}\rangle
  \]
  
  - From asymptotics obtain T-matrix or S-matrix and construct observables.

Capture cross sections

- Starting from incoming particle channel
  \[
  \sigma_{\text{cap}} = \frac{4\pi}{k^2} \frac{1}{(2I_{p_1}+1)(2I_{i}+1)} \frac{c}{\hbar} \sum_{J_{\text{tot}} \alpha_i} (2J_{\text{tot}}+1) |T^\text{lm} |^2
  \]

- T-matrix equivalent integral expression
  \[
  T^\text{lm}_{\gamma\alpha} = -\frac{1}{\hbar c} \sum_{\alpha} \langle \xi^{(-)}(r) | \alpha | V_{\gamma\alpha} \psi_{\alpha}(R) \rangle
  \]

  - No potential in photon wave!
  \[
  T^\text{lm}_{\gamma\alpha} = \frac{1}{\hbar c} \sum_{\alpha} \langle F_\Lambda(0,k_r) | \alpha | V_{\gamma\alpha} \psi_{\alpha}(R) \rangle,
  \]
  \[
  = \frac{1}{\hbar c} \sum_{\alpha} \int dr \, \frac{1}{r \hbar} F_\Lambda(0,k_r) \phi_\alpha^* V_{\gamma\alpha} \phi_\alpha \psi_{\alpha}(R).
  \]

  - All analysis ignored gauge invariance... \( \mu = -1, 0, +1 \)
Photon gauge invariant plane wave

- incoming photon along z axis:
  \[ \nabla \cdot A = \nabla \cdot Z = 0 \]
  \[ A_0 = Z_0 = 0 \]
  
  Polarization vectors \( \xi_{\pm 1} \) cover all

\[ \xi_{\mu}, e^{i k \cdot r} = \sum_{JM, \Lambda m, \mu} i^J Y_{\Lambda}^m (r) \xi_{\mu} \langle \Lambda m, 1 \mu | JM \rangle F_{\Lambda} (0, kr) / r \]
\[ \times \frac{4\pi}{k} \sum_{\Lambda_i m_i} Y_{\Lambda_i}^{m_i} (k) \ast \langle \Lambda_i m_i, 1 \mu_i | JM \rangle. \]

Photon gauge invariant plane wave

- incoming photon along z axis:
- splitting into magnetic and electric parts

\[ \nabla \cdot A = \nabla \cdot Z = 0 \]
\[ A_0 = Z_0 = 0 \]

Polarization vectors \( \xi_{\pm 1} \) cover all

\[ A_{\mu} = \xi_{\mu}, e^{i k \cdot r} = \mu \sqrt{2\pi} \sum_{J} \sqrt{2J + 1} i^J [A_{J\mu}(r; M) + i \nu A_{J\nu}(r; E)] \]

\[ A_{JM}(r; E) = \frac{J + 1}{2J + 1} \frac{1}{(kr)^{1 - J}} F_{J - 1}(0, kr) Y_{J - 1, J}^{M}(\hat{r}) \]
\[ - \frac{J}{2J + 1} \frac{1}{(kr)^{J + 1 - 1}} F_{J + 1}(0, kr) Y_{J + 1, J}^{M}(\hat{r}). \]

\[ A_{JM}(r; M) = (kr)^{-J} F_{J}(0, kr) Y_{J, J}^{M}(\hat{r}) \]
EM field normalization and other properties

- The vector field is normalized like a plane wave

\[ \int A_{JM}(\mathbf{r}; e) \cdot A_{JM'}(\mathbf{r}; e') \, d\mathbf{r} = \delta(k - k') \delta_{JJ'} \delta_{MM'} \delta_{ee'} \]

- They are related by:

\[ A_{JM}(\mathbf{r}; M) = \frac{1}{ik} \nabla \times A_{JM}(\mathbf{r}; E) \]

\[ A_{JM}(\mathbf{r}; E) = \frac{1}{ik} \nabla \times A_{JM}(\mathbf{r}; M), \]

\[ \nabla \cdot A_{JM}(\mathbf{r}; e) = 0 \quad \text{for} \quad e = M \text{ or } E. \]

\[
A_{JM}(\mathbf{r}; E) = \frac{(-1)^{J}}{\sqrt{2J+1}} \sum_{\lambda} Y_{\lambda}^{J}(\hat{r}) \sum_{l=-J}^{J} \left( \frac{J-1}{2J+1} (-1)^{l} \right)^{1/2} Y_{l}^{M}(\mathbf{r}) Y_{J+1,\lambda}^{l}(\hat{r}).
\]

Selection rules

- Electric and magnetic transitions are classified not according to their spatial part \( \Lambda \) but their multipolarity: the total angular momentum \( J_{\gamma} \) that is transferred to or from the nucleus by the E1 and M1 photons have \( J_{\gamma} = 1 \), whereas the E2 and M2 have \( J_{\gamma} = 2 \), etc.

- When the additional parity change from the current operator is taken into account, electric transitions change nuclear parities according to \((-1)^{J_{\gamma}} \), and magnetic transitions by \((-1)^{J_{\gamma}-1} \). Parities are not changed by M1, E2 and M3 transitions, for example, but are changed by E1 and M2 multipoles.

- M0 and E0 transitions are not allowed by angular momentum selection rules.
Final state photons

- For photon in exit channel – generalize $k$ in $z$-axis for any direction:

$$\xi_{\mu} e^{ik\cdot r} = \mu \sqrt{2\pi} \sum_{JM} \sqrt{2J+1} i^J [A_{JM}(r; M) + i\mu A_{JM}(r; E)] D_{M\mu}^J(R_{k_i})$$

- Vector T-matrix can be written in this general form too:

$$T^\mu(k_\gamma, k_i; E) = \mu \sqrt{2\pi} \sum_{JM} \hat{j} i^{-J} D_{M\mu}^J(R_{k_i}) [A_{JM}(r; M) | V_{\gamma p} | \Psi(R; k_i)]$$

$$T^\mu(k_\gamma, k_i; E) = -i\mu^2 \sqrt{2\pi} \sum_{JM} \hat{j} i^{-J} D_{M\mu}^J(R_{k_i}) [A_{JM}(r; E) | V_{\gamma p} | \Psi(R; k_i)],$$

Longitudinal part of the field

- Does not satisfy the transverse gauge

$$A_{M}^{\nu}(r; \text{long}) = \frac{1}{k} \sqrt{\frac{J}{2J+1}} \hat{k}_\gamma \Psi_{1/2, J}^{M}(\hat{r})$$

- Differs only previous by coefficients

$$A_{M}^{\nu}(r; \gamma) = \frac{1}{k} \sqrt{\frac{J}{2J+1}} \hat{k}_\gamma \Psi_{1/2, J}^{M}(\hat{r})$$

- Long wavelength approximation (neglect $L=J+1$)

$$kr \ll 1$$
Electric transitions – standard form

- long wavelength approximation
  \[
  A_\text{JM}(r; \xi) \approx \sqrt{\frac{J+1}{J}} A_\text{JM}(r; \text{long}) = \sqrt{\frac{J+1}{J}} k \frac{1}{kr} F_J(0, kr) Y_j^M(\hat{r})
  \]

- Siegert’s theorem: uses continuity equation to transform current into charge density
  \[
  T_{J\ell}^{(E)} = \frac{s_J}{i\kappa J} \left( \nabla f_J(r) \left| V_{\gamma p} \right| \Psi(\mathbf{R}, k_f) \right) = i\kappa J \left( \nabla f_J(r) \right) \Phi_b \left| \hat{j}_q(r) \right| \Psi.
  \]
  Integration by parts
  \[
  T_{J\ell}^{(E)} = -i\kappa J \left( \Phi_b f_J(r) \right) \left| \nabla \cdot \hat{j}_q(r) \right| \Psi.
  \]

- Continuity eq + Schrodinger eq
  \[
  T_{J\ell}^{(E)} = -i\kappa J \left( \Phi_b f_J(r) \right) \left| \nabla \cdot \hat{j}_q(r) \right| \Psi = \frac{i\kappa J}{\hbar} \left( \delta(r-r_i) H - H^\dagger \delta(r-r_f) \right) \left| \Psi \right|
  \]

- long wavelength approximation
  \[
  F_J(0, kr) \approx \frac{1}{(2J+1)!!} (kr)^{J+1} \quad \text{thus} \quad f_J(r) \approx \frac{k^{J+1}}{(2J+1)!!} r^J Y_j^M.
  \]

- Siegert theorem gives a matrix element proportional to multipole operator
  \[
  \left< \Phi_b q f_J M | \Psi \right> \approx \frac{k^J}{(2J+1)!!} \left< \Phi_b q^J Y_j^M | \Psi \right>
  \]

- putting it back in the T matrix
  \[
  T_{J\ell}^{(E)} = -i\sqrt{4\pi} \sqrt{2J+1} \sqrt{\frac{8\pi \hbar c (J+1)}{k^3 J}} \left( \Phi_b q i^J f_J M | \Psi \right)
  \]
Extensions and limitation of framework

- T-matrix

\[ T_{\gamma\alpha} = -\frac{1}{k} \frac{q}{\hbar c} \sum_{\alpha} \frac{8\pi \hbar c (J+1)}{Jk} \frac{\langle F_{\gamma} \phi_{\alpha} \rangle}{(2J+1)!!} (kr)^{J+1} \]

- \( \Psi(R) \) – separation of two bodies whereas \( Z(r) \) – distance from center of mass non-local equation!

Extensions and limitations
- coupled channel framework can describe \( A+B \to (AB)\) to many channels
- only two bodies = photon+AB
- decay process \( AB \to A'+B' \) needs separate formulation (Hauser-Feshbach)

Capture cross sections and B(EL)

- electric multipole operator appearing in Coulomb inelastic scattering

\[ M_{JM}^0 = q r^J Y_J^M \]

- transition strength function from scattering to bound

\[ \frac{1}{h} \sum_{m_i M_i} \left| \langle \Phi_i^{m_i} \left| M_{JM}^0 \left| \frac{2}{\pi} \psi_i^{M_i} \right| \right. \right|^2 = (2J_i+1) \frac{d}{dE} B(E_J, i \to \gamma). \]

- photons in long wavelength approximation

\[ q_f(r) \simeq \frac{k_f+1}{(2J_i+1)!!} M_{JM}^0 \]

- capture cross section

\[ \alpha_{\text{cap}} = \frac{16\pi^3}{k_i^2} \frac{(2J_i+1) J+1}{(2J_i+1)(2J_i+1) \frac{1}{(2J_i+1)!!}^2} \frac{d}{dE} B(E_J, i \to \gamma). \]
Photo-absorption cross section

- Using detailed balance this can be obtained from capture

\[
\sigma_{\text{photo}} = \frac{k^2 (2J_p+1)(2J_l+1)}{k^2 2(2J_b+1) \sigma_{\text{cap}}} \\
= \frac{(2\pi)^3 (J+1)}{J((2J+1)!!)^2} \frac{k^{2J+1}}{2J+1} \frac{d\sigma(EJ_p \rightarrow \gamma)}{dE} \\
= \frac{(2\pi)^3 (J+1)}{J((2J+1)!!)^2} \frac{k^{2J+1}}{2J+1} \frac{d\sigma(EJ_\gamma \rightarrow p)}{dE}.
\]

- work out specific form for E1 – homework 6
- work out specific properties for protons – homework 8
- work out long wave approximation for magnetic field - homework 9

examples

![Example Output]

14\text{C}

NAMELIST
&FRESCO
  &homm=0.100&match=100
  &theta=0.045&absorb=-1
  &ebich=0.0000&ebich=50

APARTITION &name=neutron: mass=1.0007 zp=0 new=1 name='14C' mass=1.00032 zp=6
  &APARTITION &name=gamma: mass=0 new=1 name='14C' mass=0.00032 zp=6
  &STATES &rho=0.90000 &ep=0.00000 &theta=0.00000
  &STATES &rho=0.90000 &ep=0.00000 &theta=0.00000

&PART
  &Pot type=1 shape=0 q[1]=14.0000 0.0000
  &Pot type=1 shape=0 q[1]=57.0000 1.70.7
  &Pot type=1 shape=0 q[1]=0.0000 1.7
  &Pot type=2 shape=0 q[1]=14.0000 0.0000
  &Pot type=2 shape=0 q[1]=55.7700 1.2200
  &Pot type=2 shape=0 q[1]=5.0000 1.2200

&OVERLAP
  &overlap k1=1 k2=1 k1=2 k2=2

&SOURCE n=1 km=1 km=1 km=2 n=1 km=1 km=2

Fig. B.5. Neutron capture cross section for 14\text{C} as a function of neutron energy as calculated with the input of Box B.6.
Fig. 1.13. (a) The S-factors for the cross sections of $^{14}$N$p$,$y$,$^{15}$O reaction, to the ground state, and 6.18 MeV and 6.79 MeV excited states of $^{15}$O. The curves are hybrid R-matrix fits discussed in Appendix B. (b) The S-factor for the $^{14}$N$p$,$y$,$^{15}$O reaction.